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ON THE PROJECTIONS OF REPRESENTATION SPACES OF THE
SYMMETRY GROUP ON THE MINKOWSKI SPACE

Jan Rzewuski *)
CERN -- Geneva

ABSTRACT

In this note we generalize the projection of the representation space of the symmetry group $SU(2,2) \times U(2)$ on the Minkowski space to arbitrary internal symmetries $U(m)$. The procedure involves certain restrictions on the co-ordinates of the representation space. Representations of the symmetry group in the restricted space and in the corresponding restricted Hilbert space are constructed.

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*) On leave of absence from the Institute of Theoretical Physics, Wroclaw University, Cybulskiego 36, 50-205 Wroclaw, Poland.

The problem of projecting the representation space of the full symmetry group of the physical system on the Minkowski space arises in attempts to provide a common geometrical basis for internal as well as external symmetries^{1),2)} [cf. also Refs 3)-5) for further literature].

Already the lowest non-trivial linear representation space of the symmetry group provides such a basis. The physical Minkowski space must be embedded in this representation space in a way which is consistent with the group. This means that transformations of the group in the representation space should induce, via the projection, correct transformations of the Minkowski space. In particular, the last one should be invariant with respect to internal symmetries.

We assume that the physical symmetry group G is the direct product $G = G_1 \times G_2$ of the external and internal symmetry group in accordance with a theorem by Coleman-Mandula and Lopuszanski [cf., e.g. Ref. 6)]. G is the conformal group or its Poincaré subgroup whereas a sufficiently general candidate for G_2 (internal symmetries) is $U(m)$.

Since we shall be interested in the most economic extension of Minkowski space, we start with the lowest linear representation space of G which is $\mathbb{C}^{4m}$. The co-ordinates $\xi_{a;\alpha}$ ($\xi_{a;\alpha} \in \mathbb{C}$, $a = 1, \dots, 4$, $\alpha = 1, \dots, m$) of this space form a complex $4 \times m$ matrix the elements of which transform with respect to $G = G_1 \times G_2$ according to

$$\xi'_{a;\alpha} = q^{(1)}_a{}^b q^{(2)}_\alpha{}^\beta \xi_{b;\beta}. \quad (1)$$

Let us first consider the case when $G_2 = GL(2, \mathbb{C})$ or one of its subgroups [in the uninteresting case $G_2 = GL(1, \mathbb{C})$ no projection on M_4 exists]. In this case the solution is known⁷⁾. One first uses the Penrose projection $\mathbb{C}^8 \rightarrow M_4^{\mathbb{C}}$ 2)

$$Z_\mu = x_\mu - i y_\mu = i \lambda \frac{\xi_{a;1} (\gamma^\mu \gamma_\mu)^{ab} \xi_{b;2}}{\xi_{a;1} [\gamma^\mu (1 - \gamma_5)]^{ab} \xi_{b;2}} \quad (2)$$

in which the co-ordinates of a complex Minkowski space $M_4^{\mathbb{C}}$ appear as ratios of antisymmetric forms $[(\gamma_\mu)_a{}^b]$ are Dirac matrices, $(\gamma^T)^{ab} = -(\gamma^T)_{ab}$ is a transposition matrix rising and lowering the indices according to $(\gamma^\mu \gamma_\mu)^{ab} = (\gamma^T)^{ac} (\gamma_\mu)_c{}^b$ and satisfying $(\gamma^T)_{ac} (\gamma^T)^{cb} = -\delta_a^b$, $(\gamma^T)^{ab} = -(\gamma^T)_{ba}$, $(\gamma^\mu \gamma_\mu)^{ab} = -(\gamma^\mu \gamma_\mu)^{ba}$. The notation used here is explained in full detail in Ref. 8).

From (2) it is seen that the z_μ are invariant with respect to $G_2 = GL(2, \mathbb{C})$ because the antisymmetric forms in the numerator and denominator of (2) are multiplied under $G_2 = GL(2, \mathbb{C})$ by the common factor $\det GL(2, \mathbb{C})$. It can also be shown [cf., e.g. Refs 2)-4) or 9)] that conformal linear transformations $G_1 = SU(2, 2)$ of the complex variables $\xi_{a;\alpha}$ given by the generators

$$\begin{aligned} d &= -\frac{i}{2} \gamma_5, \\ P_\mu &= -\lambda^{-1} \gamma^+ \gamma_\mu, & \gamma^\pm &= \frac{1}{2} (1 \pm \gamma_5), \\ K_\mu &= -\lambda \gamma^- \gamma_\mu, \\ M_{\mu\nu} &= \frac{i}{4} [\gamma_\mu, \gamma_\nu], \end{aligned} \quad (3)$$

