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ON THE ZERO MODES OF THE INTERNAL DIRAC OPERATOR
IN KALUZA - KLEIN THEORIES*

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ABSTRACT

It is shown that the internal Dirac operator in Kaluza-Klein theories has zero modes for a large variety of cases. This implies in particular that spinors need not acquire a mass of the order of Planck's mass in such theories. For SU_2 as gauge group, an explicit zero mode solution is given.

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One of the biggest obstacles presently confronting the re-formulation of gauge theories as Kaluza-Klein theories [1-7] is that if fermions are included within the framework of a consistent dimensional reduction procedure then they apparently acquire a mass that is bounded from below by Planck's mass, i.e., 10^{19} GeV. In this letter it is demonstrated that this need not necessarily be so.

I begin by re-formulating the problem in a technical language. For the convenience of the reader I repeat some known material [1-3]. Readers interested in more technical aspects of the calculations involved may consult reference [1]. More information on the mathematical background can be found in particular in references [2] and [3]. In the Kaluza-Klein approach with an underlying geometry of a principal fibre bundle, the Dirac Lagrangian is written as

$$\int d\mathbf{y} \left\{ \bar{\psi}(\mathbf{x}, \mathbf{y}) \Gamma^{\mu} (d_{\mu} + f A_{\mu}^S d_S) \psi(\mathbf{x}, \mathbf{y}) + \bar{\psi}(\mathbf{x}, \mathbf{y}) \Gamma^S D_S \psi(\mathbf{x}, \mathbf{y}) \right\}, \quad (1)$$

where a Fierz-Pauli term has been suppressed as it need not concern us here. A mass term could be added to (1), but it is not the troublesome one, as it can always be suppressed. Notations in equation (1) are the following. (Compare also reference [1].) Early Greek indices $\alpha, \beta, \dots$ run from 1 to 4, late Latin indices $s, t, \dots$ from 5 to $4+N$, N being the order of the gauge group. The x are Minkowskian, the y internal coordinates. The latter describe the gauge group as a manifold, which is what is meant by saying that the underlying geometry is that of a principal fibre bundle. Spinors ψ are to be defined on the $(4+N)$ - dimensional manifold and therefore are $\psi^{(4+N)/2}$ - dimensional objects. They depend on Minkowskian and internal coordinates. The d_s are partial differentiations in the group space in an orthonormal basis. For a non-Abelian gauge group the basis is anticommuting

$$[d_s, d_t] = if_{st}^v d_v, \quad (2)$$

where f_{st}^v are the structure constants of the gauge group. The D_s are covariant derivatives with respect to group coordinates in the Riemannian sense, defined on the spinors. Using the internal elements of the Clifford algebra Γ^s , we may write

$$\Gamma^S D_S \psi = \left(\Gamma^S d_S + \frac{1}{16i} f_{st}^v \Gamma^s [\Gamma^v, \Gamma^t] \right) \psi, \quad (3)$$

where one uses the fact that the connection coefficients of the group space are, up to a known factor, the structure constants of the group. The Clifford algebra is normalized in the usual way:

$$\{\Gamma^\alpha, \Gamma^\beta\} = 2\eta^{\alpha\beta}, \{\Gamma^\alpha, \Gamma^\beta\} = 2g^{st} = -2\delta^{\alpha\beta}, \{\Gamma^\alpha, \Gamma^\beta\} = 0, \eta = \text{diag}(+---).$$

The constant f is related to the gravitational constant κ , $f = \sqrt{\kappa} \sim 10^{-19} \text{ GeV}^{-1}$.

The dimensional reduction procedure can now be specified by the requirements that the term in curly brackets in (1) depends on y at most in the form of an overall weight factor and that the first half of the expression (1) is a normal minimal coupling term. The second half of expression (1) can then be given the form of a mass term [1]. The observation relevant to us is that the fermion mass thus generated is of the order of Planck's mass.

