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AND POSITRON GAS IN A STRONG
MAGNETIC FIELD. PROPAGATION OF
ELECTROMAGNETIC WAVES**

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POLARIZATION OF RELATIVISTIC ELECTRON AND POSITRON
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Abstract: Structural properties of the polarization operator, Green function and electromagnetic eigenmodes for relativistic electron-positron gas placed in a magnetic field at a non-zero temperature are established basing on the most general consideration of unitarity and the properties under gauge and Lorentz transformations, charge conjugation, inversion of space-time and 4-velocity of the medium. Polarizations of the eigenmodes are described in the general form. Connection between the charge asymmetry and the appearance of elliptic polarization is formally established. For each eigenmode one of the principal axes of the polarization ellipse lies in the plane containing the external magnetic field and the direction of the wave propagation while the other is orthogonal to this plane. The polarization operator is calculated within the one-loop approximation of the Furry picture within the technique of the temperature Green functions ^[1] with the subsequent analytical continuation of the frequency. The results obtained are valid also for a uniformly moving medium.

I. INTRODUCTION.

Investigation of thermodynamical and kinetical characteristics of the relativistic gas in a strong external field can be interesting not only in itself (e.g. in an astrophysical context) but also as a mean to clarify more principle questions about, say, spontaneous appearance of external field accompanying the phase transition, about the nonphysical and infra-red singularities etc. In these aspects the consideration of theories other than quantum electrodynamics is also of interest.

In the present paper we deal with the electron-positron placed into external magnetic field $\vec{B}$. The latter is now taken as small. The temperature $T = \beta^{-1}$ and the chemical potential μ ($-\mu$) of electrons (positrons) are different from zero. To guarantee the stability in the charge-asymmetric case $\mu \neq 0$ one assumes that there exists an ion lattice, which compensates the charge and whose effect is disregarded in every other respect. We study the polarization operator Π of this system, i.e. the quantity that is responsible for screening of weak electromagnetic fields and determines the laws of propagation of electromagnetic waves and (after extension to imaginary frequency $-k_0 \rightarrow i k_4$) also contributes into the thermodynamical potential.

In Secs. 2,3 the investigation is carried out in the most general form.

The establishing of structural properties of $\Pi_{\mu\nu}$ on the basis of general considerations (Sec.2) is helpful since firstly, as it has been seen in other

examples [2] and will be seen in the present paper, it allows far-reaching predictions without any (inevitably approximate) calculations and, secondly, makes it possible to control the correctness of such calculations. The latter being highly complicated, such a control does not at all seem to be out of place.

For the sake of the utmost utilization of the properties following from the relativistic structure of the underlying interaction we introduce, according to Fradkin [1], the vector u_μ of the 4-velocity of the medium which makes it possible to regard $\Pi_{\mu\nu}$ as a tensor in the Minkowskian space-time. Then the general tensor structure established in terms of the matrices which can be built on characteristic vectors and tensors that enter in the problem. For this purpose we have used the transversality of the tensor $\Pi_{\mu\nu}$ that results from the gauge invariance, the property of $\Pi_{\mu\nu}$ to contain the external field and the chemical potential so that the sum of their powers be even [1] that follows from the charge symmetry of the underlying interaction, and also the PT-invariance of the latter. Unitarity^{requires} that the polarization operator be represented as a Hermitian matrix until the absorption is explicitly taken into account. On this basis the conclusion is made that the coefficients with which the symmetric (antisymmetric) matrices enter in the decomposition of $\Pi_{\mu\nu}$ are real (imaginary). The explicit introduction of the 4-velocity vector u_μ into the density matrix allows us to conclude that the parity of the polarization operator under the reflection of λ , in the rest frame

coincides with that under the reflection of the electric charge. This circumstance eliminates some matrix structures and leads us to important conclusion that antisymmetric structures in $\Pi_{\mu\nu}$ contain only odd powers of the chemical potential μ . It is these structures that are responsible for the appearance of elliptically polarized eigenmodes and the Far effect. Then $\Pi_{\mu\nu}$ is diagonalized in the general form (Sec.3) which allows us to obtain an expression for the Green function to write three dispersion equations and to describe polarization properties of the three eigenmodes. A more detailed consideration is given to the cases of electromagnetic wave propagation along and across the external magnetic field.

In Sec.4 the polarization operator in a magnetic field is calculated as a loop Furry diagram in Fradkin's technique of temperature Green functions with the subsequent analytic continuation $K_4 \rightarrow iK_0$. This technique is described in Ref.[1], basing on the results of Refs.[3].

2. TENSOR STRUCTURE OF THE POLARIZATION OPERATOR.

GENERAL CONSIDERATION.

In the presence of an external electromagnetic field given by a time-independent potential A_μ^{ext} the polarization operator $\Pi_{\mu\nu}$ can be defined through the inverse photon Green function $D_{\mu\nu}^{-1}$ in the external field A_μ^{ext}

$$D_{\mu\nu}^{-1}(x,y) = \frac{\delta^2 \Gamma}{\delta A^\mu(x) \delta A^\nu(y)} \Big|_{A_\mu = A_\mu^{ext}} \quad (1)$$

by the relation

$$D_{\mu\nu}^{-1}(x,y) = \left(g_{\mu\nu} \frac{\partial^2}{\partial x_\rho \partial x^\rho} - \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \right) \delta^{(4)}(x-y) - \Pi_{\mu\nu}(x,y) \quad (2)$$

In Eq. (1) Γ is the effective electromagnetic action, i.e. the generating functional of one-particle-irreducible vertex functions containing only photon legs. It is connected with the generating functional $Z(J)$ of the Green functions [1] (J is the electric current source) by the Legendre transformation

$$\Gamma(A) = A_\mu J^\mu - \ln Z(J), \quad A_\mu = \frac{\delta \ln Z}{\delta J^\mu} \quad (3)$$

Then Eq. (1) is the consequence of the definition of the photon Green function

$$D_{\mu\nu}(x, y) = \frac{\delta^2 \ln Z}{\delta J^\mu(x) \delta J^\nu(y)} \Big|_{J=J^{ext}} \quad (4)$$

From Eq. (3) it is seen that the Maxwell equations of motion for the mean electromagnetic field A_μ have the form of the Euler - Lagrange equation for the action $\Gamma(A)$:

$$\frac{\delta \Gamma}{\delta A^\mu} = J_\mu \quad (5)$$

In formulae (1)-(5) the lowering and the lifting of indices are fulfilled with the aid of the metric tensor $g_{\mu\nu}$. Formulae (1)-(5) have direct meaning in the temperature (thermodynamical) perturbation theory where the generating functional $Z(J)$ is explicitly built [1]. In this case $-\beta \leq x_4 \leq \beta$ and the metrics are "mixed": in the χ_μ -space the scalar product is Euclidean while for currents and fields the scalar product is determined by the Minkowski metric $g_{00} = -g_{ii} = -1$. (In other words, if we do not distinguish between upper and lower indices, then the fourth components

of the fields, currents and the 4-velocity to be introduced below are purely imaginary while those of co-ordinates are real. For all tensors and vectors, including $\pi_{\mu\nu}$, the definition holds that each fourth component differs by i-factor from the corresponding zeroth component: $\pi_{\alpha\beta\dots\gamma_4} = i \pi_{\alpha\beta\dots\gamma_0}$. According to this the matrix indices μ, ν in Eq. (2) originate from the differentiation with respect to the fields pertaining to the Minkowskian space $\mu, \nu = 0, 1, 2, 3$ while the differential operator in the first term in Eq. (2) must be understood as $\frac{\partial^2}{(\partial x^\rho)^2}$, $\rho = 1, 2, 3, 4$. The photon Green functions obtained from $Z(J)$ by differentiating with respect to J and defined in the temperature technique at discrete values of the Fourier-variable $k_4 = \frac{2\pi n}{\beta}$, may be analytically continued as $k_4 \rightarrow i k_0 + \epsilon$ one obtains Fourier-transforms of advanced (retarded) time-dependent Green functions if the continuation starts with positive (negative) values of ϵ . The possibility and singleness of such a continuation is guaranteed [1] by spectral representations for these functions. By performing the above mentioned analytical continuation in all the terms of the expansion of the generating functional $Z(J)$ in powers of J we determine the generating functional $Z(J)$ of the time-dependent functions^{x)}. After that $g_{\mu\nu}$ is the metric tensor of the Minkowskian space throughout equations (1)-(5). This is how they will be understood hereafter.

^{x)} The existence of the generating functional of the time-dependent functions is appealed to in Ref. [4].

