INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

OUTLINE OF A NONLINEAR,
RELATIVISTIC QUANTUM MECHANICS OF EXTENDED PARTICLES

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ABSTRACT

A quantum theory of intrinsically extended particles similar to de Broglie's theory of the Double Solution is proposed. A rational notion of the particle's extension is enthroned by realizing its internal structure via soliton-type solutions of nonlinear, relativistic wave equations. These droplet-type waves have a quasi-objective character except for certain boundary conditions which may be subject to stochastic fluctuations. More precisely, this assumption amounts to a probabilistic description of the center of a soliton such that it would follow the conventional quantum-mechanical formalism in the limit of zero particle radius. At short interaction distances, however, a promising nonlinear and nonlocal theory emerges. This model is not only capable of achieving a conceptually satisfying synthesis of the particle-wave dualism, but may also lead to a rational resolution of epistemological problems in the quantum-theoretical measurement process. Within experimental errors the results for, e.g., the hydrogen atom can be reproduced by appropriately specifying the nature of the nonlinear self-interaction. It is speculated that field theoretical issues raised by such notions as identical particles, field quantization and renormalization are already incorporated or resolved by this nonlocal theory, at least in principle.

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I. INTRODUCTION AND MOTIVATION

In recent years the conviction has apparently increased among most physicists that all fundamental interactions between particles - including gravity - can be comprehended by appropriate extensions of local gauge theories.

The principle of gauge invariance was first formulated in 1928 by Hermann Weyl [157] with the aim to give a proper connection between Dirac's relativistic quantum theory [37] of the electron and the Maxwell-Lorentz's theory of electromagnetism. Later Yang and Mills [167] as well as Sharp were able to extend the latter theory to a nonabelian gauge theory with $SU(2)$ as local symmetry group. This became necessary in order to account for the postulated "internal" degrees of freedom - such as isospin and strangeness - of the then newly observed baryons. After the recent acclaim of the Weinberg-Salam model of weak interactions and some achievements of the so-called quantum chromodynamics (QCD) (see Ref. 91 for a review) a "Grand unification" of all particle forces seems to be in reach [130].

However, it should be noted that in order to apply the methods of quantum field theory (QFT), in particular those of renormalization [12, 109], all these gauge models necessarily have to make reference to structureless elementary fields.

Besides several electrons and neutrinos, quarks* [56] - which nowadays come in at least three colours - intermediate vector bosons, gluons and Higgs fields have to be postulated as elementary building blocks of matter. Furthermore, the elusive quarks as well as the gluons (which appear to be late revenge of Pauli, who despite of his invention of the neutrino, was known for his aversion against introducing hypothetical particles into physics) supposedly constituting hadrons need to be confined in stable hadrons by a yet unknown mechanism. Otherwise, they would have been observed at a detectable rate.

With "grand unified theories" becoming more into fashion the number of constituent fields has become "positively baroque" [130]. This is admitted even by the standards of the "orthodox view" in particle physics.

In order to dissolve the dilemma of quark-lepton proliferation, the further speculation [113] has been put on top of the former, that quarks and gauge fields are themselves composite and in "reality" built out of preons, these themselves of prepreons etc.!! With regard to this retreat to preons and

*James Joyce's "Three quarks make a muster mark" [80] should be compared with Wilhelm Busch's earlier statement "Nichts als Quark!" [25, p.60].

prepares the questions may be allowed if high-energy physics intends to mimic Lakatos' ironical example of a research program which is typical for a nonfalsifiable theory? [85, p.100].

For one of the founders of quantum mechanics (QM), Heisenberg, the quest for elementarity "has a tolerably clear meaning only if the particle can be divided into pieces with a small amount of energy, much smaller than the rest of the mass particle itself". Furthermore, the heuristic value of higher symmetries "could probably be compared with the heuristic value of cycles and epicycles in Ptolemy's astronomy" [69].

