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The two-dimensional Coulomb gas
in the critical region

by

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1. Introduction

In a previous paper¹ we derived by statistical mechanical graph expansion equations for the equilibrium properties of the two dimensional Coulomb gas. These equations were again obtained by extending results of another paper² by us where we considered a simple system which we from graph expansions showed could be interpreted in terms of a chemical equilibrium. I.e. a system consisting of atoms A and B which tend to bind into molecules AB, may be regarded as one of species A, B, and AB the effective densities of which are governed by the chemical equilibrium. In the same way oppositely charged ions of an ionic system tend to form neutral dipolar pairs. The added complication of the latter system is the long range nature of the Coulomb interaction. As we have pointed out before¹ our derivations are closely related to the renormalization technique of Quantum Field Theory and may thus throw some light upon the physics behind this technique. E.g. in the same way that we replace the total densities of the different ions with effective densities of free ions, the field theory replaces the bare mass of a particle with a renormalized mass.

Previously¹ we did only a simple incomplete analysis of our expressions in the case of two dimensions to which we will stick in the present analysis too. We found explicitly the pressure p in the ideal case of un-cut logarithmic Coulomb interaction. Above some collapse temperature we recovered the known exact result.³ Below this temperature all ions collapse into neutral pairs, and we verified that p in this region is nothing but the ideal gas pressure of the neutral pairs, which had been suggested, but not shown before.³

By cutting the Coulomb interaction inside a certain radius one obtains a situation which is more interesting due to its equivalence to a variety of other systems in two dimensions,^{4,5,6} to which others have tried to make approximations. This cut-off in the interaction will prevent the above mentioned collapse, but still ions will tend to form dipolar pairs. Neglecting the dielectric effects from these pairs we could perform a simple analysis of our expressions. The result of this computation was that the density of free ions became zero as the temperature went down to half the previous collapse temperature. Thus there would be a phase transition at this temperature below which there would be no free ions to create a Debye shielding. Such a phase transition has also been suggested by approximations performed by others.^{4,5,6}

In the present paper we will do a more careful analysis of our expressions including the dipolar effect of the neutral pairs. This makes the analysis much more hard to perform as the expressions will have an unavoidable singular behavior close to the transition point. Due to the dielectric properties the transition temperature will change with density. However, to leading singular order we are able to obtain explicit results which mainly confirm those of Kosterlitz and Thouless^{4,5} obtained by use of more direct physical arguments.

2. Analysis of the correlation function

Of special interest is the properties of the pair correlation function in the vicinity of the critical point when dielectric effects are taken into account. Due to the singular nature of the expressions they become hard to analyze in this region. We will here find the leading singular behavior. To establish the equations to be analyzed we will need the equations of ref. [1] . When referring to those equations we will precede them by an I. The system we consider is a symmetric electrolyte of oppositely charged ions.

In terms of the chain bond $C_{ij}(r)$ and the vertex function $\mu_{ij}(r)$ the pair correlation function $h_{ij}(r)$ is given by (I.8) and (I.9) ($i,j=1,2$)

$$\rho_{0i}\delta_{ij} + \rho_{0i}\rho_{0j}\tilde{h}_{ij}(k) = \tilde{\mu}_{ij}(k) + (\tilde{\mu}\tilde{C}\tilde{\mu})_{ij} \quad (1)$$

with

$$\tilde{C}(k) = \frac{\tilde{v}(k)}{1 - \tilde{\mu}(k)\tilde{v}(k)} \quad (2)$$

Here the ρ_{0i} ($i=1,2$) is the density of ions i . The tilde denotes Fourier transform, and $\tilde{C}$, $\tilde{v}$, and $\tilde{\mu}$ are matrices in these equations. The $v_{ij}(r)$ is the potential bond (I.5)

$$v_{ij}(r) = -\beta\phi_{ij}(r) \quad (3)$$

where $\phi_{ij}(r)$ is the cut Coulomb interaction for which we may use form (I.3) to be more explicit.

$$\phi_{ij}(k) = \frac{2\pi q_i q_j}{k^2 + \epsilon^2} \frac{1}{1 + (kr_0)^2} \quad (4)$$

The $q_i = \pm q$ is the ionic charge. The r_0 represents the small r cut-off of the Coulomb interaction, and ϵ^{-1} is a large r cut-off. We want ^{to study} the limit $\epsilon r_0 \rightarrow 0$ with $\rho_{0i} r_0^2$ fixed. As explained between (I.2) and (I.3) an $\epsilon > 0$ is strictly speaking needed in two dimensions to perform calculations properly after which the limit $\epsilon \rightarrow 0$ is taken. So we will keep the ϵ when considered necessary.