Now by linearizing Eq. (5) near the value $A_\mu = A_\mu^{ext}$ that satisfies the equation $(\delta\Gamma/\delta A_\mu)_{A=A^{ext}} = J_\mu^{ext}$ we obtain for the deviation $a_\mu = A_\mu - A_\mu^{ext}$ the equation

$$D_{\mu\nu}^{-1} a^\nu = J_\mu - J_\mu^{ext} \quad (6)$$

where $D_{\mu\nu}^{-1}$ is given by (1). This is the equation for electromagnetic oscillations near the value $A_\mu = A_\mu^{ext}$ induced by the deviation of the current from the value $J = J^{ext}$. At $J = J^{ext}$ eq. (6) is the equation for free oscillations

$$D_{\mu\nu}^{-1} a^\nu = 0 \quad (7)$$

Important properties of the operator $D_{\mu\nu}^{-1}$ follow from Eq. (1). With the introduction of the 4-velocity of the medium u_μ , $u_\mu u^\mu = -1$ according to Fradkin [1], Γ becomes a functional containing only scalar combinations of u_μ , ∂_μ and the field A_μ . The absence of pseudoscalar terms in $\Gamma(A)$ follows from P- and T-invariance of the elementary interaction underlying the statistical theory, in our case it is the electron-photon interaction. Since Γ is a scalar, $D_{\mu\nu}^{-1}$ and $\Pi_{\mu\nu}$, determined according to (1), (2), are tensors. From Eq. (1) it follows that $D_{\mu\nu}^{-1}(x, y) = D_{\nu\mu}^{-1}(y, x)$. For external fields of the form $A_\mu^{ext}(x) = \frac{1}{2} F_{\mu\nu} x^\nu$, where the strength tensor $F_{\mu\nu}$ is constant, all the gauge-invariant quantities including $\Pi_{\mu\nu}$ are translational-invariant [5]: $D_{\mu\nu}^{-1}(x, y) = D_{\mu\nu}^{-1}(x-y)$. Therefore for the Fourier-transform of $D_{\mu\nu}^{-1}$ with respect to the difference $x-y$ we have the property

$$D_{\mu\nu}^{-1}(k) = D_{\nu\mu}^{-1}(-k) \quad (8)$$

Further on, the unitarity of the theory requires that the effective action functional $\Gamma(A)$ of the electromagnetic field remain real as long as the field A in $\Gamma(A)$ is such that it cannot be absorbed by transforming, say, into electron-positron pairs or through any other absorption channel. When $A_\mu = A_\mu^{ext} + a_\mu$, where A_μ^{ext} corresponds to a constant electric field which is not absorbed, and a_μ is a small correction corresponding to a "photon" field which can be absorbed $\Gamma(A)$ has an imaginary part due to the contribution of the term $\int dx dy D_{\mu\nu}^{-1}(x,y) a^\mu(x) a^\nu(y)$. Since $a_\mu(x)$ is real, we see that $D_{\mu\nu}^{-1}$ is a real tensor no longer than the possibility of photon absorption is disregarded. By virtue of the property that the fact that the tensor $D_{\mu\nu}^{-1}$ is real leads to hermiticity its Fourier-transform

$$D_{\mu\nu}^{-1}(k) = D_{\nu\mu}^{-1*}(k) \quad (9)$$

The same statement concerns the polarization operator too. We shall build the polarization operator as a Hermitian matrix tensor indices in Minkowskian space bearing in mind that this requirement is valid for a transparency region (if any). In words $\Pi_{\mu\nu}(k)$ is a Hermitian matrix as long as the absorption of the photons is disregarded. In the practice of specific calculations this means that the matrix $\Pi_{\mu\nu}(k)$ remains Hermitian as long as the spectral integrals that represent it are taken as the principal value.

Finally, the gauge invariance leads to transversality of the polarization operator

$$k^\nu \Pi_{\nu\mu} = \Pi_{\mu\nu} k^\nu = 0 \quad (10)$$

and of all the rest of the one-particle-irreducible photon vertex functions. Up to the terms which vanish when differentiated with respect to A the functional $\Gamma(A)$ contains A

only through the gauge-invariant combination $\phi_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

In this case the k_μ -transversality of the derivatives of $\Gamma(A)$ with respect to A is guaranteed. With the 4-velocity vector

u_μ taken into consideration, $\Gamma(A)$ is, generally speaking, a function of scalar combinations of the vectors A_μ , ∂_μ , u_μ , such that in the frame $\vec{u} = 0$ they contain A_μ via $\phi_{\mu\nu}$ only. Since none of the nonzero scalars can vanish due to a choice of the Lorentz frame we conclude that at $\vec{u} \neq 0$, Γ contains the field A only via the scalar combinations of ∂_μ , u_μ and $\phi_{\mu\nu}$. This proves the transversality of $\Pi_{\mu\nu}$ in arbitrary Lorentz frame, which fact is however evident enough.

Thus, to determine the general structure of $\Pi_{\mu\nu}$ we must build various Hermitian transverse matrices making use of the tensor $F_{\mu\nu} = \phi_{\mu\nu}^{\text{ext}}$ and the vectors u_μ and k_μ . Below we consider the tensor $F_{\mu\nu}$ corresponding to a constant magnetic field of strength B in the rest system of the medium (in a moving system there arises, generally, an electric field, too). In invariant form these restrictions are written as follows (the reader is asked to consult Appendix whenever he wishes to understand what sense relativistic invariant entities met throughout the paper take in the rest frame);

$$F_{\mu\nu} u^\nu = 0$$

(11)

$$\tilde{F} = -\frac{1}{4} F_{\mu\rho} F^{\rho\mu} > 0, \quad \mathcal{G} = -\frac{1}{4} F_{\mu\rho} F^{*\rho\mu} = 0 \quad (12)$$

Here the dual tensor in the Minkowskian space is defined as

$F_{\rho\mu}^* = \frac{1}{2} \epsilon_{\rho\mu\lambda\kappa} F^{\lambda\kappa}$. We shall act by analogy with Ref [2]. All the sixteen independent tensors in Minkowskian space can be built by multiplying the four vectors k_μ , $F_{\mu\rho} k^\rho$, $F_{\mu\rho}^2 k^\rho$, u_μ where by definition $F_{\mu\rho}^2 \equiv F_{\mu\nu} F^\nu_\rho$. Among these sixteen there exist ten tensors symmetric under permutation of indices. By taking for $\Pi_{\mu\nu}$ their linear combination with ten coefficients and substituting it into (10) we obtain for the coefficients a set of four equations. Its solution leaves six coefficients independent. On gathering the corresponding matrices we obtain $\Pi_{\mu\nu}$ as a combination of six symmetric tensors $\psi_{\mu\nu}^{(1,2,3,4,7,8)}$ (see Eq. (14), (15) below). By dealing in a similar way with the remaining six antisymmetric matrices we get from (10) a set of four equations for the coefficients at them. The determinant of this system vanishes (irrespective of condition (11)), and therefore instead of two there arise three antisymmetric matrices $\psi_{\mu\nu}^{(5,6,9)}$. Thus we obtain

$$\Pi_{\mu\nu}(k) = \sum_i \pi_i (u k)^2, k F^2 k, \tilde{F}, k^2 \psi_{\mu\nu}^{(i)} \quad (13)$$

where $\psi_{\mu\nu}^{(1,2,3,4,9)}$ are matrices which are even with respect to the field F

$$\begin{aligned}
 \Psi_{\mu\nu}^{(1)} &= k^2 g_{\mu\nu} - K_\mu K_\nu, \quad \Psi_{\mu\nu}^{(2)} = F_{\mu\lambda} k^2 F_{\nu\sigma} K^\sigma \\
 \Psi_{\mu\nu}^{(3)} &= -k^2 \left(g_{\mu\lambda} - \frac{K_\mu K_\lambda}{k^2} \right) F_\rho^\lambda F^{\rho\eta} \left(g_{\eta\nu} - \frac{K_\eta K_\nu}{k^2} \right) \\
 \Psi_{\mu\nu}^{(4)} &= (u_\mu - K_\mu (uK)/k^2) (u_\nu - K_\nu (uK)/k^2) \\
 \Psi_{\mu\nu}^{(9)} &= (u_\mu - K_\mu (uK)/k^2) F_{\nu\sigma}^2 K^\sigma - (u_\nu - K_\nu (uK)/k^2) F_{\mu\lambda}^2 K^\lambda + \frac{(KF^2K)}{k^2} (K_\mu u_\nu - K_\nu u_\mu)
 \end{aligned} \tag{14}$$

and $\Psi_{\mu\nu}^{(5,6,7,8)}$ are matrices odd in the field F :

$$\begin{aligned}
 \Psi_{\mu\nu}^{(5)} &= (uK) (K_\mu F_{\nu\lambda} K^\lambda - K_\nu F_{\mu\lambda} K^\lambda + k^2 F_{\mu\nu}) \\
 \Psi_{\mu\nu}^{(6)} &= u_\mu F_{\nu\lambda} K^\lambda - u_\nu F_{\mu\lambda} K^\lambda + (uK) F_{\mu\nu} \\
 \Psi_{\mu\nu}^{(7)} &= [k^2 F_{\mu\lambda}^2 K^\lambda - K_\mu (KF^2K)] F_{\nu\rho} K^\rho + [k^2 F_{\nu\lambda}^2 K^\lambda - K_\nu (KF^2K)] F_{\mu\rho} K^\rho \\
 \Psi_{\mu\nu}^{(8)} &= (uK) \left[(u_\mu - K_\mu (uK)/k^2) F_{\nu\lambda} K^\lambda + (u_\nu - K_\nu (uK)/k^2) F_{\mu\lambda} K^\lambda \right] \\
 \Psi_{\mu\nu}^{(1,2,3,4,7,8)} &= \Psi_{\nu\mu}^{(1,2,3,4,7,8)}, \quad \Psi_{\mu\nu}^{(5,6,9)} = -\Psi_{\nu\mu}^{(5,6,9)}
 \end{aligned} \tag{15}$$

$$(KF^2K) \equiv K^\mu F_{\mu\nu}^2 K^\nu \tag{16}$$

In accordance with Eq. (9) the coefficients at the antisymmetric matrices $\pi_{5,6,9}$ in Eq. (13) are purely imaginary in the transparency region. The above construction fixes the matrices (14), (15) to the accuracy of a scalar factor only. The only factor not absorbed in π_0 in Eq. (13) is (uK) . We fixed it in Eqs. (14), (15) so that in accordance with Eq. (8) the symmetric matrices might be even in K and antisymmetric be odd.