(acting on the first index) induce the correct non-linear conformal transformations of the complex Minkowski variables

$$\begin{aligned} D z_\lambda &= i z_\lambda, \\ P_\mu z_\lambda &= i g_{\mu\lambda}, \\ K_\mu z_\lambda &= -i g_{\mu\lambda} z^2 + 2i z_\mu z_\lambda, \\ M_{\mu\nu} z_\lambda &= -i g_{\mu\lambda} z_\nu + i g_{\nu\lambda} z_\mu, \end{aligned} \quad (4)$$

[λ in (3) is a parameter with dimension of length].

It is seen from (4) that the real part x_μ of z_μ transforms with respect to translations (P_μ) like a vector whereas the imaginary part y_μ is translationally invariant (behaves like a co-ordinate difference). We are obliged, therefore, to identify x_μ with the co-ordinates of the real Minkowski space M_4 .

It can be shown^{4),9)} that projection (2) is the only projection in terms of antisymmetric forms consistent with the group $\mathcal{P} \times GL(2, \mathbb{C})$, where $\mathcal{P} \subset SU(2, 2)$ is the Poincaré subgroup of the conformal group. Consistency with the whole conformal group is a consequence of consistence with $\mathcal{P}$. It can also be shown^{4),9)} that the two other projections in terms of symmetric and Hermitean tensors are identical with (2) in the case when these tensors are simple (i.e., when they are represented by bilinear forms).

To approach the homogeneous space of the group in the case when $G_2 = SL(2, \mathbb{C})$ or $G_2 = SU(2)$, we parametrize the projection $\mathbb{C}^8 \rightarrow M_4$, provided by the real part of (2), by introducing the four real conformal invariants

$$N_{;\mu} := (\sigma_{\mu})^{\dot{a}\dot{b}} \xi_{a;\alpha}^* f^{\dot{a}\dot{b}} \xi_{b;\beta} \quad (5)$$

where σ_i ($i=1,2,3$) are Pauli matrices, $\sigma_0 = 1$ and $f^{\dot{a}\dot{b}}$ is the Hermitean 4×4 matrix with eigenvalues $+1, +1, -1, -1$, determining $SU(2,2)$ [in representation (3) $f^{\dot{a}\dot{b}} = i(\gamma_4)^{\dot{a}c}(\gamma_5)_c^{\dot{b}}$]. We first split the variables $\xi_{a;\alpha}$ into two parts

$$\xi^{\pm} = \gamma^{\pm} \xi \quad (6)$$

by means of the projectors

$$\gamma^{\pm} = \frac{1}{2} (1 \pm \gamma_5). \quad (7)$$

From (3) it follows that ξ^- is translationally invariant. It is seen from (2) that

$$Z_{\mu} = i\lambda \frac{\xi_{a;1}^+ (\gamma^{\mu} \gamma_5)^{ab} \xi_{b;2}^- + \xi_{a;1}^- (\gamma^{\mu} \gamma_5)^{ab} \xi_{b;2}^+}{2 \xi_{a;1}^- (\gamma^{\mu})^{ab} \xi_{b;2}^-} \quad (8)$$

is a linear combination of ξ^+ with coefficients depending on ξ^- . Relations (8) can be inversed with respect to ξ^+ and we obtain

$$\xi^+ = -i\lambda^{-1} Z^{\mu} \gamma_{\mu} \xi^-. \quad (9)$$

It is seen that we can perform a change of variables

$$\{\xi_{a;\alpha}\} \longleftrightarrow \{x_{\mu}, y_{\mu}, \xi_{a;\alpha}^-\} \quad (10)$$

in which y_{μ} and $\xi_{a;\alpha}^-$ represent the translationally invariant parameters of the particular embedding of M_4 in $\mathbb{C}^8$.

