CAPTURE OF COOLING ELECTRONS BY COOL PROTONS

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Simplified formulae for the capture of low-energy electrons by stationary protons are averaged over Maxwellian and "flattened" Maxwellian electron velocity distributions. The latter distribution is more nearly appropriate for electron beams used in accelerator proton-beam cooling experiments. Flattening increases the capture rate by a factor of about two. Similar formulae for the capture of antiprotons by protons are mentioned.

1. INTRODUCTION

In electron cooling of proton beams,¹ a proton sometimes captures an electron and proceeds as a neutral hydrogen atom. The observation of such atoms yields information about electron and proton beams.² The process is most important in the late stages of cooling when the protons are nearly at rest in the electron gas. The theory of electron capture by protons has of course been worked out.³ It results in elaborate expressions requiring computer evaluation. Here we use simplified approximate formulae, adequate for the low relative electron-proton energies of particular interest in electron cooling of accelerator proton beams, that are easily integrated over Maxwellian and "flattened Maxwellian" electron velocity distributions.

2. FIRST APPROXIMATION

Spitzer⁴ quotes for the capture cross section of a stationary nucleus for electrons of nonrelativistic velocity V_e

$$\sigma(V_e) = A \sum_{n=1}^{\infty} \frac{v_0}{v_n} \frac{h\nu_0}{E} \frac{g_n}{n^3}, \quad (1)$$

where $A = 2^4 3^{-3/2} h e^2 / (m_e^2 c^3) = 2.11 \times 10^{-22} \text{ cm}^2$; h is Planck's constant; e is the electron charge; m_e is the electron mass; c is the velocity of light; $h\nu_0 = Z^2 \alpha^2 m_e c^2 / 2$ is the ground state binding energy, 13.6 eV for hydrogen; $h\nu_n$ is radiated photon energy in capture to level $n = h\nu_0 / n^2 + E$; the electron kinetic energy $E = m_e V_e^2 / 2$; α is the fine-structure constant $= 1/137$; and

g_n "is a correction factor, generally about equal to one." For the moment, we follow Spitzer in setting

$$g_n = 1. \quad (2)$$

Then

$$\sigma = A \left(\frac{h\nu_0}{E} \right)^2 \sum_1^{\infty} \frac{1}{n(n^2 + h\nu_0/E)}. \quad (3)$$

That is [see, for example, Abramowitz and Stegun⁵ (6.3.16)],

$$\sigma = \frac{A}{2} \frac{h\nu_0}{E} \left\{ \psi \left(1 + i \sqrt{\frac{h\nu_0}{E}} \right) + \psi \left(1 - i \sqrt{\frac{h\nu_0}{E}} \right) + 2\gamma_1 \right\}, \quad (4)$$

where ψ is the digamma function

$$\psi(z) = (\Gamma(z))^{-1} (d/dz) \Gamma(z),$$

and γ_1 is Euler's constant

$$\gamma_1 = \gamma \cong 0.5772. \quad (5)$$

Later we will give γ_1 a different value. Using the asymptotic form of ψ [Abramowitz and Stegun⁵ (6.3.18)], then

$$(E/(h\nu_0))\sigma = A[\gamma_1 + \ln \sqrt{(h\nu_0/E)}], \quad (6)$$

neglecting terms of order $E/(h\nu_0)$ on the right. The expression is accurate in so far as

$$E/(h\nu_0) \ll 1. \quad (7)$$

For $E/(h\nu_0)$ about 0.2 the error is about one per cent.

The rate of capture per proton is

$$\alpha_r n_e, \quad (8)$$

where n_e is the number of electrons per unit volume and $\alpha_r = \langle V_e \sigma(V_e) \rangle$, where the angular brackets denote averaging over the electron velocity distribution.

If one says simply that the distribution is characterized by some typical thermal velocity V_T and takes $V_e = V_T$ everywhere, then from (6)

$$\alpha_r = AZ^2 \alpha^2 c^2 V_T^{-1} [\gamma_1 + \ln(Z\alpha c/V_T)]. \quad (9)$$

For very small V_T , the constant γ_1 is negligible in comparison with the logarithm. Then

$$\alpha_r \approx 19.4 Z^2 \alpha^2 c^2 V_T^{-1} \ln(Z\alpha c/V_T) \quad (10)$$

A formula equivalent to this is quoted by Budker and Skrinisky⁶ [their Eq. (2.9)]. We try now, at least for well-defined thermal distributions, to be a little more accurate.