We are in a position now to use the Furry theorem in a

version corresponding to a moving frame of reference. The correlation and thermodynamical functions in this frame correspond to averaging with the density matrix

$$\rho = \exp\{-\beta(P_\mu u^\mu - \mu N)\} \quad (17)$$

where β and μ are scalar parameters equal to the inverse temperature and the chemical potential in the rest frame, P_μ is the four-momentum operator and N is the operator of the difference between the number of electrons and positrons, which is the Lorentz-scalar

$$\begin{aligned} P_\mu &= \int P_{\mu\nu}(x) d\sigma^\nu \\ N &= \frac{1}{e} \int j_\mu(x) d\sigma^\mu \end{aligned} \quad (18)$$

Here

$$\begin{aligned} P_{\mu\nu}(x) &= P_{\mu\nu}^{(0)}(x) - g_{\mu\nu} j_\alpha A^\alpha \\ j_\mu(x) &= \frac{ie}{2} (\bar{\Psi}(x) \gamma_\mu \Psi(x) - \Psi(x) \gamma_\mu^T \bar{\Psi}(x)) \end{aligned} \quad (19)$$

and $P_{\mu\nu}^{(0)}$ is the part of the energy-momentum tensor without the interaction. In eq. (18) $d\sigma^\nu$ is the element of the spacelike hyperplane orthogonal to the 4-velocity u_μ of the medium. Of course, we might by applying the Gauss theorem transform to the integrals over d^3x since $P_{\mu\nu}$ and j_μ are subject to the continuity relations. These relations, however, do not hold for $P_{\mu\nu}^{(0)}$ separately. So, bearing in mind to develop in what follows perturbation expansions which refer to quantities constructed with the help of $P_{\mu\nu}^{(0)}$ as nonperturbed we must give up the intention to integrate over d^3x since otherwise

we would face the situation which we do not want, indeed, when each perturbation term is not relativistically covariant while the total sum is. Therefore we retain the $d\mathcal{S}^\mu$ -integration henceforward and develop the temperature technique [1] with the relativistic invariant $P_\mu u^\mu$ (equal to the rest energy) instead of the Hamiltonian of the usual theory. To do so we must pass to the Schrödinger representation so that the operators do not depend on the time-like variable parallel with u_μ , (i.e. the time in the co-moving frame or the proper time), and quantize on the space-like surface $\mathcal{S}$, orthogonal to u_μ . Then the temperature co-ordinate X_4 replaces the proper-time and thus ^{is} a relativistically invariant quantity. In the thermodynamical perturbation theory thus modified for moving media the free propagators are altered to contain u_μ while the interaction vertices remain unaffected except for the factor $\text{sgn } u_0$ which is a projection of the vector u_μ onto the direction orthogonal to the hyperplane $\mathcal{S}$. This factor is acquired by each charge e in the diagrams for $\Pi_{\mu\nu}$ due to the fact that the interaction enters into $P_{\mu\nu}$ only through $g_{\mu\nu}$. Now we refer to the diagrams in the picture where the interaction-free density matrix without the chemical potential $S_0 = \exp\{-\beta P_\mu^{(0)} u^\mu\}$ is taken as the unperturbed density matrix. Within this picture the Furry theorem requires that (at least in the rest frame) in each closed loop diagram the sum of knots with the μ -“charge” and those with the e -charge be even [1]. From this we conclude that the coefficient functions $\Pi_{1,2,3,4,9}$ by the matrices even in F are even in μ and the coefficients $\Pi_{5,6,7,8}$ by the

odd in F matrices are odd in $\mathcal{M}$ since we do not have at our disposal any scalar that would be odd with respect to the field $F_{\mu\nu}$ (the quantity $uF^{\star K}$ is a pseudoscalar and is excluded by the P- and T-parity):

$$\pi_{1,2,3,4,9}(\mu) = \pi_{1,2,3,4,9}(-\mu); \pi_{5,6,7,8}(\mu) = -\pi_{5,6,7,8}(-\mu) \quad (20)$$

Now consider the properties of the loop diagrams under the change of the sign of the vector $\mathcal{U}_\mu$. In the frame where $\vec{\mathcal{U}} = 0$ we have usual free propagators in which the fermion and boson energy values $E_k = (\vec{k}^2 + m^2)^{1/2}$, $\omega_k = (k^2)^{1/2}$ are multiplied by $\text{sgn } \mathcal{U}_0$ since it is this factor that arises in front of the Hamiltonian in the density matrix and the above square roots are the eigenvalues of its free part which determines the free propagators. As soon as the sign of the vector $\mathcal{U}_\mu$ changes, these roots change their signs, too. It may be easily seen by direct inspection that in this case the fermion and boson free temperature Green functions (both with zero and nonzero μ) do not change. Hence when the sign of $\mathcal{U}_\mu$ changes, all alterations in the loop diagrams are due to the interaction only: therefore if the inversion $\mathcal{U}_\mu \longleftrightarrow -\mathcal{U}_\mu$ is accompanied by the inversion $A_\mu \longleftrightarrow -A_\mu$, the value of the loop diagram remains invariant. This circumstance altogether forbids the terms $i = 7, 8, 9$ of the polarization operator (13)

$$\pi_{7,8,9} = 0 \quad (21)$$

since for them the sum of powers of $\mathcal{U}_\mu$ and the external field F is odd. Thus we have obtained that the part of $\pi_{\mu\nu}$ which is antisymmetric in indices is odd with respect to the external (magnetic) field while the symmetric part is even. This is a relativistic form of the Onsager symmetry relations of the kinetic coefficients. ⁽⁶⁾ As a matter of fact we have presented a dynamical derivation of these relations in any order of perturbation theory.

Eqs. (21) are confirmed by the one-loop calculations with $\mu \neq 0$ and the magnetic field kept exactly (see the last section).

Note that the discarded matrices $\psi_{\mu\nu}^{(7,8)}$ are produced by the differentiation indicated in (1) due to the dependence of the effective Lagrangian $\mathcal{L}(x)$ (connected with Γ by the relation $\Gamma = \int \mathcal{L}(x) d^4x$) on the invariant $\phi_{\mu\rho} \partial^\mu \phi_\nu^\rho \mathcal{U}^\nu$ while $\psi_{\mu\nu}^{(9)}$ is due to the invariant $\phi_{\mu\rho}^2 \phi_\nu^\epsilon \mathcal{U}^\nu \partial^\mu \phi_\epsilon^\rho$. The rest of the matrices $\psi_{\mu\nu}^{(a)}$ arise due to the dependence of $\mathcal{L}$ on the invariants ^{x)} $\phi_{\mu\rho} \phi^{\rho\mu}$, $(\phi_{\mu\rho} \phi^{*\rho\mu})^2$,

^{x)} This way to obtain the covariant structure of $\pi_{\mu\nu}$ was used in quantum field theory by Bialynicka-Birula and Bialynicki-Birula [7]. However, these authors omitted the dependence of $\mathcal{L}$ on the derivative-containing invariants. (Note, that one may put $\phi_{\mu\nu} = \text{const}$ no sooner than the functional

$\phi_{\mu\rho} \partial^\mu \partial^\nu \phi_\nu^\rho$, $\partial_\mu \phi^{\mu\nu} u_\nu$, $\phi_{\mu\rho}^2 \partial^\mu \phi_\nu^\rho u^\nu$. The invariants $\phi_{\mu\rho} \partial^\mu \phi_\nu^\rho u^\nu$, $\phi_{\mu\rho}^2 \phi_\nu^\sigma u^\nu \partial^\mu \phi_\sigma^\rho$ in which the sum of the powers of u and ϕ is odd must be omitted from the effective Lagrangian directly on the grounds of the above analysis the 4-velocity reflection properties,

Now that eq. (21) is established the following statement becomes true: all the symmetric tensors enter in Eq. (13) with the coefficients even in μ and the antisymmetric ones have odd in μ coefficients. With the view of what has been said above this means that in the transparency region the purely imaginary terms $\pi_{5,6,7}$ are odd in μ and vanish in the charge-symmetric case $\mu = 0$ (the so called "hot vacuum"). Below we shall see that it is these terms that are responsible for elliptically polarized Bose eigen-excitations. Supposing the medium is absent, $u_\mu = 0$, then we are left with the matrices $\psi_{\mu\nu}^{(1,2,3)}$ characteristic of the vacuum polarization by a magnetic field. In another limiting case $F_{\mu\nu} = 0$ there remain only two (symmetric) matrices $\psi_{\mu\nu}^{(1,3)}$ and the polarization operator is even in μ which is in agreement with the result contained in Ref [1].

3. BOSE-EXCITATIONS. GENERAL CONSIDERATION.

To find the Bose-excitation spectrum is to solve equation

(differentiation in (1) has been performed). As is seen from comparison with Ref. [2] this has not lead to the loss of matrix structures of the polarization operator but the expressions for the coefficients π_i are lacking the derivatives with respect to the invariant $\phi_{\mu\rho} \partial^\mu \partial^\nu \phi_\nu^\rho$. It would be more accurate to write the effective Lagrangian as a differential operator that acts on the unity

$$\mathcal{L}(x) = \mathcal{L}(\square, \phi_{\mu\rho} \partial^\mu \partial^\nu \phi_\nu^\rho, \phi_{\mu\nu} \phi^{\mu\nu}, (\phi_{\mu\nu} \phi^{*\mu\nu})^2) \cdot 1$$

for the excitation field a_μ . Simultaneously the poles of the Green function $D_{\mu\nu}$ are found. To fulfill this program it is convenient [2] to diagonalize the polarization operator

$$\Pi_{\mu\nu}(k) = \sum_{i=1}^4 \mathcal{H}_i(k) \frac{b_\mu^{(i)} b_\nu^{*(i)}}{(b_\mu^{(i)} b^{*\mu(i)})} \quad (22)$$

Here $\mathcal{H}_i$, functions of all the scalars, are eigen-values and $b_\mu^{(i)}$ are eigenvectors of the polarization operator

$$\Pi_{\mu\nu} b^{(i)\nu} = \mathcal{H}_i b_\mu^{(i)} \quad (23)$$

Then due to the orthogonality $b_\mu^{*(i)} b^{\mu(j)} = 0$ for $i \neq j$ the Green function obtained by inversion of Eq. (2) is represented as follows

$$D_{\mu\nu}(k) = \sum_{i=1}^4 \frac{1}{k^2 - \mathcal{H}_i(k)} \frac{b_\mu^{(i)} b_\nu^{*(i)}}{(b_\mu^{(i)} b^{*\mu(i)})} \quad (24)$$

and the vector-potential of the excitation as

$$a_\mu(k) = \sum_{i=1}^4 \delta(k^2 - \mathcal{H}_i(k)) b_\mu^{(i)}(k) \quad (25)$$

This implies that the Laplacian $\square$ and $\partial^\mu \partial^\nu$ act on all the functions $\phi_{\mu\nu}(x)$ to the right of them in the power series expansion of $\mathcal{L}$. The quantity $\mathcal{L}$ being written in such a way, eq. (1) gives rise to dependence of the polarization operator, say, in a constant magnetic field $F_{\mu\nu}$, on k^2 and kF^2k .