It can be shown further that y_{μ} is connected with $r_{;\mu}$ (we use from now on the semicolon whenever it is necessary to distinguish vectors with respect to external and internal symmetries) by means of an inversive relation

$$\tilde{r}_{\mu;\nu} = -2\lambda^{-1} y^{\mu} r_{\mu;\nu} \quad (11)$$

in which the coefficients $r_{\mu;\nu}$ depend on ξ^- only and satisfy the orthogonality relations^{7),8)},

$$\begin{aligned} r_{\mu;\lambda} r^{\mu}_{\nu} &= 4 g_{\lambda\nu} |\xi_{a;1}^- (\gamma^T)^{ab} \xi_{b;2}^-|^2, \\ r_{\mu;\lambda} r^{\mu}_{\nu} &= 4 g_{\mu\nu} |\xi_{a;1}^- (\gamma^T)^{ab} \xi_{b;2}^-|^2. \end{aligned} \quad (12)$$

We obtain the desired parametrization by replacing $y_{\mu;}$ by $r_{\mu;}$ and $\xi_{a;\mu}^-$

$$\{\xi_{a;\alpha}\} \longleftrightarrow \{x_{\mu;}, r_{\mu;}, \xi_{a;\alpha}^-\}. \quad (13)$$

The submanifolds

$$\gamma_1 := \frac{1}{4} r_{\mu;\mu} r^{\mu}_{\mu} = \text{const} \quad (14)$$

and

$$\begin{aligned} \gamma_1 &= \text{const}, \quad \gamma_2 := r_{j;0} = \text{const} \\ \gamma_1 + \gamma_2^2 &\geq 0 \end{aligned} \quad (15)$$

of $\mathbb{C}^8$ are invariant with respect to $SU(2,2) \times SL(2,\mathbb{C})$ or $SU(2,2) \times SU(2)$ respectively. It is certain, therefore, that they contain the homogeneous manifolds of these groups and, if there are no other independent invariants, coincide with these manifolds. They also contain in a consistent and unique way the Minkowski space and provide therefore the most economic embedding.

In the case when we restrict the conformal symmetry $SU(2,2)$ to its Poincaré subgroup, another invariant appears, namely $|\xi_{a;1}^- (\gamma^T)^{ab} \xi_{b;2}^-|^2$. We obtain, therefore, 14 or 13 dimensional submanifolds of $\mathbb{C}^8$ containing M in a way consistent with the groups $\mathcal{P} \times SL(2,\mathbb{C})$ or $\mathcal{P} \times SU(2)$ respectively.

Due to the fact that $r_{\mu;\mu}$ is invariant with respect to a phase transformation $\xi \rightarrow \xi' = e^{i\phi} \xi$, the above statements are valid also in the case of $G_2 = U(2)$.

Let us go over now to the general case $G_2 = GL(m, \mathbb{C})$ or one of its subgroups. Now, the representation space is $\mathbb{C}^{4m}$ and the co-ordinates $\xi_{a;\alpha}$ form a $4 \times m$ complex matrix. We can, therefore, construct, according to (2), $1/2 m(m-1)$ complex vectors

$$Z_\mu^{(a,\beta)} = \frac{i\lambda}{2} \frac{\xi_{a;\alpha} (\gamma^\mu \gamma_\mu)^{ab} \xi_{b;\beta}}{\xi_{a;\alpha}^- (\gamma^\mu)^{ab} \xi_{b;\beta}^-}, \quad (16)$$

which, according to the statement made after formula (2), transform in the correct way (4) if the $\xi_{a;\alpha}$ undergo a conformal transformation $G_1 = SU(2,2)$. In fact, we have infinitely many such vectors given by the basic antisymmetric forms appearing in (16)

$$Z_\mu(\kappa) = \frac{i\lambda}{2} \frac{\kappa^{\alpha\beta} \xi_{a;\alpha} (\gamma^\mu \gamma_\mu)^{ab} \xi_{b;\beta}}{\kappa^{\alpha\beta} \xi_{a;\alpha}^- (\gamma^\mu)^{ab} \xi_{b;\beta}^-}. \quad (17)$$

Moreover neither (16) nor (17) are invariant with respect to $G_2 = GL(2, \mathbb{C})$ or its subgroups $SL(m, \mathbb{C})$ or $U(m)$ because the antisymmetric bilinear forms

$$\xi_{a,a_2;\alpha_1,\alpha_2} := \begin{vmatrix} \xi_{a_1;\alpha_1} & \xi_{a_1;\alpha_2} \\ \xi_{a_2;\alpha_1} & \xi_{a_2;\alpha_2} \end{vmatrix} \quad (18)$$

provide a $\binom{4}{2} \times \binom{m}{2}$ dimensional irreducible representation of $GL(4, \mathbb{C}) \times GL(m, \mathbb{C})$ and, therefore, the $z_\mu(\kappa)$ transform into each other like ratios of linear combinations of vectors of such a representation. This situation is unsatisfactory because the physical space M_4 must be invariant with respect to internal symmetries.