Therefore, a refreshing attempt of redirecting the course of the history of particle physics has been undertaken by Barut [4]. In this theory all matter is made up of the absolutely stable particles: protons, electrons, neutrinos and photons. Strong interactions are entirely of a magnetic type. When treated non-perturbatively, magnetic forces are supposed to become very strong at short distances. This approach, to a certain extent, seems to work well. However, questions about the nature of the stable particles themselves remain unanswered. Also it mainly rests upon the notions of quantum electrodynamics (QED) [11] which also has been deeply questioned:

In the introduction to a perspective paper, Born, the founder of the statistical interpretation of quantum mechanics critically comments that "existing attempts to apply the quantum theory to the electromagnetic field are open to serious objections". "... relativistic invariance cannot be derived simply from the symmetry of the formulae in the four world co-ordinates, but must be artificially imposed and demonstrated by a complicated proof. Further, it is not a self-contained theory of electromagnetic field but a superposition of Maxwell's electromagnetic field on the material field of Schrodinger or Dirac, in which the elementary particles occur as point-charges. Thus there is no idea of the radius of the particle, and consequently no rational motion of mass, not to mention a theory of the ratio of the mass of a proton to that of an electron. In addition to these fundamental difficulties there are others, such as that of infinitely great "Nullpunktsenergie", which is avoided by an artificial modification of the formalism". (Citation from Ref. 21 without the underlining)

The question which is raised here by Born concerns the problematic regularization procedures used in the perturbative treatment of QED. The occurring difficulties have been circumvented by renormalization theory but not eliminated. It is, e.g., not known if the renormalized perturbation series of

the scattering matrix converges for high energies [12, p.374]. Therefore, even the proponents of canonical relativistic quantum field theory in which leptons (and quarks) are regarded as intrinsically geometrical points* admit [11, p.5] the possibility that local field theory is a correct description of microphysics only if limited to distances bigger than 10^{-13} cm.

Consequently, a foundation of the notion of particle extension being compatible with (general) relativity as well as with the basic features of QM strikes us as the most crucial point of departure from existing theories.

To be sure, the concept of a finite size of a particle is an old one, being first proposed by Abraham and Lorentz in the case of an electron. Later on, an extensible model of electron and the muon - regarded as the next excited state - has been devised by Dirac [38] (see also [6]), anticipating to some extent the "bag" models [63] in QCD.

Intuitively as well as physically justifiable by nuclear models a first step toward the inclusion of spatial extension is to regard a particle as a relativistic fluid droplet. This proposal has been considered before (see Ref. 17 and references therein) and in connection with the quark hypothesis taken up recently [155]. The conceptual difficult problem is that of a reconciliation of such mechanical models with quantum theory.

*The absurdity of the point (-particle) concept from a intuitive point of view has probably never been more vividly expressed than 1890 by Wilhelm Busch: "Noch gewandter und windiger als wir, und das will doch was sagen, trieben die nur gedachten, die rein mathematischen Punkte ihre terpsichorischen Künste. Sie waren aber dermaßen schüchtern, dass sie immer kleiner und kleiner wurden, je mehr man sie ansah; ja, einer verschwand gänzlich, als ich ihn schärfer in's Auge faßte.

Gelungene Burschen, diese Art Punkte! Der alte Brenneke, mein Mathematiklehrer, pflegte freilich zu sagen: "Wer sich keinen Punkt denken kann, der ist einfach zu faul dazu!" Ich hab's oft versucht seitdem. Aber just dann, wenn ich denke, ich hätt'ihn, just dann hab ich gar nichts. Und überhaupt, meine Freunde! Geht's uns nicht so mit allen Dingen, denen wir gründlich zu Leibe rücken, dass sie grad dann, wenn wir sie mit dem zärtlichsten Scharfsinn erfassen möchten, sich heimtückisch zurückziehen in den Schlupfwinkel der Unbegreiflichkeit, um spurlos zu verschwinden, wie der bezauberte Hase, den der Jäger nie treffen kann? Ihr nickt; ich auch". [25, p.18]

Not only the locality of conventional QFT may be questioned but also its inherently linear structure, finding its foundational expression in the "superposition principle". This is an anomalous incident in physics. In all other known cases it has to be regarded as an approximation to the full, nonlinear theory.