From this it is seen that to find the dispersion law for the wave whose vector-potential is $b_\mu^{(i)}(k)$ one must solve the equation

$$k^2 = \mathcal{H}_i(k) \quad (26)$$

Owing to the transversality (10) one of the eigenvalues, say, with $i = 4$ is zero

$$\mathcal{H}_4 = 0, \quad b_\mu^{(4)} = k_\mu \quad (27)$$

This wave is purely longitudinal in the four-dimensional sense and the electromagnetic field strength for it is zero. Therefore it is sufficient to sum over the values $i = 1, 2, 3$ in (22)-(25) (the longitudinal part in $D_{\mu\nu}$ remains arbitrary) and to solve, correspondingly, three dispersion equations (26).

The more traditionally exploited quantity, the refractive index n , defined as the ratio of the velocity of light in the vacuum to its phase velocity in the medium, for each eigenmode in (26) is

$$n_i(k_0, \vec{k}) = \frac{|\vec{k}|}{k_0} = \sqrt{1 + \frac{\mathcal{H}_i(k_0, \vec{k})}{k_0^2}}$$

We do not find it convenient, however, to use this quantity in the present context.

Now we proceed to study the solutions of equations (7) using the representation (13)-(15), (21) for $\tilde{T}_{\mu\nu}$. First of all we introduce four mutually orthogonal basis vectors in the Minkowski space

$$\begin{aligned} C_\mu^{(1)} &= k^2 F_{\mu\lambda}^2 k^\lambda - k_\mu (k F^2 k) \\ C_\mu^{(2)} &= F_{\mu\lambda}^* k^\lambda, \quad C_\mu^{(3)} = F_{\mu\lambda} k^\lambda \\ C_\mu^{(4)} &= k_\mu \end{aligned} \quad (28)$$

and normalize $C_\mu^{(1,3,4)}$ to unity and $C_\mu^{(2)}$ to minus unity:

$$\begin{aligned} a_\mu^{(1)} &= C_\mu^{(1)} (-k^2 (KF^2 k) (KF^{*2} k))^{-1/2} \\ a_\mu^{(2)} &= -C_\mu^{(2)} (KF^{*2} k)^{-1/2} \\ a_\mu^{(3)} &= C_\mu^{(3)} (-KF^2 k)^{-1/2} \\ a_\mu^{(4)} &= C_\mu^{(4)} (k^2)^{-1/2} \end{aligned} \quad (29)$$

Note that in the physical range of momenta and frequencies (real $\vec{k}$ and k_0) the value of $kF^2 k$ is negative. ^(see Appendix) The vector $C_\mu^{(3)}$ is space-like while the vectors $C_\mu^{(1,2)}$ may be either time-, or space-, or light-like in different regions of the momentum space K_μ (see Table 1).

The charge symmetric case $\mu = 0$.

For the case $\mu = 0$ with the use of (13), (14) we obtain:

$$\begin{aligned} \pi_\mu^\nu a_\nu^{(1)} &= p a_\mu^{(1)} + q a_\mu^{(2)} \\ \pi_\mu^\nu a_\nu^{(2)} &= -q a_\mu^{(1)} + s a_\mu^{(2)} \\ \pi_\mu^\nu a_\nu^{(3)} &= t a_\mu^{(3)}, \quad \pi_\mu^\nu a_\nu^{(4)} = 0 \end{aligned} \quad (30)$$

where

$$\begin{aligned} p &= k^2 \pi_1 + (KF^{*2} k) \pi_3 - \frac{(2k)^2 (KF^2 k)}{(KF^{*2} k) k^2} \pi_4 \\ q &= \frac{(2k)(2F^* k)}{(KF^{*2} k)} \sqrt{-\frac{KF^2 k}{k^2}} \pi_4 \\ s &= k^2 \pi_1 - \frac{(2F^* k)^2}{(KF^{*2} k)} \pi_4 \\ t &= k^2 \pi_1 - (KF^2 k) \pi_2 + 2 F k^2 \pi_3 \end{aligned} \quad (31)$$

Remind that in the present case $\pi_{5,6,7} = 0$ due to (20). To avoid possible misunderstanding we emphasize that the scalars $kF^{*2}k$ and $(uF^{*}k)^2$ are not independent and reduce to those mentioned as arguments of π_i in formula (13). The pseudoscalar $(uF^{*}k)$ is independent (see Appendix).

From (30) it is seen that $a_{\mu}^{(3)} = b_{\mu}^{(3)}$ is an eigenvector the same as in the quantum field theory [2] with external magnetic field:

$$\mathcal{H}_3 = t, \quad b_{\mu}^{(3)} = a_{\mu}^{(3)} \quad (32)$$

From (30) it follows that in the representation on the basic vectors $a_{\mu}^{(1,2)}$ the polarization operator is the matrix

$$\begin{pmatrix} p & q \\ -q & s \end{pmatrix} \quad (33)$$

Diagonalization of this matrix gives the following eigenvalues

$$\mathcal{H}_{1,2} = \frac{p+s \pm \sqrt{(p-s)^2 - 4q^2}}{2} \quad (32a)$$

and eigenvectors

$$b_{\mu}^{(1,2)} = (s - \mathcal{H}_{1,2})a_{\mu}^{(1)} + qa_{\mu}^{(2)} = qa_{\mu}^{(1)} + (p - \mathcal{H}_{1,2})a_{\mu}^{(2)} \quad (34)$$

which on the solution of the dispersion equations (26) with $i = 1, 2$, take, respectively, the form

$$b_{\mu}^{(1,2)} = (s - \kappa^2)a_{\mu}^{(1)} + qa_{\mu}^{(2)} = qa_{\mu}^{(1)} + (p - \kappa^2)a_{\mu}^{(2)} \quad (35)$$

Equalities in (34), (35) are valid up to normalization. Hence it appears that as distinct from the eigenmode $b_\mu^{(3)} \delta(k^2 - \mathcal{H}_3)$, the polarizations of modes 1,2 cannot be described in a closed form since for this purpose one must know the scalar functions p , q , S , what is possible only approximately. In this respect the situation is similar to the quantum field theory with arbitrary combination of constant electric and magnetic fields [2] and differs from the case of the pure magnetic field, when all the three four-polarization vectors of the eigenmodes are described in the final form and ~~do~~ not depend on any particular approximation ^{x)}. The reason for this may be as follows. The fourpolarization $a_\mu^{(2)}$ is a pseudovector and, therefore, to mix it with the vector $a_\mu^{(1)}$ or $a_\mu^{(3)}$ one should have a pseudoscalar at one's disposal. In our case such a pseudoscalar is the invariant $\alpha F^* k$, that enters as a factor in

q (see Eq. (31)). In Ref.[2] it is the pseudoscalar y (12). In the purely magnetic case without medium there is no such pseudoscalar. What has been said does not explain, however, why $a_\mu^{(1)}$ and $a_\mu^{(3)}$ are not mixed.

Figure [1a] presents the polarization directions of electric $\vec{e}^{(1,2,3)}$ and magnetic $\vec{h}^{(1,2,3)}$ fields in the waves with four-potentials $a_\mu^{(1,2,3)}$. According to the formulae which we rewrite from Ref.[8] they are

^{x)} According to Refs. [2,8] they are $a_\mu^{(1,2,3)}$. In a particular approximation the polarizations corresponding to two of these vectors were found also in Ref.[9].

$$\begin{aligned} \vec{e}^{(1)} &= -\vec{k}_\perp k^2 k_0, & \vec{h}^{(1)} &= [\vec{k}_\perp \times \vec{k}_\parallel] k^2 \\ \vec{e}_\perp^{(2)} &= \vec{k}_\perp k_\parallel^2, & \vec{e}_\parallel^{(2)} &= \vec{k}_\parallel (k_\parallel^2 - k_0^2), & \vec{h}^{(2)} &= [\vec{k}_\perp \times \vec{k}_\parallel] k_0 \quad (35a) \\ \vec{e}^{(3)} &= k_0 [\vec{k}_\perp \times \vec{k}_\parallel], & \vec{h}_\perp^{(3)} &= -\vec{k}_\perp k_\parallel^2, & \vec{h}_\parallel^{(3)} &= \vec{k}_\parallel k_\perp^2 \end{aligned}$$

Eqs. (35a), as well as Fig. 1, relate to the reference frame which is at rest or may move along the external magnetic field $\vec{B}$.

Here $\vec{k}_\perp$ and $\vec{k}_\parallel$ are the wave vector components across and along $\vec{B}$. It is seen that in mode 3 (32) $\vec{e} \vec{h} = \vec{e} \vec{k} = \vec{h} \vec{k} = \vec{e} \vec{B} = 0$, i.e. this is a transverse planely-polarized wave with its electric field orthogonal to the plane where $\vec{B}$ and $\vec{k}$ lie. In a reference frame moving along $\vec{B}$ with such a velocity that the momentum $\vec{k}$ is orthogonal to $\vec{B}$ (such a frame exists in all the cases except for parallel propagation $\vec{k} \parallel \vec{B}$) the vector $a_\mu^{(4)}$ corresponds to a purely longitudinal electric wave $\vec{e} \parallel \vec{k}$ and the vector $a_\mu^{(2)}$ to a purely transverse electromagnetic one $\vec{e} \vec{k} = \vec{h} \vec{k} = 0$. Vectors (34) correspond to neither purely longitudinal nor to purely transverse waves even in this frame. However in this frame the electric fields corresponding to the waves $a_\mu^{(1)}$ and $a_\mu^{(2)}$ are mutually orthogonal. The only thing that can be said about (34) in the general case is that the modes $b_\mu^{(1)}$ and $b_\mu^{(2)}$ are planely polarized ($\mathcal{E}_{1,2}$ are supposed to be real, the possibility of complex $\mathcal{E}$ (32a) in the transparency region is discussed below) since the vectors (34) are either purely real or purely imaginary (it is seen in Table 1 that the vectors $b_\mu^{(1,2)}$ can at the utmost acquire the common

factor i) and that for these modes the vector $\vec{h}$ is orthogonal to the plane where the vectors $\vec{k}$, $\vec{B}$, $\vec{e}$ lie.