To get a closed set of equations for given β , ρ_{0i} and $\phi_{ij}(r)$ eqs. (I.13) – (I.17) are needed too.

$$\mu_{ij}(r) = \mu_{ij}(r) - \rho_{0i} \delta_{ij} \delta(\vec{r}) = \rho_i \rho_j f_{ij}(r)$$

$$f_{ij}(r) = \exp(C_{ij}(r)) - C_{ij}(r) - 1$$

$$K_{ij} \rho_i \rho_j = \rho_{ij} \tag{5}$$

$$\rho_{0i} = \rho_i + \sum_j \rho_{ij}$$

$$K_{ij} = \int [\exp(C_{ij}(r)) - \frac{1}{2}(C_{ij}(r))^2 - C_{ij}(r) - 1] d\vec{r}$$

The K_{ij} may be interpreted as the constant of chemical equilibrium, while the ρ_i is the density of free unbound ions i . Due to the symmetry of the problem we may simplify our equations to avoid matrices, i.e. we get effectively a one component system. [The $\rho_{01} = \rho_{02}$ and $v_{11} = v_{22} = -v_{12}$ which imply $\rho_1 = \rho_2$ and $C_{11} = C_{22} = -C_{12}$]. Thus we introduce (I.28) and (I.30)

$$\rho_e = \rho_1 + \rho_2, \quad \rho_n = \rho_{11} + \rho_{22} + 2\rho_{12}, \quad \rho_0 = \rho_e + \rho_n = \rho_{01} + \rho_{02}$$

$$v(r) = v = v_{11}, \quad C(r) = C = C_{11}$$

$$\mu(r) = \mu_{11} + \mu_{22} - 2\mu_{12} = \rho_0 \delta(\vec{r}) - \rho_e^2 \left[\frac{1}{2} (e^{-C} - e^C) + C \right] \quad (6)$$

$$K = \frac{1}{4} (K_{11} + K_{22} + 2K_{12}) = \int \left[\frac{1}{2} (e^{-C} + e^C) - \frac{1}{2} C^2 - 1 \right] d\vec{r}$$

With this replacement Eq.(2) will stay unchanged except that now the $\tilde{C}$, $\tilde{\mu}$ and $\tilde{v}$ will become scalar quantities. The equation of chemical equilibrium becomes

$$K \rho_e^2 = \rho_n \quad (7)$$

The ρ_e may be interpreted as the density of free ions while ρ_n will be twice the density of pairs. Eqs.(6) and (7) will now replace Eqs.(5).

By use of (3), (4), and (6) the $\tilde{v}$ may be substituted in (2) to give

$$-\tilde{C}(k) = \frac{2\pi\beta q^2}{k^2(1+(\epsilon r_0)^2) + k^4 r_0^2 + \epsilon^2 + \chi_0^2 - \Pi(k)} \quad (8)$$

where

$$\chi_0^2 = 2\pi\beta q^2 \rho_0 \quad (9)$$

and

$$\Pi(k) = 2\pi\beta q^2 (\rho_0 - \tilde{\mu}(k)) = -2\pi\beta q^2 \rho_e^2 \int d\vec{r} e^{i\vec{k}\vec{r}} [\sinh(C) - C] \quad (10)$$

Near the critical point the $-C(r)$ will become large for small r . Thus the integrals giving $\Pi(0)$ and K may be put equal, i.e.

$$\chi_e^2 = 2\pi\beta q^2 \rho_e = 2\pi\beta q^2 (\rho_0 - \rho_n) = \chi_0^2 - 2\pi\beta q^2 K \rho_e^2 = \chi_0^2 - \Pi(0) \quad (11)$$