A way out of this difficulty is provided by imposing on the co-ordinates $\xi_{a;\alpha}$ constraints which would reduce the representation space of G , and simultaneously, the variety of z_μ . We shall consider here $4(m-2)$ linear independent constraints

$$\lambda_{ia} = 0, \quad i = 3, 4, \dots, m, \quad a = 1, \dots, 4, \quad (19)$$

where

$$\lambda_{ia} = c_i^\alpha \xi_{a;\alpha}, \quad (20)$$

and c_i^α are $(m-2)$ independent vectors. Conditions (19) are invariant with respect to the group G in the sense that

$$\lambda'_{ia} := c_i'^\alpha \xi'_{a;\alpha} = g_a^{(1)b} \lambda_{ib}, \quad (21)$$

where

$$c_i'^\alpha = c_i^\beta (q^{(2)-1})_\beta^\alpha. \quad (22)$$

It follows from (21) that

$$\lambda_{ia} = 0 \iff \lambda'_{ia} = 0. \quad (23)$$

We shall prove in this note that conditions (19) imply that all the points $z^{(\alpha,\beta)}$, $\alpha, \beta = 1, \dots, m$, defined in (16) as well as the points $z_\mu(\kappa)$ defined in (17) coincide and are invariant with respect to the group $G_2 = GL(2, \mathbb{C})$ of internal symmetries. The proof is immediate but we shall postpone it for convenience to a later place, first deriving some general results concerning representations of G in the restricted space.

To this end, we introduce new variables

$$\eta_{a;A} = \xi_{a;A}, \quad a = 1, \dots, 4, \quad A = 1, 2, \quad (24)$$

$$\lambda_{ia} = c_i^\alpha \xi_{a;\alpha}, \quad a = 1, \dots, 4, \quad i = 3, \dots, m.$$

We can assume, without loss of generality, that the subdeterminant $\det\{c_i^{\bar{A}}\}$ ($i, \bar{A} = 3, \dots, m$) of the matrix $\{c_i^\alpha\}$ is different from zero

$$\det\{c_i^{\bar{A}}\} \neq 0. \quad (25)$$

We can solve, therefore, Eqs (24) with respect to $\xi_{a;\alpha}$ and obtain

$$\xi_{a;\alpha} = a_{\alpha}^A \eta_{a;A} + b_{\alpha}^i \lambda_{ia}, \quad (26)$$

where

$$\begin{aligned} a_A^B &= \delta_A^B, \quad b_A^i = 0, \quad A, B = 1, 2, \\ a_{\bar{A}}^B &= -b_{\bar{A}}^i c_i^B, \quad b_{\bar{A}}^i = [\det \{c_i^{\bar{A}}\}]^{-1} A_{\bar{A}}^i, \\ (B &= 1, 2, \quad i, \bar{A} = 3, \dots, m) \end{aligned} \quad (27)$$

and $A_{\bar{A}}^i$ is the adjoint determinant of order $m-3$ corresponding to the element $c_i^{\bar{A}}$ of the matrix $\{c_i^{\bar{A}}\}$. The coefficients $b_{\bar{A}}^i$ satisfy the relations (following from general properties of determinants)

$$b_{\bar{A}}^i c_i^{\bar{B}} = \delta_{\bar{A}}^{\bar{B}}, \quad c_i^{\bar{A}} b_{\bar{A}}^k = \delta_i^k. \quad (28)$$

In a new co-ordinate system $\xi'_{a;\alpha} = g_{\alpha}^{\beta} \xi_{a;\beta}$, $g \in G_2$, we have

$$\begin{aligned} \eta'_{a;A} &= \xi'_{a;A}, \\ \lambda'_{ia} &= c_i^{\alpha} \xi'_{a;\alpha} = \lambda_{ia}, \end{aligned} \quad (29)$$

and the inverse relations

$$\xi'_{a;\alpha} = a(qa)_{\alpha}^A \eta'_{a;A} + b(qa, qb)_{\alpha}^i \lambda'_{ia}, \quad (30)$$

where $a(qa)_{\alpha}^A$ and $b(qa, qb)_{\alpha}^i$ are solutions of the equations

$$\begin{aligned} (qa)_{\alpha}^A &= a(qa)_{\alpha}^B (qa)_B^A, \\ (qb)_{\alpha}^i &= b(qa, qb)_{\alpha}^i + a(qa)_{\alpha}^B (qb)_B^i. \end{aligned} \quad (31)$$