The reason for this is the following. The differentiation d_s in $\bar{\psi}^\alpha f_{\alpha\beta}^{\gamma\delta} \psi$ has to generate a constant μ together with a representation matrix T_s acting on the internal indices of the spinors, i.e., after dimensional reduction this term gives

$$\bar{\psi}(x) \Gamma^\alpha f_{\alpha\beta}^{\gamma\delta} A_s^\beta T_s \psi(x), \quad (4)$$

where the ψ are coefficients of a Fourier expansion of ψ over the group space. One then has to identify $\mu f = e$, with e the gauge coupling constant. This introduces the mass scale

$$\mu \sim 10^{19} \text{ GeV}, \quad (5)$$

which will then also appear as a factor when the dimensional reduction for $\bar{\psi}^\alpha D_s \psi$ is carried out. The only way for such a term not to appear is to require

$$\Gamma^s D_s \psi = 0. \quad (6)$$

Note that for Abelian groups $D_s = d_s$ and (6) cannot be true if a minimal coupling term is to be present. In the following we will see that the presence of non vanishing connection coefficients in the group space (i.e. the presence of two different terms in equation (3)) is essential for the existence of solutions of equation (6). Equation (6) is the statement that ψ is a zero mode solution of the differential operator $\Gamma^s D_s$, the internal Dirac operator. If (6) can be fulfilled, spinors in Kaluza-Klein theories need not have a mass of the order (5).

I now proceed to discuss solutions of (6). After dimensional reduction,

$$\bar{\psi}^\alpha D_s \psi \rightarrow \bar{\psi}^\alpha \left\{ \Gamma^s T_s + \frac{1}{16i} f_{st\nu} \Gamma^s [\Gamma^t, \Gamma^\nu] \right\} \psi \quad (7)$$

with

$$[T_s, T_\nu] = i f_{st\nu} T_w, \quad (8)$$

which means that all we are left with is to solve the algebraic equation

$$\left(\Gamma^s T_s + \frac{1}{16i} f_{st\nu} \Gamma^s [\Gamma^t, \Gamma^\nu] \right) \psi = 0. \quad (9)$$

That the differential operator is effectively an algebraic one (which can also be seen by carrying out the action of d_s in equation (3)) is a consequence of the specific geometric properties of the Lie group and the above mentioned specification for the dimensional reduction procedure (which implies that the y -dependence can be expressed through a Fourier expansion).

Note now that a natural representation of the gauge group on the spinors can be given without introducing extra internal labels. In fact, introduce

$$t_s = \frac{1}{8i} f_{st\nu} [\Gamma^t, \Gamma^\nu]; \quad (10)$$

then

$$[t_s, t_\nu] = i f_{st\nu} t_w, \quad (11)$$

and the t_s can act as representation matrices for the Lie algebra of the gauge group. The spinors ψ are then to be looked upon as multiplets of Dirac spinors; this internal degree of freedom may then be identified with the gauge degree of freedom.

Inserting (10) into (9) we find

$$\left(\Gamma^s T_s + \frac{1}{2} \Gamma^s t_s \right) \psi = 0. \quad (12)$$

If the internal degree of freedom is identified with the spinor degree of freedom as described above, then $T_s \rightarrow t_s$ and (12) can be replaced by [1]

$$\Gamma^s t_s \psi = 0. \quad (13)$$

The latter equation has none but the trivial solution however. In fact, in order to show that $\Gamma^S t_s$ has no zero eigenvalues it is sufficient to show that $(\Gamma^S t_s)^2$ has no zero eigenvalues. Now

$$(\Gamma^S t_s)^2 = 3 t^S t_s. \quad (14)$$

But $t^S t_s$ is nothing but the quadratic Casimir operator of the gauge group which is zero only for the trivial representation. Therefore the above assertion on the zero modes of $\Gamma^S t_s$ is true.