For the particular case of propagation across the external magnetic field $\vec{k} \cdot \vec{B} = 0$, in the rest frame the pseudoscalar $uF\vec{k}$ vanishes together with q (31), the vector $a_\mu^{(1)}$ cannot be combined linearly with the pseudovector $a_\mu^{(2)}$, and each of them becomes an eigenvector of $\Pi_{\mu\nu}$, as is seen from (30) as well as from (34). In Eq. (34) at $q = 0$ (we adopt that $\sqrt{(p-s)^2} = p-s$) $b_\mu^{(1)} = (s-p)a_\mu^{(1)}$, $b_\mu^{(2)} = qa_\mu^{(2)}$, $\mathcal{H}_1 = p$, $\mathcal{H}_2 = s$, in the latter case the zero factor q being excluded by normalization, which was not fixed in Eq. 34.

For another particular case of propagation along the external magnetic field $\pm \vec{k} \parallel \vec{B}$ again $q = 0$ due to the vanishing of the factor KF^2k , and again eigenmodes are described by the vectors $a_\mu^{(1)}$ and $a_\mu^{(2)}$, but now the polarization of the first mode is purely transversal while that of the second one is longitudinal [8]. From the axial symmetry of the problem (the medium is either at rest or moves along $\vec{B}$) it should be expected [8] that $\mathcal{H}_1 = \mathcal{H}_3$ at $\pm \vec{k} \parallel \vec{B}$, i.e. $p = t$, which indeed is a kinematic consequence of eq. (31) (cf. the corresponding consideration for $\mu \neq 0$ presented below).

As has been mentioned above, from hermiticity of $\Pi_{\mu\nu}$ in the transparency region it follows that the coefficients $\Pi_{1,2,3,4}$ are ^{there} real. Therefore at space-like k_μ , $k^2 > 0$

the reality of all the elements of the matrix (33) follows from (31). ^(Remind that $KF^2K < 0$) Thus, the matrix (33) is not Hermitian and reality of eigenvalues (32a) is not guaranteed. This indicates the well comprehensible fact that in the Minkowskian space-time two eigenvalues $\mathcal{H}_{1,2}$ of a Hermitian tensor $\mathcal{T}_{\mu\nu}$ may appear to be complex if the lengths of the corresponding eigenvectors vanish: $b_{\mu}^{*(i)} b^{\mu(i)} = 0$. (In this case eqs. (23), (24) are reduced to the following form:

$$\mathcal{T}_{\mu\nu} = \sum_{i=1,2} \mathcal{H}_i \frac{b_{\mu}^{(i)} b_{\nu}^{*(i)}}{(b_{\mu}^{(i)} b^{*(i)\mu})} + \mathcal{H}_3 \frac{b_{\mu}^{(3)} b_{\nu}^{*(3)}}{(b_{\mu}^{(3)} b^{*\mu(3)})} \quad (36)$$

$$\mathcal{H}_1 = \mathcal{H}_2^*, \quad \mathcal{H}_3 = \mathcal{H}_3^* \quad (37)$$

$$b_{\mu}^{*(1,2)} b^{\mu(3)} = b_{\mu}^{*(i)} b^{(i)\mu} = 0, \quad b_{\mu}^{*(i)} b^{(i)\mu} \neq 0, \quad i=1,2 \quad (38)$$

$$D_{\mu\nu} = \sum_{i=1,2} \frac{b_{\mu}^{(i)} b_{\nu}^{*(i)}}{(b_{\mu}^{(i)} b^{*(i)\mu})} \frac{1}{k^2 - \mathcal{H}_i} + \frac{b_{\mu}^{(3)} b_{\nu}^{*(3)}}{b_{\mu}^{(3)} b^{*\mu(3)}} \frac{1}{k^2 - \mathcal{H}_3} \quad (39)$$

In (36), (38), (39) the designation $\bar{i}$ is introduced, which is interpreted as follows: $\bar{1} = 2$, $\bar{2} = 1$. Eqs. (25), (26) remain unchanged. Besides, for the symmetric $\Pi_{\mu\nu}$ under consideration (when $\mu = 0$) $b_{\mu}^{(1)} = b_{\mu}^{*(2)}$. At time-like K_{μ} as a consequence of (10) the three quantities $\mathcal{H}_i$, $i = 1, 2, 3$ are eigenvalues of the Hermitian tensor Π_{kj} with the components $k, j = 1, 2, 3$ which relate to the three-dimensional subspace orthogonal to the vector K_{μ} . Since the metric in this subspace is definite, all the $\mathcal{H}_i$ are real and thus the complexity in question can arise only in the space-like region of the four-momentum K_{μ} . The complexity of $\mathcal{H}_{1,2}$ must entail the complexity of the solutions (if they exist in this region) of the dispersion equations (26) for K_0 which are not induced by any absorption processes intrinsic for the polarization operator. In contrast with all the other requirements on the polarization operator formulated in the present paper, the reality of $\mathcal{H}$ in the transparency region cannot be expected a priori from the perturbative calculations, since the latter may give rise to ghosts (cf. the ghost pole in quantum electrodynamics [10]). The presence of ghosts must be established from the results of approximate calculations and would indicate intrinsic inconsistency of the latter.

As distinct from the ghost situation when there arises a singularity in the forbidden region, in the case under consideration the singularity of $\Pi_{\mu\nu}$ (it is the root branching point in (32a) or in the subsequent eqs. (45), (47)) may lie on the real axis without contradicting to what is admissible by

causality. The (complex) poles of the Green function D (39) (solutions of the dispersion equations (26), if any) may appear lying either on the physical sheet of the K_0 -plane where they are forbidden by analyticity, or on the second sheet where they are admissible from the viewpoint of analyticity but may well be ghost singularities in the sense that they correspond to no physics and describe no real absorption. The expressions that we have obtained for $\pi_{\mu\nu}$ are cumbersome for which reason we have not yet answered the question of whether or not any of the possibilities considered above can occur in reality.

To conclude this discussion we emphasize once again that the requirements imposed on the polarization operator by unitarity do not yet guarantee that the right-hand side $\mathcal{H}_i$ of the dispersion equation be real. The appearance of imaginarity would be pregnant with complications.

The charge asymmetric case $\mu \neq 0$.

Let us pass over to the general case of $\mu \neq 0$. Equation (13) now includes antisymmetric matrices $\psi_{\mu\nu}^{(5,6)}$. Instead of Eq. (30) we have now

$$\begin{aligned}\pi_{\mu}^{\nu} a_{\mu}^{(4)} &= p a_{\mu}^{(1)} + q a_{\mu}^{(2)} + r a_{\mu}^{(3)} \\ \pi_{\mu}^{\nu} a_{\nu}^{(2)} &= -q a_{\mu}^{(1)} + s a_{\mu}^{(2)} + v a_{\mu}^{(3)} \\ \pi_{\mu}^{\nu} a_{\nu}^{(3)} &= -r a_{\mu}^{(1)} + v a_{\mu}^{(2)} + t a_{\mu}^{(3)} \\ \pi_{\mu}^{\nu} a_{\nu}^{(4)} &= 0\end{aligned}\tag{40}$$

where p, q, t as before are given by (31) and the rest of the invariant functions are

$$\begin{aligned} T &= -(uK) \left[\sqrt{k^2(KF^{*2}K)} \pi_5 + 2\tilde{F} \sqrt{\frac{k^2}{KF^{*2}K}} \pi_6 \right] \\ V &= (uF^{*}K) \sqrt{-\frac{KF^2K}{KF^{*2}K}} \pi_5 \end{aligned} \quad (41)$$

Remember that in the transparency region π_{56} are purely imaginary as a consequence of hermiticity of $\pi_{\mu\nu}$. Diagonalization of $\pi_{\mu\nu}$ reduces to that of the matrix composed of the coefficients in the right-hand side of Eq. (40). The coefficients T and V give Hermitian or anti-Hermitian contribution to this matrix in different regions of the momentum variable k_μ , as is shown in Table 1.

The eigenvalue problem is reduced to the cubic equation

$$x^3 + bx^2 + cx + d = 0 \quad (42)$$

where

$$\begin{aligned} b &= -(p+s+t) \\ c &= ps+st+pt+r^2v^2+q^2 \\ d &= -r^2s+v^2p-q^2t-ps\tilde{t}+2+q\tilde{v} \end{aligned} \quad (43)$$

All the coefficients in Eq. (43) are real in the transparency region at any sign of k^2 and KF^2K , therefore, one of the solutions of Eq. (42) is real and the other two are either real or, if not, they are mutually complex-conjugated. In the latter case the representations (36)-(39) are again valid. The meaning of the fact that the quantities $\mathcal{H}_i$ may be complex was discussed above. Now we present formulae for the solutions of Eq. (42)

assuming that all of them are real. This is guaranteed if the discriminant $P^3 + Q^2$ is negative what implies that also $P < 0$. The quantities P and Q are composed of the coefficients of (43) as follows

$$2Q = \frac{2b^3}{27} - \frac{bc}{3} + d, \quad P = \frac{3c - b^2}{9} \quad (44)$$

The solutions of Eq. (42) are:

$$\begin{aligned} x_{1,3} &= R \cos \alpha \pm \sqrt{3} R \sin \alpha - \frac{b}{3} \\ x_2 &= -2R \cos \alpha - \frac{b}{3} \end{aligned} \quad (45)$$

where $R = \sqrt{-P \operatorname{sgn} Q}$, $\alpha = (\frac{1}{3} \arccos \frac{Q}{R^3})$.