Although the latter possibility has been always admitted for QM (see, e.g. [161] footnote), only in more recent times a generalized, "convex" scheme of quantum theory has been developed [101-103] which is flexible enough to comprise nonlinear versions of it. In some sense, however, the thereby admitted nonlinear models are more akin to classical than to quantum mechanics [62, 82]. Furthermore, the usual point-particle picture seems unavoidable in relativistic generalizations [82] thereof.

Nevertheless, a promising alternative remains which may well avoid all these obstacles: The Theory of the Double Solution invented [28-30] by the founder of wave mechanics: de Broglie. The basic idea is to introduce, besides the conventional $\psi(x)$ wave with its probabilistic interpretation, a so-called "universal" nonlinear wave function $u(y)$ as "the true representation of the physical entity particle", which would be an extended wave phenomenon centered around a point ... [28, p.99].

Centered around this new paradigm attempts are made in this paper to develop a nonlinear and nonlocal and relativistic quantum mechanics (NLQM) which incorporates a natural notion of particle extension into the theory.

Section II summarizes the basic concepts of de Broglie's theory of the Double Solution in the case of spinor fields. After recapitulated the difficulties of a one-particle interpretation of the probabilistic spinor waves satisfying Dirac's equation, nonlinear spinor waves are considered on the u-level. De Broglie's ideas are specified for a universal wave function obeying a nonlinear spinor equation similar to those adopted in Heisenberg's unified field theory [66,67]. Here, due to the proposed objective character of the u-wave, the soliton solutions [90] of the (semi-) classical theory attract the main attention.

Section III exhibits in detail the droplet aspects of these u - solitons. Furthermore, the important issue is analyzed, how to associate a coordinate

*This terminology although regrettably identical, by no means should be confused with that used by Everett III, in his Many-Worlds Interpretation of Quantum Mechanics [36].

center with such relativistically extended objects. The latter point is of prime importance for a synthesis of the $u(y)$ - waves with the probabilistic $\psi(x)$ to which Section IV is devoted. Our proposal is to regard the latter as contingent conditions for the center of charge coordinates of the $u(y)$ - soliton. This together with functional relation for the mass of the soliton will lead to a nonlinear and nonlocal model of the integrodifferential-type. However, other links between u and ψ -level can be devised and are discussed in this Section.

In Section V it will be demonstrated that the usual point-particle picture of conventional QM is recovered for vanishing soliton radius. This corresponds to zero coupling constant of the nonlinear self-interaction on the u-level.

A model has to reproduce standard results, e.g., for the hydrogen atom. Section VI discusses available arguments indicating that for NLQM such an empirical agreement is achieved within experimental errors. According to further speculations, also a new theoretical basis for the Lamb-shift, usually viewed as an "experimentum crucis" of QFT, may arise.

In Section VII the hypothesis is advanced as that many-soliton solutions could yield a rational access to the concept of strictly identical particles occurring in the many-body problems of QFT.

The realization of nonlinear u-waves so far has only been exemplified with regard to a nonlinear Dirac equation. In Section VIII the geometrical reasons for this choice as given via gauge invariant principles by Weyl [158, 159] as well as connections to other models are pointed out. Other nonlinear models are discussed in the case of scalar and electromagnetic fields. If respected by nature, the nonlinear structure of Yang-Mills gauge fields [167] could provide further clues for an understanding of the internal structure of extended particles. In the NLQM-scheme this may, in particular, apply to the case of a minimal coupling to the u-waves.

In Section IX some epistemological questions with regard to the quantum-mechanical measurement problems are discussed in the new rational light provided by NLQM.

Ultimately, a nonlinear relativistic quantum mechanics has to be developed which incorporates conventional QFT but at the same time avoids the difficulties such as renormalization. Indications that this could indeed be the case for NLQM are pointed out in the final Section X.

Obviously, the proposed NLQM with its nonlinear integrodifferential equations induces such an intricate mathematical structure into physics that many of the arguments not being drawn from an extensive literature on related attempts, but originally presented in this paper can only be tentatively if not highly speculative. In many respects the work is intentionally based on rather intuitive if not naive physical concepts and therefore could (or should?) have been written in much earlier decades!