Looking for zero modes of the internal Dirac operator we are therefore left with equation (12), where now $T_s \neq t_s$. In particular, t_s acts on spinor labels and T_s acts on a separate set of internal labels. This implies in particular the commutation relations

$$[\Gamma^S, T_V] = [t^S, T_V] = 0. \quad (15)$$

Furthermore the following algebraic identities will be useful:

$$(\Gamma^S T_S)^2 = T^S T_S - 2 T^S t_S, \quad (16)$$

$$\Gamma^S T_S \Gamma^V t_V = T^S t_S \quad (17)$$

$$\Gamma^S t_S \Gamma^V T_V = 5 T^S t_S \quad (18)$$

In order to solve (12), introduce a new spinor ψ through

$$\psi = \{ \Gamma^S T_S + \alpha \Gamma^S t_S \} \varphi. \quad (19)$$

From (12), φ has to satisfy the equation

$$\{ \beta + T^S t_S \} \varphi = 0, \quad (20)$$

with

$$\beta = [T^S T_S + \frac{3}{2} \alpha t^S t_S] (\alpha - \frac{1}{2}). \quad (21)$$

Note that by properly choosing α , the parameter β can be given any desired real value.

From (20) one infers that (12) is solved if the matrix $\Gamma^S t_s$ has some real eigenvalue. Since $\Gamma^S t_s$ is a hermitian matrix, such a real eigenvalue

can always be found. Correspondingly (12) always has zero mode solutions.

I will now give an explicit zero mode solution for the case that the gauge group is SU_2 . To do this I will use

$$t_V t_S = -\frac{1}{16} \Gamma_V \Gamma_S, \quad (22)$$

a relation specific for SU_2 . Introduce now the spinor ϕ through

$$\psi = \{ \delta + T^V t_V \} \phi. \quad (23)$$

From (20), ϕ has to satisfy

$$\{ \beta \delta + (\beta + \delta) T^V t_V + (T^V t_V)^2 \} \phi = 0. \quad (24)$$

On account of (22)

$$(T^V t_V)^2 = \frac{1}{8} T^V t_V - \frac{1}{16} T^W T_W \quad (25)$$

and (24) becomes

$$\{ (\beta \delta - \frac{T^W T_W}{16}) + (\beta + \delta + \frac{1}{8}) T^W t_W \} \phi = 0. \quad (26)$$

This equation is solved if

$$\beta \delta - \frac{T^W T_W}{16} = 0 \quad (27)$$

and

$$\delta = -\beta - \frac{1}{8}. \quad (28)$$

Inserting (28) into (27) one finds

$$\beta (\beta + \frac{1}{8}) + \frac{1}{16} T^W T_W = 0, \quad (29)$$

which has the real solutions ($T^W T_W < 0$)

$$\beta = -\frac{1}{16} \pm \frac{1}{4} \sqrt{\frac{1}{16} - T^W T_W}. \quad (30)$$

Compare this with the definition of β in terms of α , which solved for

α reads

$$\alpha = - (T^s T_s + \frac{1}{2} \beta) / (\frac{3}{2} t^s t_s - \beta). \quad (31)$$

For SU_2 , with the normalizations used here, $t^s_t = -3/4$ and using (30) one finds that the denominator in (31) cannot vanish for any representation of SU_2 . We therefore have shown the following. If the underlying geometry of space-time is that of a principal fibre bundle, and if the internal degree of freedom is not the spinor degree of freedom in the sense described above, then the internal Dirac operator always has zero modes.

For SU_2 , an explicit solution of (12) is given by

$$\psi = P\phi = \{ \Gamma^s T_s + \alpha \Gamma^s t_s \} \{ \delta + T^s t_s \} \phi, \quad (32)$$

where α and δ are given above. There are two such solutions corresponding to the two values of β , equation (30). Note that $P \neq 0$ so that (32) is indeed a nontrivial solution, $\psi \neq 0$. The spinor ϕ may be chosen from any representation of SU_2 , otherwise the solution (32) does not contain any free parameters. It is determined of course only up to an overall factor.

I conclude this letter with the following remarks. Unsuccessful attempts in solving the problem under consideration have been reported in references [1] and [5] already. In reference [5] extended Kaluza-Klein theories have been studied and an attempt is made to overcome the charge-mass relation $\mu \neq e$. In reference [6] a five dimensional Kaluza-Klein theory with torsion is studied. It seems that the introduction of torsion leads to a solution of the problem as well. Without torsion however, the present method seems to be the only available one. The suggestion to study equation (6) has already been formulated in reference [7], where no solutions were given however. For the actual solution of equation (6) I have been inspired by the construction in reference [4] of a diagonalized mass matrix for spinors.

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