The χ_e will be the inverse effective shielding length due to free ions ($\epsilon \rightarrow 0$). For small k

$$\Pi(0) - \Pi(k) = (\epsilon_0 - 1)k^2 \quad (12)$$

The ϵ_0 will have the effect of a macroscopic dielectric constant, and one obtains an effective inverse temperature

$$\beta_e = \beta / \epsilon_0 \quad (13)$$

Neglecting the effect of $\epsilon_0 (\neq 1)$ the phase transition was located at $\beta q^2 = 4$. Taking ϵ_0 into account it turns out that the transition must occur at $\beta_e q^2 = 4$ if the equations give a proper behavior close to this point, and ϵ_0 stays finite. We will see that this latter problem is far from trivial. Computing the ϵ_0 via the simple minded approach of assuming (12) to hold for all k one realizes that $\epsilon_0 \rightarrow \infty$ at the transition which means $\beta \rightarrow \infty$ at this point. At the same time one finds finite β for which obviously $\chi_e \rightarrow 0$ (as $\epsilon \rightarrow 0$) which means that we should be below the transition in temperature. Thus such a simple approach will give a controversial situation close to a possible phase transition. This means that we have to be more careful to extract the singular behavior to leading order, if our equations are meaningful at all close to the transition. Obviously in a more ambitious computation we have to deal with some k -dependent dielectric constant in (12).

From (8) and (11) one concludes that $C(r)$ has a large r cut-off at the inverse shielding length $(\epsilon^2 + \chi_e^2)^{-\frac{1}{2}}$, and a small r cut-off r_0 . I.e. for larger r the $C(r)$ may be considered zero, and for smaller r it is roughly constant. The more interesting region is between these two-limits where $C(r)$ will closely resemble the

Coulomb interaction, except for a slowly varying k - or r -dependent dielectric constant. We may thus apply the results derived in Appendix A in this region, where the k and r are substituted by the variables v and η defined by (A7) and (A12) with $R=(\epsilon^2+\chi_e^2)^{-\frac{1}{2}}$. Eq.(8) may be written as

$$- \tilde{C}(k) = \frac{2\pi}{k^2 \hat{u}(v)} \quad (14)$$

with $(\epsilon \rightarrow 0)$

$$\hat{u}(v) = \frac{1}{\beta q^2} \left[1 + e^{2(v-1)s_+} e^{-2vs} + \frac{1}{k^2} (\Pi(0) - \Pi(k)) \right] \quad (15)$$

Utilizing result (A19) and Eqs.(A3) and (A4) we immediately find $(r=2Re^{-\eta s}, n=2)$

$$- C(r) = - \hat{C}(\eta) = s \int_{-\infty}^{\eta} \frac{dv}{\hat{u}(v)} \quad (16)$$

This should be valid to leading order for large $s = \ln\left(\frac{R}{r_0}\right)$ and $0 < \eta < 1$, but will be meaningful for any η due to the exponential terms in (15). In fact for $r=0$ ($\eta \rightarrow \infty$) integral (16) is nothing but the proper Fourier inversion integral replacing v with k .

To find the $\Pi(k)$ the result of Appendix B may be utilized since near the critical point the integrand of (10) will be a function like (B1) where $H(r)$ is slowly varying for $0 < \eta < 1$. Utilizing (B7) in (15) we thus obtain

$$\hat{u}(v) = s \int_v^{\infty} \hat{z}(\eta) d\eta + \frac{1}{\beta q^2} \left[1 + e^{2(v-1)s_+} e^{-2vs} \right] \quad (17)$$

where

$$\hat{z}(\eta) = \frac{-4\sigma}{(\beta q^2)^2} \left[\sinh(\hat{C}(\eta)) - \hat{C}(\eta) \right] e^{-4\eta s} \quad (18)$$

with

$$\sigma = \left(2\pi\beta q^2 \rho_e R^2 \right)^2 = \left(\frac{\chi_e^2}{\epsilon^2 + \chi_e^2} \right)^2$$

Eqs. (16)-(18) when integrated will determine $C(r)$. This may be done numerically. To do so the integral in (17) with $v = -\infty$ has to be given an apriori estimate which will be checked by the solution, and may thus be corrected by iteration. The βq^2 and $s = \ln R/r_0$ are then considered known.

However, for large $s (\rightarrow \infty)$ and in the region $0 < \eta, v < 1$ the exponentials in (17) may be neglected, and expression (18) may be simplified to a simple exponential as $-\hat{C}(\eta)$ becomes large. Outside this region the $\hat{z}(\eta)$ may be neglected.