II. BASIC CONCEPTS OF THE THEORY OF THE DOUBLE SOLUTION

Stable matter with non-zero rest mass consists exclusively out of fermions. Therefore it appears sufficient for an illustration of the principles to discuss here the spinor version of de Broglie's Theory of the Double Solution [28] only.

For comparison and because of later need we will first recapitulate the basic notion* of conventional quantum mechanics very briefly: According to Dirac [37] the principle of superposition of the quantum-mechanical state vectors is at the heart of its very foundation, in order to ascertain the usual probabilistic interpretation. Once this principle is accepted it necessitates a linear evolution equation for non-interacting ("pure") states, which in the Schrödinger representation are denoted by complex wave functions $\psi(x)$. Passing over to a relativistic QM (see [11]) these ψ become bispinor-valued functions of space-time and are required to obey the (linear!) Dirac equation

$$\left[i\gamma^\mu \left(\frac{\partial}{\partial x^\mu} + i\frac{e}{\hbar c} A_\mu(x) + i\Gamma_\mu \right) - m c/\hbar \right] \psi(x) = 0$$

(2.1)
If present, these Dirac spinors are minimally coupled to the four-potentials A_μ of the electromagnetic field (By admitting space-time dependent γ -matrices and a metric-compatible [134, 23] spinor connection Γ_μ this equation may be cast into a form which even respects Einstein's principle of general covariance.).

*A full exposition of nonrelativistic QM can be found in the classical textbooks of Dirac [37] or Landau & Lifshitz [87], whereas the mathematically oriented reader may consult Jauch [77].

It follows from (2.1) and the corresponding adjoint equation for $\bar{\psi} := \psi^\dagger \gamma^0$ that the four-current

$$\vec{J}_\psi(\psi) := c \bar{\psi} \vec{\gamma} \psi \quad (2.2)$$

is locally conserved (see Section III for more details in the general case of interaction). This allows us to define an invariant, time-independent scalar product

$$(\psi, \chi) := \int_{\text{spacelike hypersurface}} \bar{\psi} \gamma_\nu \chi \sqrt{|g|} d\Sigma^\nu \quad (2.3)$$

of two solutions ψ and χ of the free Dirac equation. Since it is also positive-definite, the space of solution $\psi(x)$ of (2.1) can be given the structure of a Hilbert space $\mathcal{H}$. In a rest frame, the expression

$$\frac{1}{c} \int_0 d^3x = \psi^\dagger \psi d^3x \quad (2.4)$$

can be rightly regarded as the probability of finding a point-like particle at the time $t = \frac{x^0}{c}$ within an area of space between $\vec{x}$ and $\vec{x} + d\vec{x}$. This is similar to the construction given in non-relativistic QM.

However, the four-vector of energy-momentum (formally identical to (3.13) with the nonlinear term in (3.9) dropped) has no lower bound, thus entailing instability of all matter. Conventionally, "the only way to remedy this catastrophic situation of an indefinite energy is [field] quantization according to Fermi-Dirac statistics" [79, p.41]. On the other hand, such a procedure is well-known to lead to a "many-particle" theory with a host of divergency problems. Although these, to some extent, can be cured in QED by complicated renormalization procedures, it is widely felt that the divergences are symptomatic of a chronic disorder in the small-distance behaviour of the theory.

Due to these conceptual difficulties it appears to be absolutely necessary to account for the observed particle extension already in the foundation of the theory. So far de Broglie's proposal of a Theory of the Double Solution [28] seems to be the only promising framework for such an undertaking.

The basic idea of this theory is to introduce - in addition to the probabilistic $\psi(x)$ - a universal* wave function $u(y)$ which should be "the true representation of the physical entity "particle", which would be an extended wave phenomenon centered around a point ...". Furthermore "... the particle must be represented, not by a true point singularity of u , but by a very small singular region in space where u would take on a very large value and would obey a non-linear equation, of which the linear equation of Wave Mechanics would be only an approximate form valid outside the singular region" (quotation from Ref. 28, p.99).