The coefficients a and b satisfy relations

$$\begin{aligned} a(hqa) &= a(ha(qa)), \\ b(hqa, hqb) &= hb(qa, qb) - a(ha(qa))hb(qa, qb), \end{aligned} \quad (32)$$

which ensure uniqueness of the functions $a(ga)$ and $b(ga,gb)$ defined in (31) on the group G_2 (we do not consider here the external symmetries G_1 because the transformation properties of η and λ under this group are obvious). One also easily verifies that $a(ga)$ and $b(ga,gb)$ are the same functions of C_1^α as a and b are of C_1^α .

From (26) and (29) we derive finally the relations between η and η' .

$$\begin{aligned} \eta'_{a;A} &= (qa)_A^B \eta_{a;B} + (qb)_A^i \lambda_{ia}, \quad \lambda'_{ia} = \lambda_{ia}, \\ \eta_{a;A} &= [q^{-1}a(qa)]_A^B \eta'_{a;B} + [q^{-1}b(qa,qb)]_A^i \lambda'_{ia}. \end{aligned} \quad (33)$$

This closes the system of mutual relations of the variables $\xi_{a;\alpha}$ and $\eta_{a;A}$, λ_{ia} in different co-ordinate systems. One has to keep in mind, however, that all these relations are local and valid only in the neighbourhood of $g = e$ for which $\det \{C_i^{\bar{A}}\} \neq 0$.

Imposing of the invariant conditions $\lambda_{ia} = 0$, ($a = 1, \dots, 4$; $i = 3, \dots, m$) replaces the linear representation in $\mathbb{C}^{4m}$ by a non-linear local representation in the space $\mathbb{C}^8 \times \mathbb{C}^{2(m-2)} = \mathbb{C}^{2(m+2)}$ of the variables $\eta_{a;A}$ and $a_{\bar{A}}^B$ given by

$$\begin{aligned} \eta'_{a;A} &= (qa)_A^B \eta_{a;B}, \\ a_{\alpha}^{\prime A} &= a(qa)_{\alpha}^A. \end{aligned} \quad (34)$$

Relation (30) now takes the form

$$\xi'_{a;\alpha} = a(qa)_{\alpha}^A \eta'_{a;A} = (qa)_{\alpha}^A \eta_{a;A}. \quad (35)$$

Introducing this into (18) we prove both statements concerning (16) and (17).

We obtain first

$$\xi'_{ab;\alpha\beta} = (qa)_{\alpha\beta}^{12} \eta_{ab;12}, \quad (36)$$

where

$$\xi'_{ab;\alpha\beta} := \begin{vmatrix} \xi'_{a;\alpha} & \xi'_{a;\beta} \\ \xi'_{b;\alpha} & \xi'_{b;\beta} \end{vmatrix}, \quad \eta_{ab;12} := \begin{vmatrix} \eta_{a;1} & \eta_{a;2} \\ \eta_{b;1} & \eta_{b;2} \end{vmatrix}, \quad (37)$$

$$(qa)_{\alpha\beta}{}^{12} := \begin{vmatrix} (qa)_{\alpha}{}^1 & (qa)_{\alpha}{}^2 \\ (qa)_{\beta}{}^1 & (qa)_{\beta}{}^2 \end{vmatrix},$$

and consequently

$$\frac{\xi'_{ab;\alpha\beta}}{\xi'_{cd;\alpha\beta}} = \frac{\eta_{ab;12}}{\eta_{cd;12}}. \quad (38)$$

From (38) the first part of the statement (coincidence) follows for $g = e$ and the second part (invariance) for arbitrary g (satisfying condition $\det \{C_i^{\bar{A}}\} \neq 0$) and $\alpha, \beta = 1, 2$.

By imposing additional invariant conditions on the matrix $\{\xi_{a;\alpha}\}$ we have obtained a unique and consistent projection $\mathbb{C}^{2(m+2)} \rightarrow M_4$ in terms of the new variables $\eta_{a;A}$ and a non-linear local representation of G in the space of the variables $\eta_{a;A}$ and $a_{\bar{A}}^B$.