The corresponding eigenvectors of $T_{\mu\nu}$ are written e.g. in the form

$$\begin{aligned} b_{\mu}^{(i)} &= [\tau(x_i - t) + r v] a_{\mu}^{(1)} + [(x_i - t)(p - x_i) - r^2] a_{\mu}^{(2)} + \\ &+ [\tau q + v(p - x_i)] a_{\mu}^{(3)} \end{aligned} \quad (46)$$

At $\tau = v = 0$, $b_{\mu}^{(i)}$ (46) convert into $b_{\mu}^{(i)}$ (34) if $x_i \neq t$ and into $a_{\mu}^{(3)}$ if $x_i = t$.

Now that the expressions for the r.-h. sides of the dispersion equations (26) and the corresponding eigenmode polarizations are obtained, it should be noted that being found from the most general properties of the theory and based on no dynamical calculations these results and the consequences of them discussed below do not tell us whether or not the solutions of Eqs. (26) are as a matter of fact realized. In other

words we investigate "kinematic vacancies" and leave alone for the present the question of which of them are filled. We know from experience that in specific approximate dynamical calculations it may turn out that for some modes there are no physical solutions of the dispersion equations (26). E.g. the book [11] by V.L.Ginzburg gives evidence of the absence of solutions for one of the three modes in some models of plasma kinetics when spatial dispersion (the dependence of $\Pi_{\mu\nu}$ on $\vec{k}$) is disregarded. In the vacuum with a magnetic field the mode 1 ($q=0, \mathcal{H}_1=p$ in (32a), (34)) [8] is absent except for the longitudinal propagation. It is to be supposed that in the general case all the kinematical possibilities are saturated, i.e. all the three modes are realized.

Now we pass over to the investigation of the kinematical information contained in the solutions (46). Even at real $\mathcal{H}$ the eigenvectors (46) are complex in the domain of transparency. With the aid of Table 1 we establish that in this domain for

$$kF^{*2}k > 0$$

$$i \operatorname{Im} b_{\mu}^{(i)} = [+q + v(p - \mathcal{H}_i)] a_{\mu}^{(3)}$$

$$\operatorname{Re} b_{\mu}^{(i)} = [q(\mathcal{H}_i - t) + +v] a_{\mu}^{(1)} + [(\mathcal{H}_i - t)(p - \mathcal{H}_i) - v^2] a_{\mu}^{(2)} \quad (46a)$$

For $kF^{*2}k < 0$ the designations of the real and imaginary parts in these formulae must change places. One case reduces to the other by changing the common normalization of the vectors

$b_{\mu}^{(i)}$, therefore, they prove to be in fact undistinguishable. Complexity of the polarization 4-vectors in the domain of transparency implies that eigenmodes are elliptically

polarized. For the elliptic polarization the antisymmetric matrices $\psi_{\mu\nu}^{(5,6)}$ in the expansion of the Hermitian polarization operator are directly responsible and as has been proved above these matrices appear at the same time to be odd in the chemical potential (charge-odd). Now we describe the elliptic polarizations. Since the vector $\vec{h}$ of the magnetic field in the wave is orthogonal to the wave vector $\vec{k}$ for all 4-polarizations $a_{\mu}^{(i)}$ (this follows simply from the fact that $h_m^{(i)} = \epsilon_{mnl} k_n a_l^{(i)}$, $m, n, l = 1, 2, 3$ and physical is due to the absence of magnetic charges), it draws an ellipse in the plane orthogonal to the direction of propagation. The first semiaxis of the ellipse lies in the plane containing the external field $\vec{B}$ and the wave vector $\vec{k}$ (plane (Bk)) on the line orthogonal to $\vec{k}$ ($\vec{h}^{(3)}$ -direction in Fig. 1a). The second semiaxis is perpendicular to this plane ($\vec{h}^{(1,2)}$ -direction in Fig. 1a). This is valid for all the three modes $b_{\mu}^{(i)}$ (46). Since the electric field vector $\vec{e}$ of the wave lies in the (Bk) -plane for the 4-polarizations $a_{\mu}^{(1,2)}$ is orthogonal to this plane for $a_{\mu}^{(3)}$ we conclude from (46a) that at each spatial point $\vec{x}$ the vector $\vec{e}$ circumscribes an ellipse in the plane orthogonal to the (Bk) -plane (Fig. 1b). One cannot say anything of the angle between this plane and the propagation direction $\vec{k}$ (angle γ_i in Fig. 1b) in the general case. This angle, the same as the eccentricity of the ellipse, is different for each of the three modes (46) and can be found only from approximate calculations. One

semiaxis of the ellipse lies in the (Bk) -plane. (It makes the above undetermined angle γ_i with the vector $\vec{k}$). The other semiaxis is orthogonal to this plane. The latter direction is universal for all the modes. In somewhat different terms the elliptic polarizations are presented in the book by V.L.Ginzburg (Ref. [11], § 11) among the results concerning the plasma described by classical equations in a magnetic field. It is obvious that this one and the other results of the present section are model-independent.

To sum up the consideration of propagation at arbitrary angle to the external field we say: all the three eigenmodes are elliptically polarized so that the semiaxes of the ellipses lie one in the (Bk) -plane and the other is orthogonal to it. This is valid for the vectors $\vec{E}$ and $\vec{h}$ in the wave. The polarization plane of the latter is at the same time always orthogonal to $\vec{k}$.

In the particular case of propagation across the external magnetic field $\vec{B}$ when the medium is at rest we find that $q = v = 0$ due to the vanishing of the pseudoscalar $\mathcal{U}F^*k$ (see Eqs. (31), (41)). From Eq. (40) we have for the eigenvectors and eigenvalues

$$b_{\mu}^{(2)} = a_{\mu}^{(2)}, \quad \mathcal{E}_2 = S$$

$$b_{\mu}^{(1,3)} = (\mathcal{E}_{1,3} - t) a_{\mu}^{(1)} + r a_{\mu}^{(3)}, \quad \mathcal{E}_{1,3} = \frac{(p+t) \pm \sqrt{(p-t)^2 - 4r^2}}{2} \quad (47)$$

Here the eigenmode $b_{\mu}^{(2)}$ is the plane-polarized transversal wave, for which the vector $\vec{E}$ is parallel to $\vec{B}$. The eigenmodes $b_{\mu}^{(1,3)}$ are polarized elliptically since at $(\kappa F^* k) k^2 > 0$ the coefficient t is purely imaginary and the vector $a_{\mu}^{(1)}$ is

real whereas at $(KF^*k)k^2 < 0$ vice versa (see Table 1).

The ellipse circumscribed by the vector $\vec{E}$ lies in a plane orthogonal to $\vec{B}$, and one of its semiaxes coincides with the propagation direction $\vec{k}$. It is worth noting that the vector $\vec{h}$ does not circumscribe an ellipse; it is parallel to the external field $\vec{B}$ because the vector potential $a_{\mu}^{(1)}$ has $\vec{h} = 0$ for the propagation across $\vec{B}$. In the model of plasma kinetics without spatial dispersion that is considered in Ref. [11], § 11 two modes are dynamically realized which in our case coincide with $b_{\mu}^{(2)}$ and with one of the modes $b_{\mu}^{(1,3)}$. From the formulae presented in Ref. [11] one concludes that these modes are polarized just in the way described above.

In another particular case of propagation along the external field $\vec{B}$ we have $KF^2K = 0$. Here the medium may either move along $\vec{B}$ or be at rest since the transition to the reference frame moving along the external magnetic field does not change the latter. Now again $q = v = 0$ and besides $t = p$ (see Eqs. (31), (41)). We again have Eq. (47) but with $p = t$

$$b_{\mu}^{(2)} = a_{\mu}^{(2)}, \quad \mathcal{L}_2 = S \quad (48)$$

$$b_{\mu}^{(1,3)} = (\mathcal{L}_{1,3} - t)a_{\mu}^{(1)} + \tau a_{\mu}^{(3)}, \quad \mathcal{L}_{1,3} = t \pm \sqrt{-\tau^2}$$

The mode $b_{\mu}^{(2)}$ is a purely electric longitudinally polarized wave (see Eq. (35a)). The fact that $KF^2K = 0$ leads to the validity of the equality $KF^*k = 2\tilde{f}k^2$ under which τ is guaranteed to be imaginary in the domain of transparency due to the positivity of $\tilde{f} = \frac{B^2}{2}$ (see Eq. (41) or Table 1). Therefore $\mathcal{L}_{1,3}$ remain real. With this circumstance in mind we change the norms

lization of $b_\mu^{(1,3)}$ by multiplying it by the factor $\pm (-\gamma^2)^{-1/2} (kF^2 k)^{1/2}$ and then have from Eq. (48)

$$b_\mu^{(1,3)} = \frac{C_\mu^{(1)}}{k_2} \pm i a_\mu^{(3)}, \quad \left(\frac{C_\mu^{(1)}}{k_2} \parallel \vec{B}, \frac{C_\mu^{(3)}}{k_2} \perp \vec{B} \right) \quad (48)$$

where $a_\mu^{(1,3)}$ are given by Eq. (28) and in this case they are light-like. The vectors $\vec{C}_\mu^{(1)}$ and $\vec{C}_\mu^{(3)}$ are spacelike unit vectors which lie on a plane orthogonal to $\vec{B}$ and are real. They correspond to two transversal waves polarized in mutually orthogonal planes. On calculating the field strengths for the vector-potentials $\vec{C}_\mu^{(1)}, \vec{C}_\mu^{(3)}$ one can see that the lengths of the vectors $\vec{E}$ and $\vec{H}$ coincide in the two waves. Therefore the eigenmodes (48a) are circularly polarized in the opposite directions. As to the dispersion laws they are different for these two polarizations $\mathcal{H}_1 \neq \mathcal{H}_3$. (This phenomenon is known to be the basis of the Faraday effect). As distinct from the charge-symmetric case $\mu = 0$, with the plane-polarized modes discussed above now the axial symmetry of the problem (remember that $\vec{k} \parallel \vec{B}$) does not require the equality $\mathcal{H}_1 = \mathcal{H}_3$ (although again $p = t$) since the clockwise or counterclockwise rotations with respect to the external field $\vec{B}$ are physically different once the system is noninvariant under the charge reflection. The axial symmetry resulted only in the fact that the polarization is circular and not elliptic since otherwise the large semiaxis would distinguish some direction in the plane orthogonal to $\vec{B}$ and $\vec{k}$.