In our proposal of a nonlinear, relativistic quantum mechanics (NLQM) this point of view will be adopted to quite an extent. More specifically, it will be assumed that the internal structure of extended particles arises from a set of semi-classical Dirac spinors

$$u(y) = \{ u^{(i)}(y) \mid i = 1, \dots, f \} \quad (2.5)$$

Here the index i may account for possible "internal" degrees of freedom. These may correspond to the hypothetical colour or flavour degrees of freedom as in the conventional quark model. Preferably, however, they should be associated with the "flavours" distinguishing the absolutely stable fermions, which in Barut's theory [4] are the building blocks of matter. In order to allow for localized, soliton-type [90] configurations it will be of prime importance to postulate a nonlinear, relativistic spinor equation

$$[iL^\mu (\frac{\partial}{\partial y^\mu} + iB_\mu(y)) + NL(u, \psi) - \mu c/\hbar] u(y) = 0$$

for the dynamics of the internal structure. (2.6)

There are many choices for a nonlinear self-interaction in (2.6). Weyl [158, 159] has given powerful geometric reasons for the inclusion of a nonlinearity

$$NL(u, \psi) - NL(\psi) = -\frac{3}{8} \ell^{*2} \bar{u} L^5 L_y u L^5 L^y \quad (2.7)$$

*The "universality" refers to a novel construction of identical particles (Sec. VII) and excitations thereof possible in nonlinear models.

of the axial-vector type. These arguments are based on an Einstein-Cartan-type gravity with spin and torsion [65]. If the matrices L^μ , L^5 are chosen to be space-time dependent generalizations of the familiar Dirac matrices γ v tensored with $U(x)$ -vector-operators λ_j , $j = 1, \dots, f$ (generalized Gell-Mann matrices [56]) Weyl's $SL(2, \mathbb{C})$ (Lorentz-) invariant spinor equation can be generalized [93] to a $GL(2f, \mathbb{C}) \subset U(f) \otimes SL(2, \mathbb{C})$ gauge model in curved space-time. Then $B_\mu(y)$ denotes a $g_L(2f, \mathbb{C})$ -valued gauge connection which also incorporates a Yang-Mills type gauge field A_μ^λ (see Sec. VIII) as well as the spinor connection Γ_μ originating from the curved space-time. This theory of which further details have been given elsewhere [97] embodies also the fundamental spinor equation of Heisenberg's unified field theory [66-68, see also 78] which originally incorporated only isotopic spin as a further symmetry. The term (2.7) introduces the modified Planck length

$$\ell^* := (8\pi\hbar G/c^3)^{1/2} \quad (2.8)$$

of (possibly strong [73]) gravity as a fundamental length into particle physics.

The dynamics of our model (2.6, 2.7) can be comprised into the Dirac-Weyl Lagrange density

$$\begin{aligned} \mathcal{L}_{D-W} = & \hbar c \sqrt{|q|} \left[i \bar{u} L^\mu (\partial/\partial y^\mu + iB_\mu) u \right. \\ & \left. - \frac{3}{16} \ell^{*2} \bar{u} L^5 L_y u \bar{u} L^5 L^y u - (\mu c/\hbar) \bar{u} u \right] \end{aligned} \quad (2.9)$$

at least, if we neglect its connection with the ψ -function introduced later for a completion of the model.

Irrespective of the specific nature and origin of the self-interaction (see Sec. VIII for a discussion of other suggestions) we will adopt, with de Broglie [29, preface], the hypothesis that the relativistic equation for the propagation of the u wave contains a basically nonlinear self-interaction. Otherwise, according to an argument due to Einstein (see Ref. 30, p.44) there will in general be no guidance for a region with high (singular) field values by the regular solutions of the linear equation.