3. SECOND APPROXIMATION

We consider the Maxwell distribution

$$f(\mathbf{v}_e) = (2\pi kT/m_e)^{-3/2} \exp(-E/(kT)) \quad (11)$$

and the corresponding "flattened" distribution

$$f(\mathbf{v}_e) = (2\pi kT/m_e)^{-1} \exp(-E/(kT)) \delta(V_{ex}), \quad (12)$$

obtained by suppressing one of the components of velocity. Then

$$\alpha_r = \langle V_e \sigma(V_e) \rangle = \frac{\int_0^\infty dV_e V_e^{n-1} V_e \sigma(V_e) \exp(-E/(kT))}{\int_0^\infty dV_e V_e^{n-1} \exp(-E/(kT))}, \quad (13)$$

where $n = 3$ for Maxwell and $n = 2$ for "flattened." With

$$z = E/(kT) \quad (14)$$

as integration variable and using (6),

$$\begin{aligned} \alpha_r \sqrt{(m_e kT/2)/(h\nu_0 A)} &= \int_0^\infty dz z^{n/2-1} z^{-1/2} e^{-z} \\ &\quad \times \frac{(\gamma_1 + \frac{1}{2} \ln(h\nu_0/(kT)) - \frac{1}{2} \ln z)}{\int_0^\infty dz z^{n/2-1} e^{-z}} \end{aligned} \quad (15)$$

$$\begin{aligned} &= \{\gamma_1 + \frac{1}{2} \ln(h\nu_0/(kT)) \\ &\quad - \frac{1}{2} \psi(n/2 - \frac{1}{2})\} \Gamma(n/2 - \frac{1}{2}) / \Gamma(n/2) \end{aligned} \quad (16)$$

Finally then, with $n = 3$ and 2 respectively,

$$\begin{aligned} \alpha_r(\text{Maxwell}) &= \frac{\sqrt{2} ch\nu_0 A}{\sqrt{(m_e c^2 kT)}} \frac{2}{\sqrt{\pi}} \{\gamma_1 + \frac{1}{2} \gamma \\ &\quad + \frac{1}{2} \ln(h\nu_0/(kT))\} \end{aligned} \quad (17)$$

$$\begin{aligned} \alpha_r(\text{flattened}) &= \frac{\sqrt{2} ch\nu_0 A}{\sqrt{(m_e c^2 kT)}} \sqrt{\pi} \{\gamma_1 + \frac{1}{2} \gamma \\ &\quad + \frac{1}{2} \ln(h\nu_0/(kT)) + \ln 2\}. \end{aligned} \quad (18)$$

For the Maxwell distribution, we can compare with the more accurate results of Spitzer.⁴ He integrated (1) term by term over electron velocities and summed the resulting series numerically. He quotes values of a function $\phi(\beta)$ defined by

$$\alpha_r = \beta \phi(\beta) 2A \sqrt{(2kT/(m_e \pi))} \quad (19)$$

$$\beta = h\nu_0/(kT) \quad (20)$$

From (17), with $\gamma_1 = \gamma$,

$$\phi(\beta) = \ln \sqrt{\beta} + 3\gamma/2 \quad (21)$$

$$\doteq \ln(2.377\sqrt{\beta}). \quad (22)$$

This accurately reproduces Spitzer's values for large β (low electron temperature) and even for $\beta = 5$ ($kT = 2.72$ eV when $Z = 1$), it is too low by less than 2%.

4. THIRD APPROXIMATION

However, so far we have made the approximation (2). To judge the accuracy of this we compare the results of (5) and (6) with values quoted by Bates et al.³ The comparison in Table I shows errors of 10–15%. The work of Seaton³ suggests that considerable improvement results on replacing (6) by

$$\sigma = A \left(\frac{h\nu_0}{E} \right) \left\{ \ln \sqrt{\frac{h\nu_0}{E}} + \gamma_1 + \gamma_2 \left(\frac{E}{h\nu_0} \right)^{1/3} \right\}, \quad (23)$$

with

$$\gamma_1 = 0.1402 \quad \gamma_2 = 0.525. \quad (23')$$

This is indeed so, as shown in the table.

The expression (23) is again readily integrated over thermal distributions; (16) is replaced by

$$\left\{ \ln \sqrt{\frac{h\nu_0}{kT}} - \frac{1}{2} \psi \left(\frac{n}{2} - \frac{1}{2} \right) + \gamma_1 \right. \\ \left. + \gamma_2 \frac{\Gamma \left(\frac{n}{2} - \frac{1}{6} \right)}{\Gamma \left(\frac{n}{2} - \frac{1}{2} \right)} \left(\frac{kT}{h\nu_0} \right)^{1/3} \right\} \frac{\Gamma \left(\frac{n}{2} - \frac{1}{2} \right)}{\Gamma \left(\frac{n}{2} \right)}, \quad (24)$$

so that

$$\alpha_r(\text{Maxwell}) = \frac{\sqrt{2}ch\nu_0 A}{\sqrt{m_e c^2 kT}} \frac{2}{\sqrt{\pi}} \left\{ \gamma_1 + \frac{1}{2} \gamma \right. \\ \left. + \Gamma \left(\frac{4}{3} \right) \gamma_2 \left(\frac{kT}{h\nu_0} \right)^{1/3} \right. \\ \left. + \ln \sqrt{\frac{h\nu_0}{kT}} \right\}, \quad (25)$$

$$\alpha_r(\text{flattened}) = \frac{\sqrt{2}ch\nu_0 A}{\sqrt{m_e c^2 kT}} \sqrt{\pi} \left\{ \gamma_1 + \frac{1}{2} \gamma \right. \\ \left. + \Gamma \left(\frac{5}{6} \right) \frac{\gamma_2}{\sqrt{\pi}} \left(\frac{kT}{h\nu_0} \right)^{1/3} \right. \\ \left. + \ln \left(2 \sqrt{\frac{h\nu_0}{kT}} \right) \right\}. \quad (26)$$