With

$$\begin{aligned} u &= u(x) = \hat{u}(v) \quad , \quad x = -sv \\ z &= z(x) = \hat{z}(\eta) \quad , \quad x = -s\eta \end{aligned} \quad (19)$$

Eqs. (17) and (18) may be differentiated by use of (16) to obtain

$$\begin{aligned} \frac{du}{dx} &= z \\ \frac{dz}{dx} &= \left(4 - \frac{1}{u} \right) z \end{aligned} \quad (20)$$

It turns out that these differential equations are precisely those of ref. [5] (by simple substitution of variables) and thus also corresponds to those of ref. [4] (upon which ref. [5] builds). Solution of (20) with appropriate boundary conditions will lead to $C(r)$ and other quantities of interest. Dividing Eqs.(20) with each other one obtains the simple differential equation

$$\frac{du}{dz} = 4 - \frac{1}{u} \quad (21)$$

which solved yields

$$4u - \ln u = z + \text{const.} \quad (22)$$

This directly relates u and z . Substituted back in (20) the u in terms of x may be obtained by integration which can not be performed analytically. However, Eq.(22) may be expanded around its minimum at $u = \frac{1}{4}$ to get

$$z = 8 \left[\left(u - \frac{1}{4} \right)^2 \pm a^2 \right] \quad (23)$$

by which more explicit results may be obtained. (The a is some constant.) This inserted in (20) gives a differential equation for u that may be integrated in a simple way. With plus sign in (23) we obtain

$$\text{Arccotg} \left(\frac{u - \frac{1}{4}}{a} \right) = -\tau$$

with

$$\tau = 8a(x+x_0) \quad (24)$$

where x_0 is some constant of integration. Eq.(24) inverted yields

$$u = \frac{1}{4} - a \cotg(\tau) \quad (25)$$

We will also need the minus sign in (23). The solution for this latter case is now easily obtained from (25) by substituting a with ia to obtain

$$u = \frac{1}{4} - a \coth(\tau) \quad (26)$$

(It may be mentioned that in (25) "Arccotg" might have been replaced with "Arctg" with a corresponding replacement in (26). This latter possibility would give another solution in (26), which, however, will not be of interest in the present case as it will yield a negative z). With (25) and (26) inserted in (23) we find for z

$$z = \frac{8a^2}{\sin^2 \tau} \quad \text{with } + \text{ sign in (23)} \quad (27)$$

$$z = \frac{8a}{\sinh^2 \tau} \quad \text{with } - \text{ sign in (23)}$$

With our approximations for large s the $C(r)$ is most easily obtained from (18) with (27) inserted, i.e. with $\beta q^2 \approx \frac{1}{4}$ ($r=2Re^x$)

$$-C(r) = -4x + \ln(8z) - \ln \sigma \quad (28)$$

Our solution contains constants a and x_0 which have to be determined. There will be three different cases to consider; one with plus sign in (23) and two with minus signs. These two latter cases are for $u > \frac{1}{4}$ and $u < \frac{1}{4}$ respectively. These two regions are separated by the non relevant region around $u = \frac{1}{4}$ where $z < 0$.

At the large cut off radius $r=2R$ or $x=0$ the $C(r) \approx 0$. With (27) inserted in (28) this means $\tau \sim a$, or $x \sim 1$ ($\sigma \sim 1$ as $\epsilon \rightarrow 0$). For large s we thus may put $x_0=0$ or

$$\tau = 8ax \quad (29)$$

As we will see this boundary condition will work only when s stays finite, which will be in two of the above three cases. In the third case we must be more careful and remember the ϵ in Eq.(4).

The region of interest for x will be $-s < x < 0$, so from (29) the τ will be negative. We now may make a simple connection to the temperature. The $x = -s$ corresponds to $r=r_0$ and for small r the $C(r)$ behaves like the potential $v(r)$. Thus from consideration of (16) and (17) we find for the two cases (25) and (26)

$$\frac{1}{\beta q^2} = u(-s) = \frac{1}{4} + a \cotg(8as) \quad (30)$$

and

$$\frac{1}{\beta q^2} = u(-s) = \frac{1}{4} + a \coth(8as) \quad (31)$$

respectively. Note that (30) means $\beta q^2 \gtrsim 4$ while (31) means $\beta q^2 < 4$ only. Eqs.(30) and (31) represent temperatures above the critical one as the s will turn out to be finite. Thus Eq.(30) shows that there will be supercritical temperatures such that $\beta q^2 > 4$. This reflects the effect of neutral dipolar pairs for finite densities.