It is common for nonlinear equations such as (2.6) that the superposition principle in general does not hold. This is not be an undesirable feature of a model which aims at a rational notion of the particle's extension. Let us discard for the moment the internal symmetries and construct - in a rest frame - a soliton-type solution

$$u_s(y) = \frac{4}{\ell^*} \left(\frac{2\pi\mu c}{3\hbar} \right)^{1/2} R_s(s) e^{-i\omega t \mu c^2/\hbar} \quad (2.10)$$

Characteristically such a solution is singular in the limit $\ell^* \rightarrow 0$ of the coupling constant occurring in the nonlinear term (2.7) implying that the "soliton" persist for a rescaling of the "strength" of the interaction! There $R_s(s)$ denotes a dimensionless spinor field depending solely on the dimensionless radial coordinate $s = |\vec{y} - \vec{x}|/\ell_u$ having $\vec{x}$ as origin. ($\ell_u = \hbar/\mu c$ denotes the reduced compton wave length). It follows from the Clifford algebra of the Dirac matrices [11] that the charge conjugated spinor field

$$u_A(y) = u^c(y) := i L^2 u_s^*(y) \quad (2.11)$$

is then also a solution of (2.6) which may be termed an anti-soliton. However, the superposition $u = u_s + u_A$ of positive and negative frequency solutions with overlapping domains of high field values is, in general, not a solution of a nonlinear differential equation (a "weak" superposition principle can be established, though see Sec. VII). This can be taken as an indication that the arguments which prevented a "one-particle" interpretation of the linear Dirac equation within the realm of a semi-classical (not "second" quantized) theory do not apply to our nonlinear model. This important point will be discussed further after the proposed NLQM is fully presented.

III. DROPLET PROPERTIES OF THE U-SOLITONS

The considerable, empirical success of the droplet model of the nucleus (see, e.g., the famous paper of Bohr and Wheeler [20]) encourage our attempt to extract and stress those properties of the universal wave function $u(y)$ which are in common with relativistic spin fluids. As an intuitive approximation to a viable model one would like to think of extended particles as classical fluid

droplets. Admittedly, this view-point has already a long history. The paper of Bohm and Vigier [17] along with its references to earlier works may be used as a guide to the general framework of relativistic hydrodynamics. The phenomenological implications for particle interactions together with the thermodynamical aspects are recently discussed [27].

As realized by de Broglie [29], it is an important conceptual advance not to consider a generic relativistic fluid but to rely on the "hydrodynamical aspects" of the semi-classical Dirac theory in this construction. When applied to NLQM, the postulated non-linearities in the wave equation of the universal spinor $u(y)$ add further crucial features to this picture: Due to the localization of the u-soliton, the derived "fluid body" will also be concentrated in space and therefore will give the impression of a ("smeared") droplet without further assumptions. Furthermore, these droplets may become mechanically stabilized upon imposing a charge quantization condition on the corresponding u-solitons.

In order to unveil this structure in more detail, Finkelstein et al.

[53] as well as more recently Rañada [119], have undertaken extensive numerical studies. Their results show that nonlinear spinor equations of the Heisenberg-Pauli-Weyl type (2.6, 2.7) admit regular, radially localized solutions with finite energy in flat space-time. Based on the above mentioned works, it is an easy exercise to prove that the radial part of the stationary soliton solutions (2.9) acquires asymptotically the form

$$R_s(s) \sim C_\omega \begin{bmatrix} \frac{1}{1-\omega} \left(\frac{1}{3} e^{-\sqrt{1-\omega^2}s} \right)' & \chi_m \\ -\frac{1}{3} e^{-\sqrt{1-\omega^2}s} & \chi_{-m} \end{bmatrix} \quad (3.1)$$

More specifically (compare with Mielke [98]) this result holds for $\lambda = 1$ which corresponds to the quantum number $j = \frac{1}{2}$ of total angular momentum. In (3.1) χ_m are spin-weighted, spherical harmonics [124]. Valid at large distances from the center of the soliton, for $|\omega| < 1$ these asymptotic solutions clearly exhibit a Yukawa-type exponential decrease familiar from the old meson theories of strong interactions [11, Sec. 10.2]. In order to make such stationary solitons stable against decay, Finkelstein et al. [52] have suggested to impose the charge quantization condition

The zero components

$$S^{(i)}(y) := \frac{1}{c} \sqrt{|g|} S^{o(i)} = \sqrt{|g|} \bar{u}^{(i)*} u^{(i)} \quad (3.6)$$

In particular, are viewed as the semi-classical densities of a (possibly) multicomponent (only hadronic?, compare with Ref. 155) fluid, but, in sharp contrast to the ψ -level, not as probability densities.