TABLE I

Comparison of Different Calculations of the Capture Cross Section

$h\nu_0/E$	49.0	100	196	400
$E(Z = 1, \text{ eV})$	0.278	0.136	0.069	0.034
$\sigma(\text{Bates et al.}^3)$ (10^{-21} cm^2)	23.0	53.7	119	272
$\sigma[(5)(6)]$	26.1	60.8	133	302
$\sigma[(23)(23')]$	23.1	53.9	119	271

Numerically, with kT in eV , $Z = 1$,

$$\alpha_r(\text{Maxwell}) = \frac{1.92}{\sqrt{kT}} \left\{ \ln \frac{5.66}{\sqrt{kT}} \right. \\ \left. + 0.196(kT)^{1/3} \right\} 10^{-13} \text{ cm}^3/\text{sec}, \quad (27)$$

$$\alpha_r(\text{flattened}) = \frac{3.02}{\sqrt{kT}} \left\{ \ln \frac{11.32}{\sqrt{kT}} \right. \\ \left. + 0.140(kT)^{1/3} \right\} 10^{-13} \text{ cm}^3/\text{sec}. \quad (28)$$

Seaton³ gives a formula equivalent to (27). His error estimates imply accuracy to within a few per cent for $kT < 1 \text{ eV}$. The formula is good within one per cent in comparison with results quoted by Massey et al.³ (their table 14.4) for $0.02 \text{ eV} \leq kT \leq 5 \text{ eV}$. We suppose that (28) will be of comparable accuracy.

Simple approximations to these formulae are

$$\alpha_r(\text{Maxwell}) = 3.79(kT)^{-0.678} 10^{-13} \text{ cm}^3/\text{sec} \quad (29)$$

$$\alpha_r(\text{flattened}) = 7.88(kT)^{-0.645} 10^{-13} \text{ cm}^3/\text{sec}. \quad (30)$$

These reproduce (27) and (28) to within one per cent for

$$0.07 < kT < 0.70$$

and to within ten per cent for

$$0.01 < kT < 3.0,$$

with kT always in eV .

These formulae, together with (8), apply in the first instance in the frame of reference in which the electron gas is at rest. Retaining formulae

(27)–(30) with T defined in the rest system of the electron gas, in the laboratory system the mean capture rate per proton is

$$\alpha_e n_e d\gamma^{-2},$$

where n_e is the electron density in the lab system, $\gamma = 1/\sqrt{(1 - V_0^2/c^2)}$, $\gamma = (1 - V_0^2/c^2)^{-1/2}$, V_0 is the mean electron velocity in the lab, and d is the fraction of ion orbit to which cooling is applied.

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APPENDIX $p + \bar{p} \rightarrow (p\bar{p}) + \gamma$

We can also compare (23) with calculations⁷ on the radiative capture of antiprotons by protons, taking account of the Coulomb interaction only between the particles. To adapt the formulae, we have to replace m_e by the reduced mass

$$m_e \rightarrow \mu = m_p/2$$

in A and $h\nu_0$, and interpret E as center-of-mass-system kinetic energy. Numerically then, with E in eV,

$$\sigma = 3.12 \cdot 10^{-24} \frac{1}{E} \left\{ \ln \sqrt{\frac{12500}{E}} + 0.14 + 0.525 \left(\frac{E}{12500} \right)^{1/3} \right\} \text{cm}^2 \quad (\text{A1})$$

Brazzi, Fiorentini and Pitzurra⁷ give the cross section in units of the square of the Bohr radius

$$a_0^2 = 0.280 \cdot 10^{-16} \text{cm}^2.$$

They measure energy in units of

$$\alpha^2 m_e c^2 = 27.2 \text{ eV}.$$

With

$$E = T\alpha^2 m_e c^2$$

(A1) becomes

$$10^8 \sigma / a_0^2 = \frac{0.409}{T} \left\{ \ln \sqrt{\frac{459}{T}} + 0.14 + 0.525 \left(\frac{T}{459} \right)^{1/3} \right\}. \quad (\text{A2})$$

This agrees with the numerical results of Brazzi et al. to within the accuracy (some ten per cent) of their calculations (private communication), which are for the range

$$1 > T > 10^{-3}.$$

We would expect (A2) to be quite accurate in this range, where E is indeed small compared with the relevant

$$\begin{aligned} h\nu_0 &= \alpha^2 m_p c^2 / 4 \\ &= 459 \alpha^2 m_e c^2. \end{aligned}$$

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