For applications, it is important to construct representations of G in spaces of functions which take into account conditions (19) (first quantization). We start with the infinitesimal representation of G in $L^2(\mathbb{C}^{4m})$ which is given by the infinitesimal generators

$$X := \frac{\partial}{\partial \xi_{a;\alpha}} X_a^b \xi_{b;\alpha}, \quad X' := \frac{\partial}{\partial \xi_{a;\alpha}} X'_\alpha{}^\beta \xi_{a;\beta}, \quad (39)$$

for G_1 and G_2 respectively, where X_a^b and $X'_\alpha{}^\beta$ are the corresponding generators in $\mathbb{C}^{4m}$.

To introduce condition (19) we carry out the change of variables described in (24), (26) and the corresponding change in derivatives

$$\begin{aligned} \frac{\partial}{\partial \xi_{a;A}} &= \frac{\partial}{\partial \eta_{a;A}} + \frac{\partial}{\partial \lambda_{ia}} C_i^A, & \frac{\partial}{\partial \xi_{a;\bar{A}}} &= \frac{\partial}{\partial \lambda_{ia}} C_i^{\bar{A}} \\ \frac{\partial}{\partial \eta_{a;A}} &= \frac{\partial}{\partial \xi_{a;\alpha}} a_\alpha^A, & \frac{\partial}{\partial \lambda_{ia}} &= \frac{\partial}{\partial \xi_{a;\alpha}} b_\alpha^i, \end{aligned} \quad (40)$$

to obtain

$$X = \frac{\partial}{\partial \eta_{a;A}} X_a^b \eta_{b;A} + \frac{\partial}{\partial \lambda_{ia}} X_a^b \lambda_{ib} \quad (41)$$

$$\begin{aligned} X' &= \frac{\partial}{\partial \eta_{a;A}} (X'_a)^B \eta_{a;B} + \frac{\partial}{\partial \eta_{a;A}} (X'_b)_A^i \lambda_{ia} + \\ &\quad \frac{\partial}{\partial \lambda_{ia}} (cX'_a)_i^B \eta_{a;B} + \frac{\partial}{\partial \lambda_{ia}} (cX'_b)_i^k \lambda_{ka}. \end{aligned} \quad (42)$$

To restrict the representation to functions on the plane $\lambda_{ia} = 0$ it is convenient to use the relation

$$C_i^\alpha X_\alpha'^\beta \frac{\partial}{\partial C_i^\beta} = - \frac{\partial}{\partial \lambda_{ia}} (cX'_a)_i^B \eta_{a;B} - \frac{\partial}{\partial \lambda_{ia}} (cX'_b)_i^k \lambda_{ka}, \quad (43)$$

by means of which we can rewrite the generator (42) in the form

$$X' = \frac{\partial}{\partial \eta_{a;A}} \{ (X'_a)_A^B \eta_{a;B} + (X'_b)_A^i \lambda_{ia} \} - C_i^\alpha X_\alpha'^\beta \frac{\partial}{\partial C_i^\beta}. \quad (44)$$

The functions on the plane $\lambda_{ia} = 0$ can be written in the form

$$\varphi(\eta, c) := \int f(\xi_{a;\alpha}) \prod_{a=1}^4 \prod_{i=3}^m \delta(\lambda_{ia}) d\lambda_{ia} = f(a_\alpha^A \eta_{a;A}) \quad (45)$$

[the integral representation does not involve the assumption (25) and is, therefore, more general]. The generators (41) and (44) when restricted to such functions are

$$X|_{\lambda=0} = \frac{\partial}{\partial \eta_{a;A}} X_a^b \eta_{b;A} , \quad (46)$$

$$X'|_{\lambda=0} = \frac{\partial}{\partial \eta_{a;A}} (X'_a)^B \eta_{a;B} - C_i^\alpha X'_\alpha{}^\beta \frac{\partial}{\partial C_i{}^\beta} .$$

The dependence on C_i^α of the function $\phi(\eta, c) = f(a_\alpha^A \eta_{a;A})$ occurs through the variables $a_{\bar{A}}^A$ and we can put, therefore, for the last term in the second expression (46)

$$-C_i^\alpha X'_\alpha{}^\beta \frac{\partial}{\partial C_i{}^\beta} = [(X'_a) - (a X'_a)]_{\bar{A}}^B \frac{\partial}{\partial a_{\bar{A}}^B} . \quad (47)$$

Second quantization in the case of $G_2 = GL(2, \mathbb{C})$ or its subgroups was given a preliminary treatment in Ref. 10), invariant differential operators in the same case were investigated in Ref. 7). The present construction is intended to provide a basis to extend those investigations to the physically more interesting cases of $SU(3)$ and higher internal symmetries.

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