Following Bohm and Vigier [17] its local flow four-velocity may be covariantly defined by

$$v^\mu(y) := \frac{c S^\mu}{\sqrt{S^\nu S_\nu}} \quad (3.7)$$

On account of (3.7) it satisfies

$$v^\mu v_\mu = c^2 \quad (3.8)$$

as required by relativity.

If we continue to use the more familiar notion of Riemannian space-times-although our choice (2.7) of the nonlinearity naturally emerges from an Einstein-Cartan space-time with spin and torsion - the stress-energy content of our semiclassical spin fluid is expressed by the symmetric, combined [65, 81] tensor field

$$\tilde{T}_{\alpha\beta}(u) = \Sigma_{(\alpha\beta)} - \frac{3}{16} \hbar c \ell^{*2} g_{\alpha\beta} \bar{u} L^\nu u \bar{u} L^\nu L_\beta u \quad (3.9)$$

where

$$\Sigma_{\alpha\beta} = \frac{1}{2} \hbar c [\bar{u} L_\alpha D_\beta u - (\overline{D_\alpha u}) L_\beta u], \quad (3.10)$$

$$D_\mu := \partial_\mu + i \beta_\mu$$

is the canonical stress-energy tensor.

$$Q(u) := \frac{e}{c} \int_0 j_\nu(u_\nu) \sqrt{|g|} d\Sigma^\nu = Ne \quad (3.2)$$

$$N = 0, \pm 1, \pm 2, \dots$$

on these states. There, as in following formulas, the integration has to be performed over a space-like hypersurface B , whereas

$$j_\nu = S_\nu + \Gamma_{\mu\sigma}^\mu F^\sigma_\nu + \Gamma_{\mu\sigma}^\sigma F^\mu_\nu - i[A^\mu, F_{\mu\nu}] \quad (3.3)$$

denotes the four-current being locally conserved

$$\partial^\nu (\sqrt{|g|} j_\nu) = 0 \quad (3.4)$$

even in the presence of the classical gauge fields considered in Section VIII. The u-solitons contribute through the generalized spinor current

$$S_\nu := c \bar{u} L_\nu u \quad (3.5)$$

As will be shown later, such a "very crude way" of quantization is equivalent to postulating [76] the old Bohr-Sommerfeld quantization conditions. Besides fixing some otherwise arbitrary initial conditions the claim is that u-solitons with quantized charge are stable in the sense of Ljapunov [52, 128]. However, necessary as well as sufficient conditions for the stability of stationary solitons have as yet only been worked out in detail for complex scalar fields [84].

A hydrodynamical interpretation of the universal wave function $u(y)$ will now be specified which - if necessary - can also incorporate the "internal" degrees of freedom postulated in current quark models of particles.

In conformity with the canonical formalism of relativistic quantum mechanics [11], but deviating from its conventional interpretation we may regard the expression (3.5) as the four-current of a spin fluid. As required, the extended current (3.3) is locally conserved, see equation (3.4).

Since the local law

$$\partial^\alpha (\sqrt{|g|} \tilde{\sigma}_{\alpha\beta}) = 0 \quad (3.11)$$

of energy-momentum conservation holds in the absence of local gauge fields A_μ equ.(3.9) can be regarded as the stress-energy tensor which, in the language of hydrodynamics, contains the tensions keeping the body of fluid together. Moreover, (3.9) is the classical part of that tensor by which the "matter distribution" governs via the right-hand side of Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{\ell^2}{\hbar c} (\tilde{\sigma}_{\mu\nu}(u) + \langle 0 | T_{\mu\nu}(u) | 0 \rangle) \quad (3.12)$$

the metric tensor $g_{\mu\nu}$ of the pseudo-Riemannian background-manifold. (This curved background would be important for the microscopical level too, if the strong gravity hypothesis [97] holds.)