To determine both a and s we also will need the equation of chemical equilibrium. Combining (6) and (18) for large $-C(r)$ yields when neglecting $r < r_0$ for simplicity

$$\begin{aligned} K &\approx \pi \int_r^R e^{-C(r)} r \, dr = 4\pi R^2 \int_{-s}^0 e^{-C(r)} e^{2x} dx \\ &= \frac{1}{2\pi \rho_e^2 R^2} \int_{-s}^0 z(x) e^{-2x} dx \approx \frac{z(-s)}{4\pi \rho_e^2 r_0^2} \end{aligned} \quad (32)$$

(for small a and large s ; $R=r_0 e^s$). From (7), (27), and (29) one then finds for the density $\frac{1}{2}\rho_n$ of neutral pairs

$$\rho_n r_0^2 = \rho_e^2 K r_0^2 = \frac{2}{\pi} \frac{a^2}{\sin^2(8as)} \quad (33)$$

and

$$\rho_n r_0^2 = \frac{2}{\pi} \frac{a^2}{\sinh^2(8as)} \quad (34)$$

respectively. Eqs. (30), (31), (33), and (34) will determine the a and $s = \ln(R/r_0)$ where $R^{-2} = \epsilon^2 + \chi_e^2$ ($\epsilon \rightarrow 0$) and $\chi_e^2 = 2\pi\beta q^2 \rho_e$.

By use of (30) and (31) Eqs. (33) and (34) may be further simplified to

$$\frac{\pi}{2} \rho_n r_0^2 = \left(\frac{1}{\beta q^2} - \frac{1}{4} \right)^2 \pm a^2 \quad (35)$$

From this the a may be explicitly expressed in terms of ρ_n and β . Likewise the s may be expressed in terms of the same quantities by use of (30) or (31) depending upon whether the plus or minus sign in (35) will be needed. From s the density ρ_e of free ions may be found via Eqs. (11), (A7) and $R^{-2} = \epsilon^2 + \chi_e^2$ ($\epsilon \rightarrow 0$).

There will be a third case as (31) can only give $\beta q^2 < 4$ ($s > 0$). This will be the situation with subcritical temperatures where $s \rightarrow \infty$. In this case it is necessary to keep ϵ finite and take the limit $\epsilon \rightarrow 0$ at the end. Thus expression (18) for σ will be used in (28). Again $C(r) \approx 0$ at the upper cut-off $r=2R$ or $x=0$. I.e. from the second of expressions (27), (24), and (28) this means

$$\frac{8a}{\sinh(8ax_0)} = \frac{\chi_e^2}{\epsilon^2 + \chi_e^2} \quad (36)$$

Eqs. (31) and (34) will likewise turn into

$$\frac{1}{\beta q^2} = \frac{1}{4} - a \coth(8a(x_0 - s)) \quad (37)$$

$$\rho_n r_0^2 = \frac{2}{\pi} \frac{a^2}{\sinh^2(8a(x_0 - s))} \quad (38)$$

respectively, while Eq.(35) will stay unchanged when the minus sign is used. We consider the situation of small χ_e or large s which again means that x_0 large as $x_0 - s > 0$ since obviously $x_0 + x \neq 0$ in the region $-s < x < 0$ of interest. Thus from (36) we conclude $\chi_e \ll \epsilon$ as the only possibility, i.e. $\chi_e \rightarrow 0$ as $\epsilon \rightarrow 0$. So at subcritical temperatures

$$\rho_e = 0 \quad (39)$$

or $s = \infty$ which means that there will be no ions that may be

interpreted as free, and there will be no Debye shielding of the Coulomb interaction. The quantities a and $x_0 - s$ may be determined from Eqs.(35) and (37).

The line of critical points is obtained with $a=0$ in (35) and $\beta q^2 > 4$. One sees that the critical temperature is lowered (β increases) with increasing ρ_n . As mentioned before this may be interpreted as the effect of an increasing dielectric constant from neutral dipolar pairs. When $\rho_n \rightarrow 0$ this effect disappears and $\beta q^2 \rightarrow 4$ at critical point which has been concluded before.¹

We may obtain the leading singular of the correlation function or $C(r)$ for large r at the critical point. We may then take the limit $a \rightarrow 0$ of (28) approaching from supercritical temperature (i.e. $\sigma=1$), but such that $\beta q^2 > 4$. Use of (30) then leads to $8as > \frac{\pi}{2}$. Accordingly

$$s \rightarrow \infty \quad \text{as} \quad a \rightarrow 0 \quad (40)$$

for $\beta q^2 > 4$ which means that the line of critical points is approached.