Concluding the general part of this article we present the contribution of the polarization operator and the Green function into the thermodynamical potential [1] if the

representations (22), (24) are used

$$\Omega = \Omega_0 - \frac{V}{(2\pi)^3 \beta} \sum_{\mathbf{k}_4} \int d^3k \int_0^e \frac{de}{e} \sum_{i=1}^3 \frac{\partial \mathcal{H}_i}{k^2 - \mathcal{H}_i} \quad (49)$$

Here Ω_0 is the thermodynamical potential in the magnetic field calculated in our previous paper [12] V is the volume. The applicability of eq. (49) is however not limited to the case of the magnetic field. Note that the above discussed hypothetical possibility (37) for $\mathcal{H}_i$ to be complex in the domain of transparency does not lead to complexity of (49).

4. ONE-LOOP APPROXIMATION

The one-loop approximation for the polarization operator in external field is the expression

$$\pi_{\mu\nu}(k_4, \vec{x}, \vec{y} | A^{ext}) = \frac{e^2}{\beta} T \sum_{\substack{p_4 = \frac{(2n+1)\pi}{\beta}}} \gamma_\mu G(p_4, \vec{x}, \vec{y} | A^{ext}) \gamma_\nu G(p_4 + k_4, \vec{y}, \vec{x} | A^{ext}) \quad (50)$$

where G is the temperature Green function of electron in an external, in this case magnetic, field A^{ext} . It is presented e.g. in Ref. [2]. Detailed calculations of $\pi_{\mu\nu}$ (50) may be found in a separate paper by one of the present authors (H.P.R.) [13] where the resulting expressions for $\pi_{\mu\nu}$ in components are given and the properties established in Sec. 2 of the present paper are traced in detail. Here we present the result of the calculation of (50) directly for the scalar functions p, s, t, q, r, v met in Secs. 2, 3 through which the eigenvalues $\mathcal{H}_i$ are expressed.

The calculations were carried out for the medium at rest and the axis 3 was chosen to be parallel with the direction of the magnetic field ^{whose} ^{is} equal to B. After the calculations the analytical continuation $k_4 \rightarrow i k_0$ was performed leading to the advanced polarization operator [1]. In the formulae below we have used the designations $Z_1 = K_3^2 - K_0^2$, $Z_2 = K_1^2 = K_1^2 + K_2^2$ for combinations of the electromagnetic wave momentum components, $\mathcal{E}_\ell(p) = \sqrt{m^2 + p_3^2 + 2eB\ell}$ for the energy of the electron at the Landau level with the number ℓ , whose momentum along the magnetic field is p_3 . Besides, we introduce the notation

$$\alpha_{\ell\ell'} = Z_1 + 2eB(\ell' - \ell), \quad \lambda_{\ell\ell'} = Z_1 + 2eB(\ell + \ell')$$

The common "dispersion" denominator is given by

$$D_{\ell\ell'}(p) = (2p_3K_3 + \alpha_{\ell\ell'} + 2\mathcal{E}_\ell^2(p)K_0 + i0)(2p_3K_3 + \alpha_{\ell\ell'} - 2\mathcal{E}_{\ell'}^2(p)K_0 - i0)$$

The vanishing of $D_{\ell\ell'}$ is responsible for the absorption induced by electron-positron pair production and for the absorption induced by the transition of electron (positron) into excited state and/or by its acceleration. Fermi electron and positron distributions are also met in the expressions below:

$$n_i^\mp(p) = [1 + \exp(\mathcal{E}_i(p) \mp \mu)\beta]^{-1}$$

Let $I_i = p, s, t$ for $i = 1, 2, 3$, resp. Then these three invariant functions which are nonzero already in the vacuum are given by

$$I_1(K_0, K_3, K_1) = -\frac{e^3 B}{4\pi^2} \sum_{\ell, \ell'=0}^{\infty} \int \frac{dp_3}{E_{\ell}(p_3)} \left[\Lambda_{\ell\ell'}^{(i)} - \frac{(2p_3 K_3 + \alpha_{\ell\ell'}) \phi_{\ell\ell'}^{(i)}}{D_{\ell\ell'}(p_3)} \right] (\eta_{\ell}^-(p_3) + \eta_{\ell}^+(p_3)) + \mathcal{H}_i^{QFT} \quad (51)$$

Where $\mathcal{H}_i^{QFT}$ are the eigenvalues of the renormalized polarization operator in vacuum in presence of the magnetic field which were calculated in Refs. [2, 8]. Nonrenormalized (diverging) expressions for $\mathcal{H}_i^{QFT}$ are obtained in the first term in eq. (51) if $(\eta_{\ell}^-(p_3) + \eta_{\ell}^+(p_3))$ is replaced by minus unity. The functions $\Lambda_{\ell\ell'}^{(i)}$ and $\phi_{\ell\ell'}^{(i)}$ in Eq. (51) are expressed as

$$\begin{aligned} \Lambda_{\ell\ell'}^{(1)} &= \frac{k^2}{z_1} F_{\ell\ell'}^{(2)}(z_2/2eB) \quad , \quad \Lambda_{\ell\ell'}^{(2)} = \frac{2eB}{z_1} (\ell - \ell') F_{\ell\ell'}^{(1)}(z_2/2eB) \\ \Lambda_{\ell\ell'}^{(3)} &= F_{\ell\ell'}^{(2)}(z_2/2eB) \quad , \quad \phi_{\ell\ell'}^{(1)} = \frac{k^2}{z_1} \left[\lambda_{\ell\ell'} F_{\ell\ell'}^{(2)}(z_2/2eB) - 4z_2 N_{\ell\ell'}^{(1)}(z_2/2eB) \right] \\ \phi_{\ell\ell'}^{(2)} &= \frac{2}{z_1} \left\{ \left[2e^2 B^2 (\ell - \ell')^2 + z_1 (2m^2 + eB(\ell + \ell')) \right] F_{\ell\ell'}^{(1)}(z_2/2eB) + 2eB(\ell\ell')^{1/2} z_1 G_{\ell\ell'}^{(1)} \right\} \\ \phi_{\ell\ell'}^{(3)} &= \lambda_{\ell\ell'} F_{\ell\ell'}^{(2)}(z_2/2eB) + 4z_2 N_{\ell\ell'}^{(1)}(z_2/2eB) \end{aligned}$$

and

$$\begin{aligned} F_{\ell\ell'}^{(1)}(x) &= \left\{ \left[L_{\ell-1}^{\ell-\ell'}(x) \right]^2 + \frac{\ell'}{\ell} \left[L_{\ell'}^{\ell-\ell'}(x) \right]^2 \right\} \frac{(\ell-1)!}{(\ell-1)!} x^{\ell-\ell'-x} e^{-x} \\ F_{\ell\ell'}^{(2,3)}(x) &= \left\{ \frac{x}{\ell} \left[L_{\ell-1}^{\ell-\ell'+1}(x) \right]^2 \pm \frac{\ell'}{x} \left[L_{\ell'}^{\ell-\ell'-1}(x) \right]^2 \right\} \frac{(\ell-1)!}{(\ell-1)!} x^{\ell-\ell'-x} e^{-x} \end{aligned}$$

$$G_{\ell\ell'}^{(1)}(x) = 2\sqrt{\frac{\ell'}{\ell}} \frac{(\ell-1)!}{(\ell'-1)!} \chi^{\ell-\ell'} L_{\ell-1}^{\ell-\ell'}(x) L_{\ell'}^{\ell-\ell'}(x) e^{-x}$$

$$N_{\ell\ell'}^{(1)}(x) = -\frac{\ell'!}{(\ell-1)!} \chi^{\ell-\ell'-1} L_{\ell-1}^{\ell-\ell'+1}(x) L_{\ell'}^{\ell-\ell'-1}(x) e^{-x}$$

Here L_n^α are the generalized Laguerre polynomials connected with the reduced hypergeometrical function as $L_n^\alpha(x) = \binom{n+\alpha}{n} F(-n, \alpha+1, x)$

The rest of invariant functions are

$$q = \frac{e^4 B^2 k_0}{\pi^2 \chi_1} \sqrt{\frac{k^2}{\chi_2}} \sum_{\ell, \ell'=0}^{\infty} (\ell-\ell') F_{\ell\ell'}^{(1)} \int_{-\infty}^{\infty} \frac{d\beta_3}{E_\ell(\beta_3)} \frac{\beta_3 \alpha_{\ell\ell'} - 2(m^2 + 2eB\ell)k_3}{D_{\ell\ell'}(\beta_3)} (\eta_\ell^-(\beta_3) + \eta_\ell^+(\beta_3))$$

$$\tau = \frac{ie^3 B k_0}{2\pi^2} \sqrt{\frac{k^2}{\chi_1}} \sum_{\ell, \ell'=0}^{\infty} \lambda_{\ell, \ell'} F_{\ell\ell'}^{(3)} \int_{-\infty}^{\infty} \frac{d\beta_3}{D_{\ell\ell'}(\beta_3)} (\eta_\ell^-(\beta_3) - \eta_\ell^+(\beta_3))$$

$$\nu = \frac{ie^3 B}{2\pi^2} \sqrt{\frac{\chi_2}{\chi_1}} \sum_{\ell, \ell'=0}^{\infty} \frac{\ell+\ell'}{\ell-\ell'} F_{\ell\ell'}^{(3)} \int_{-\infty}^{\infty} \frac{d\beta_3 (2\beta_3 \chi_1 + k_3 \alpha_{\ell\ell'})}{D_{\ell\ell'}(\beta_3)} (\eta_\ell^-(\beta_3) - \eta_\ell^+(\beta_3))$$

In the latter formula at $\ell = \ell'$ one should take

$$\frac{\ell+\ell'}{\ell-\ell'} F_{\ell\ell'}^{(3)} = 2 L_{\ell-1}^1(x) (L_\ell^0(x) + L_{\ell-1}^0(x)) e^{-x} \quad \text{As soon as}$$

a Laguerre polynomial appears with -1 for the lower index,

it must disappear from the above formulae.