Eqs. (27), (28), and (29) combined then gives

$$\begin{aligned} -C(r) &= -4x + \ln\left(\frac{8a}{\sin(8ax)}\right)^2 = \\ -4x - 2\ln x &= -4\ln\left(\frac{r}{2R}\right) - 2\ln\ln\left(\frac{r}{2R}\right) \end{aligned} \quad (41)$$

for $|x|/s$ small ($8as \rightarrow \pi$ as $a \rightarrow 0$).

Eq.(41) will also give rise to the infinite constant $R=r_0 e^s$ in the limit. One notes that the first term of (41) has the form of the Coulomb interaction. In addition (41) gives a second term that shows the singular nature of the correlation function at the critical point.

As mentioned before the dipolar pairs give rise to dielectric effects. The (large $r \rightarrow \infty$) dielectric constant will be the ratio $u(0)/u(-s)$ as dipolar pairs with separation less than $r=r_0$ and larger than $r=2R$ are neglected (see Eqs.(16), (30), and (31)). At the critical point this dielectric constant ϵ_0 will be

$$\epsilon_0 = \frac{1}{4} \beta q^2 \quad (42)$$

At subcritical temperatures it becomes using (26) ($\tau=8a(x+x_0) \rightarrow \infty$)

$$\epsilon_0 = \left(\frac{1}{4} - a \right) \beta q^2 \quad (43)$$

where a is determined explicitly in terms of β and $\rho_n = \rho_0$ ($\rho_e = 0$) by Eq.(35).

Appendix A

From the expression for the chain bond $C(r)$ one expects that it closely resembles the Coulomb interaction itself between a lower value r_0 and an upper value R . Here the r_0 will be the small r cut-off of the interaction while the R will be the resulting effective shielding length. This resemblance with the Coulomb interaction in the region $r_0 < r < R$ we want to utilize by the several Fourier transformations that are present in our computations.

For the moment we may be general and let the dimensionality n be arbitrary. The Coulomb interaction between two unit charges is then

$$G(r) = \begin{cases} \frac{1}{r^{n-2}} & , n \neq 2 \\ -\ln r = -\int \frac{1}{r} dr & , n = 2 \end{cases} \quad (A1)$$

and its Fourier transform (Γ means gamma function)

$$\tilde{G}(k) = K \frac{1}{k^2} \quad ; \quad K = \begin{cases} \frac{4\pi^{n/2}}{\Gamma(\frac{n-2}{2})} & , n \neq 2 \\ 2\pi & , n = 2 \end{cases} \quad (A2)$$

In a dielectric medium with a k -dependent dielectric constant $\epsilon(k) = [z(k)]^{-1}$ this changes to

$$\tilde{G}(k) = K \frac{z(k)}{k^2} \quad (A3)$$

This again may be reinterpreted as effectively having distance dependent charges. So instead of (A1) we may write

$$G(r) = \begin{cases} \frac{\alpha(r)}{r^{n-2}} & , \quad n \neq 2 \\ -\int \frac{\alpha(r)}{r} dr & , \quad n = 2 \end{cases} \quad (A4)$$

where $\alpha(r)$ is the "running charge strength" in the language of Quantum Field theory. Now it lies close at hand to expect that the $\alpha(r)$ must be closely related to the $z(k)$ since physically $\alpha(r)$ act as a inverse dielectric constant in r -space. Thus we find reason to expect

$$\alpha(r) \approx z(k) \quad \text{with} \quad k \approx 1/r \quad (A5)$$

We will now show that this is actually true to leading order for slowly varying $z(k)$, i.e. for $r_0 < r < R$. [In this respect it turns out that (A4) is the appropriate form for $n=2$, instead of say $G(r) = -\alpha(r) \ln r$.]

To obtain a more explicit and direct connection between $\alpha(r)$ and $z(k)$ we will need some quantity in terms of which both of them can be expressed. The idea is to express $z(k)$ as some sum or integral over powers k^{-t} where t is close to zero. Likewise $\alpha(r)$ can be expressed in terms of r^t which also will follow from Fourier transformation. Thus we propose to write

$$z(k) = \int f(t) k^{-t} dt \quad (A6)$$

Then the next question of interest will be whether there will exist $f(t)$ that may give an arbitrary $z(k)$ with some specified path of integration. The answer is positive as (A6) is nothing but a Mellin transformation that connects $z(k)$ and $f(t)$. This becomes more obvious by replacing k with

$$kR = e^{vs} \quad \text{where} \quad e^s = \frac{R}{r^0} \quad (A7)$$

such that v is of order one in the region $r_0 < k^{-1} < R$ of interest. By this replacement (A6) is nothing but the more familiar Fourier transform with imaginary parameter t . Thus we may write in general

$$z(k) = \hat{z}(v) = \int_{-i\infty+\delta}^{i\infty+\delta} \frac{dt}{2\pi i} F(t) e^{-vst} \quad (A8)$$

(where s is real). Inserted in (A3) this gives

$$\tilde{G}(k) = K \int_{-i\infty+\delta}^{i\infty+\delta} \frac{dt}{2\pi i} F(t) \frac{1}{k^2} \left(\frac{1}{kR} \right)^t \quad (A9)$$

To Fourier transform we utilize

$$\int \frac{d^n k}{(2\pi)^n} e^{i\vec{k}\vec{r}} k^{-\lambda} = \frac{1}{\pi^{n/2}} \frac{\Gamma(\frac{n-\lambda}{2})}{\Gamma(\frac{\lambda}{2})} \frac{1}{r^n} \left(\frac{r}{2} \right)^\lambda \quad (A10)$$

[See the book Generalized Functions by I.M.Gelfand and G.E.Shilov, Academic Press, New York and London 1964, Chapter II.3.3.].

By use of (A9) and (A10) we thus obtain the $G(r)$ in terms of $F(t)$ (with $\lambda=2+t$)

$$G(r) = \frac{K}{4\pi^{n/2}} \frac{1}{r^{n-2}} \int_{-i\infty+\delta}^{i\infty+\delta} \frac{dt}{2\pi i} \frac{\Gamma(\frac{n-2-t}{2})}{\Gamma(1+\frac{t}{2})} F(t) \left(\frac{r}{2R} \right)^t \quad (A11)$$

Like we did in (A7) the r may be replaced by a parameter η

$$\frac{r}{2R} = e^{-\eta s} \quad (A12)$$

With K inserted the $\alpha(r)$ defined by (A4) becomes

$$\alpha(r)=\hat{\alpha}(\eta)= \begin{cases} \frac{1}{\Gamma(\frac{n-2}{2})} \int_{-i\infty+\delta}^{i\infty+\delta} \frac{dt}{2\pi i} \frac{\Gamma(\frac{n-2-t}{2})}{\Gamma(1+\frac{t}{2})} F(t) e^{-\eta st} & \text{for } n \neq 2 \\ \int_{-i\infty+\delta}^{i\infty+\delta} \frac{dt}{2\pi i} (-\frac{t}{2}) \frac{\Gamma(-\frac{t}{2})}{\Gamma(1+\frac{t}{2})} F(t) e^{-\eta st} & \text{for } n=2 \end{cases} \quad (\text{A13})$$

Due to singularities in the complex t -plane the δ should be given a closer specification. One notes that due to the small r cut-off r_0 the $G(0)$ is finite, and likewise due to the large r cut off R the $\tilde{G}(0)$ is finite. Thus $\hat{\alpha}(\eta) \sim e^{-\eta s(n-2)}$ as $\eta \rightarrow \infty$ and $\hat{z}(v) \sim e^{2vs}$ as $v \rightarrow -\infty$. The convergence of their Fourier transforms is thus restricted to $t < n-2$ and $t > -2$ respectively, i.e.

$$-2 < \delta < n - 2 \quad (\text{A14})$$

The path of integration may be moved outside this interval given by (A14) if the contributions from poles passed are included separately. The connection between $\alpha(r)=\hat{\alpha}(\eta)$ and $z(k)=\hat{z}(v)$ may be made even more compact (but not more useful for our purpose). The integrand of (A13) apart from $e^{-\eta st}$ is the Fourier transform (with complex argument t) of $\alpha(r)$. This may again be considered as the product of two transforms where $F(t)$ is one factor. By that the $\hat{\alpha}(\eta)$ may be expressed as an integral of convolution, i.e.

$$\alpha(\eta) = s \int_{-\infty}^{\infty} dv K(\eta-v) z(v) \quad (\text{A15})$$

where

$$K(\sigma) = \begin{cases} \frac{1}{\Gamma(\frac{n-2}{2})} \int_{-i\infty+\delta}^{i\infty+\delta} \frac{dt}{2\pi i} \frac{\Gamma(\frac{n-2-t}{2})}{\Gamma(1+\frac{t}{2})} e^{-\sigma st} & \text{for } n \neq 2 \\ \int_{-i\infty+\delta}^{i\infty+\delta} \frac{dt}{2\pi i} \left(-\frac{t}{2}\right) \frac{\Gamma(-\frac{t}{2})}{\Gamma(1+\frac{t}{2})} e^{-\sigma st} & \text{for } n=2 \end{cases} \quad (\text{A16})$$