The integral for q vanishes if $(\eta_\ell^- + \eta_\ell^+)$ is replaced by unity in the integrand. This is why the invariant q does not have the QFT contribution, although it is connected with symmetric matrices.

The invariant functions τ, ν connected with the

(x) The authors are indebted to A. Cabo for demonstrating this explicitly.

antisymmetric matrices $\psi_{\mu\nu}^{(5,6)}$ (15) and responsible for the elliptic polarization are odd indeed in the chemical potential μ , since they contain the difference $n^- - n^+$. The other functions S, p, t, q are even in μ , since they contain the sum $n^- + n^+$. The finite part of $\pi_{\mu\nu}$ was separated by subtracting the QFT limit, i.e. the value of $\pi_{\mu\nu}$ at $\beta^{-1} \equiv T = \mu = 0$ and by adding subsequently the renormalized polarization operator in vacuum with the external magnetic field as it has been pointed out in Ref. [1].

If the way of calculations is gauge-invariant, i.e. a gauge-invariant intermediate regularization is used, the result of the above procedure must be finite since the divergencies are just concentrated in the field-theoretical limit. If not, the divergencies may appear also in the terms linear and quadratic in μ (of the first and zeroth power in the external field, respectively). These divergencies, however, are not transversal, they are of the type of the fourth order light-by-light scattering diagram in quantum electrodynamics. In our case they do not appear since $\pi_{\mu\nu}$ calculated in components is sure to be transverse which is demonstrated in Ref. [13].

To obtain expressions which are explicitly relativistic invariant and thus valid for any moving medium (provided that the electric field is zero in the rest system, see Eq. (11)) suffice it to perform some kinematical identifications in the above formulae.^{*)} In all the invariant functions K_3 should

^{*)} Consult Appendix to understand what follows

be replaced by $-(uF^*k)/\sqrt{2\mathcal{F}}$ and k_3^2 will then be replaced by $k^2 + (uk)^2 + (kF^2k)/2\mathcal{F} = (uk)^2 + (kF^2k)/2\mathcal{F}$ ($u^2 = -1$). After that k_0 must be replaced by $-(uK)$ and $\mathcal{E}_2 = k_1^2 + k_2^2$ by $-(kF^2k)$. The quantity $\mathcal{X}_1 = k_3^2 - k_0^2$ then turns into $k^2 + (kF^2k)/2\mathcal{F}$. Finally, $[2\mathcal{F}]^{1/2}$ should be substituted for B. When casting a backward look at the expressions obtained we notice that a somewhat simpler way could have been chosen. That is we can first calculate $\Pi_{\mu\nu}$ for zero temperature, then divide $\Pi_{\mu\nu}$ into parts symmetric and antisymmetric under permutation of the indices μ and ν and replace the degenerate $T = 0$ Fermi distribution of electrons by the sum $n^- + n^+$ for the symmetric and by the difference $n^- - n^+$ for the antisymmetric part. This is the way to obtain temperature dependence which enters through these distributions only. The idea of using calculations at zero temperature with the subsequent reproduction of temperature dependence by introducing $n^\pm$ may be found in the paper by Bakshi, Cover and Kalman [14]. However, these authors have evidently proceeded from the example of nonmagnetoactive plasma, when only the sum $(n^+ + n^-)$ is involved, and, therefore, for the polarization operator in the one-loop approximation in a magnetic field they have presented independent expressions for electrons and positrons, without dividing $\Pi_{\mu\nu}$ into symmetric and antisymmetric parts. Nevertheless their expressions could be simply reconstructed into ours, if there were

coincidence at $T=0$. Unfortunately, we do not observe this coincidence. We cannot reason why, but we can put forward two objections to the results of calculations at $T=0$ carried out in Ref. [14]:

- 1) the property (8) is not fulfilled for the component]
- 2) there is nonhermiticity in the components π_{i3}, π_{3i} (the latter may prove to be a misprint).

The authors are greatly indebted to Professor E.S. Fradkin for stimulating their interest to the problem and permanent attention to its development.

APPENDIX

In this appendix the form taken by various relativistic covariant formations met in this paper is presented in the Lorentz frame where the medium is at rest $\vec{u}=0$, $u_0=1$ and there exists only a magnetic field $B_i = \epsilon_{ijk} F_{jk}$ directed along the z -axis: $B_3 = B$, $B_{1,2} = 0$, $F_{12} = -F_{21} = B$. The other components of the tensor $F_{\mu\nu}$ vanish. For the dual tensor $F_{\mu\nu}^*$ the only components $F_{03}^* = -F_{30}^* = B$ are nonzero. The field invariant is $\mathcal{F} = \frac{1}{2} B^2$. The components of the photon momentum across and along the magnetic field being designated as $\vec{K}_\perp$ and $\vec{K}_\parallel$, $(\vec{K}_\parallel)_3 = K_3$, $(\vec{K}_\parallel)_{1,2} = 0$, $(\vec{K}_\perp)_3 = 0$,

, the momentum-containing invariants take the form

$$K^2 = \vec{K}_\parallel^2 + \vec{K}_\perp^2 - K_0^2$$

$$\frac{KF^2K}{2\mathcal{F}} = -K_\perp^2, \quad \frac{KF^{*2}K}{2\mathcal{F}} = K_\parallel^2 - K_0^2 \quad (\text{A.1})$$

The above formulae are also valid in any other frame containing only the magnetic field, i.e. in a frame where the medium moves along $\vec{B}$ ($u_3^2 - u_0^2 = -1$). Instructive also are the following relations between the invariants;

$$KF^{*2}K - KF^2K = 2K^2\mathcal{F}$$

$$(uF^*K)^2 = 2\mathcal{F}((uK)^2 - u^2K^2) - u^2(KF^2K) \quad (\text{A.2})$$

In the abovementioned frame this reduces to

$$(uF^*K) = B(K_0u_3 - u_0K_3)$$

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$KF^2K \leq 0$			
	$K^2 > 0$		$K^2 < 0, KF^{*2}K < 0$
	$KF^{*2}K > 0$	$KF^{*2}K < 0$	
s, p, t	real		Hermitian
q	real non-Hermitian	real non-Hermitian	imaginary Hermitian
r	imaginary Hermitian	real non-Hermitian	imaginary Hermitian
v	imaginary non-Hermitian	real Hermitian	real Hermitian
$a_\mu^{(1)}$	real	imaginary	real
$a_\mu^{(2)}$	real	imaginary	imaginary
$a_\mu^{(3)}$	real		

Table 1

The table is based on eqs. (29), (31), (41) and hermiticity of $\mathcal{T}_{\mu\nu}$. Real or imaginary character of the invariant functions s, p, t, q, r, v and of the basic polarization vectors is indicated for all regions of the photon momenta K_μ disregarding absorption (in the domain of transparency). Hermiticity and non-Hermiticity of the contribution from the invariant functions in the coefficient matrix of the sets (30) (40) is shown.

FIGURE CAPTIONS.

Fig.1.

a) Directions of electric and magnetic vectors in the plane-polarized waves corresponding to 4-polarizations $a_{\mu}^{(1,2,3)}$. They are eigenmodes for propagation in the external magnetic field $\vec{B}$ in vacuum [2,8], $\vec{k}$ is the photon momentum. The vectors $\vec{B}$, $\vec{k}$, $\vec{k}_{\perp}$, $\vec{h}^{(3)}$, $\vec{e}^{(1,2)}$ lie in the same plane and the vectors $\vec{e}^{(3)}$, $\vec{h}^{(1,2)}$ are orthogonal to it, $(\vec{h}^{(3)} \vec{k} = 0)$.

b) Elliptical polarization of the eigenmodes in plasma in magnetic field. Dashed is a sector of the plane orthogonal to the (BK) -plane. The sector is bounded by the ellipse drawn by the electric vector $\vec{e}^{(i)}$, and its semiaxes $\ell_{\parallel}^{(i)}$ and $\ell_{\perp}^{(i)}$. Index i labels the three eigenmodes. For transversal propagation $\vec{k} \parallel \vec{k}_{\perp}$, $\chi_{1,3} = 0$, $\chi_2 = \pi/2$ and $\ell_{\perp}^{(2)} = 0$. For longitudinal propagation $\vec{k} \parallel \vec{B}$, $\chi_{1,3} = \pi/2$, $\ell_{\perp}^{(1,3)} = \ell_{\parallel}^{(1,3)}$, $\chi_2 = 0$, $\ell_{\perp}^{(2)} = 0$. For the charge symmetric case $\mu = 0$ elliptical polarizations degenerate into plane ones; $\ell_{\parallel}^{(3)} = 0$, $\ell_{\perp}^{(1,2)} = 0$. For transversal propagation, besides, $\chi_1 = 0$, $\chi_2 = \pi/2$. For longitudinal propagation $\chi_1 = \pi/2$, $\chi_2 = 0$.

Fig 1 a

Fig 1 b