For large s ($= \ln R/r_0$) we expect small $t \rightarrow 0$ only to be important in (A13) and (A16). Thus we may write

$$\hat{\alpha}(\eta) = \int_{-i\infty+\delta}^{i\infty+\delta} \frac{dt}{2\pi i} \left(\sum_{k=0}^{\infty} b_k t^k \right) F(t) e^{-\eta st} \quad (\text{A17})$$

From this one further sees that $\alpha(\eta)$ may be expressed in terms of $\hat{z}(\eta)$ as the asymptotic series

$$\hat{\alpha}(\eta) = \sum_{k=0}^{\infty} b_k \left(-\frac{1}{s} \frac{\partial}{\partial \eta} \right)^k \hat{z}(\eta) \quad (\text{A18})$$

For large s and slowly varying $z(\eta)$ (i.e. η of order one) higher order terms may be neglected. With $b_0 = 1$ for any n we thus asymptotically obtain

$$\alpha(r) = \hat{\alpha}(\eta) = \hat{z}(\eta) = z(k) \quad (\text{A19})$$

with $k = \frac{2}{r}$. To leading order one may also put $k \approx 1/r$, and we thus have verified assumption (A5).

Appendix B.

Due to our renormalized vertices of graphs we will need the Fourier transform of a function like (with dimensionality $n=2$)

$$S(r) = \left(\frac{r}{2R} \right)^{-4} H(r) \quad (\text{B1})$$

Between the cut-off radia r_0 and R and in the vicinity of the critial point the $H(r)$ will be slowly varying. In the spirit of Appendix A we may write like Eq.(A8)

$$H(r) = \hat{H}(\eta) = \int_{-i\infty+\delta}^{i\infty+\delta} \frac{dt}{2\pi i} f(t) e^{-\eta st} \quad (B2)$$

where again η is given by (A12). The Fourier transform of $S(r)$ is now easily obtained by comparing (A9) and (A11) (with $n=2$ and $t \rightarrow t-4$)

$$\tilde{S}(k) = 4\pi \int_{-i\infty+\delta}^{i\infty+\delta} \frac{dt}{2\pi i} \frac{\Gamma(\frac{t}{2} - 1)}{\Gamma(2 - \frac{t}{2})} f(t) \frac{1}{k^2} e^{-vs(t-4)} \quad (B3)$$

Instead of (A14) the restriction on δ will be $(\delta \rightarrow \delta-4)$

$$2 < \delta < 4 \quad (B4)$$

The integrand of (B3) has poles at $t=2, 0, -2, -4, \dots$ the residues of which will give the formal power series of $\tilde{S}(k)$ in terms of k^2 . Again small $t \rightarrow 0$ will be of interest. Thus we take out separately the contribution

$$\tilde{S}(0) = 8\pi R^2 f(2) \quad (B5)$$

from the pole at $t=2$ moving the path of integration to

$$0 < \delta < 2 \quad (B6)$$

[Eq.(B5) may be checked by direct use of (B1) and (B2) integrating up to some finite r before integration with respect to t .]

Like we did in (A17) the Γ -functions may again be expanded for small t . Keeping the leading term only we obtain $(kR=e^{vs})$

$$\tilde{S}(k) - \tilde{S}(0) = 4\pi \int_{-i\infty+\delta}^{i\infty+\delta} \frac{dt}{2\pi i} \left(-\frac{2}{t}\right) f(t) k^2 R^4 e^{-\nu s t} =$$

$$-8\pi k^2 R^4 s \int_{\nu}^{\infty} \hat{H}(\eta) d\eta = -\frac{\pi}{2} k^2 s \int_{\nu}^{\infty} r^4 S(r) d\eta \quad (B7)$$

$$= -\frac{\pi}{2} k^2 \int_0^{2/k} r^3 S(r) dr$$

(with $r=2R e^{-\eta s}$). For $k \rightarrow 0$ (B7) becomes nothing but the expected leading power k^2 . For finite k the integral in (B7) has a cut-off $2/k$. This represents the k dependent susceptibility of the physical system of interest. It confirms the assumption that only dipoles where the charges have a separation less than $r=2/k$ contribute.

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