Since a "droplet" built from an u-soliton constitute a localized object, the expression

$$P_\alpha(u) := \frac{1}{c} \int_D (\tilde{\sigma}_{\alpha\nu}(u) + t_{\alpha\nu}^{LL}) \sqrt{|g|} d\Sigma^\nu \quad (3.13)$$

of the four-vector of energy-momentum of an isolated gravitational system [141, §88] can be employed. If present, the gravitational energy is known to contribute to (3.13) through the Landau-Lifshitz pseudotensor $t_{\mu\nu}^{LL}$.

The total angular momentum of our relativistic spin fluid is given by

$$J_{\alpha\beta} = \int_D (\tau_{\alpha\beta\gamma} + \gamma_{[\alpha} \Sigma_{\beta]\gamma}) \sqrt{|g|} d\Sigma^\gamma \quad (3.14)$$

For Dirac fields the occurring spin tensor

$$\tau_{\alpha\beta\gamma} = \frac{\hbar c}{4} \bar{u} L_{[\alpha} L_{\beta} L_{\gamma]} u \quad (3.15)$$

is completely antisymmetric.

Furthermore, for an isolated system - being in our case constituted by a soliton solution for $u(y)$ with an exponentially decreasing, Yukawa-type radial localization - permanently located at the origin of a quasi-Galilean coordinate system the Einstein relation

$$P_\mu(u) P^\mu(u) = m^2 c^2 \quad (3.16)$$

holds.

For a later reconciliation with quantum mechanics it is of prime importance to construct a representative point of an extended object as given in our case by an u-soliton or - "droplet".

Since we are dealing with charged fluids, one method to obtain such a point is to define the "center of charge-coordinates" of a $u(y)$ -soliton through

$$X^\mu(y) := \frac{\int_D y^\mu j_\nu(u) \sqrt{|g|} d\Sigma^\nu}{\int_D j_\sigma(u) \sqrt{|g|} d\Sigma^\sigma} \quad (3.17)$$

Compared to others (see, e.g., Ref. 17) our definition (3.17) employs a generally covariant formalism. Despite that it depends on the choice of a space-like hypersurface D ("slice" through space-time) for the integration, which is not unexpected physically. Nevertheless, (3.17) allows for the deduction of an unambiguous determined four-velocity.

Generalizing other related definitions [29, Sec. II.8] the centroids of an $u(y)$ -lump may be introduced via

$$X^\mu(y) := \frac{1}{c \rho_g \rho^\sigma} P^\sigma \int_D y^\mu [\tilde{\sigma}_{\alpha\nu}(u) + t_{\alpha\nu}^{LL}] \sqrt{|g|} d\Sigma^\nu \quad (3.18)$$

Again a generally covariant notation is employed. In a rest frame, X^μ pass over into the "center of mass-coordinates" given e.g. in MTW [107, p.161].

In an excellent review, Dixon [39] has presented an elaborated formulation for the mass centre of extended bodies in general relativity. There the basic idea is to use

IV. SYNTHESIS OF THE PARTICLE-WAVE DUALITY

As in the Theory of the Double Solution, we intend in NLQM to take the particle-wave duality rather literally. An $u(y)$ -soliton or "droplet" is assumed to represent the localized extension and consequently also possible ("excited") internal degrees of freedom, whereas for $\psi(x)$ remains the task to represent the probability aspects of an elementary particle. "Obviously the wave equation for ψ (which is a fictitious wave that simply expresses a probability) must be linear, for the principle of superposition, which is a necessary consequence of the statistical significance of ψ ..." [28, p.221]. Although not entirely necessary, it appears rather natural to assume in NLQM that the wave equation for ψ acquires the same structure as that for $u(y)$, except that nonlinear terms and a coupling due to internal degrees of freedom are omitted. For fermions, we are then lead to the usual Dirac equation as already given by (2.1).

The conceptual most difficult issue is how to reconcile the assumption of an internal particle fluid with the probabilistic foundation of conventional QM?

In a discussion of an objection of Einstein against his guidance formula, de Broglie supposed (at least for a collection of particles) that "the ψ wave is then associated with the center of gravity of the system" (de Broglie, 1960) [28, p.137] but did not develop this idea much further.

Our synthesis of the particle-wave duality within the framework of a Double Solution Theory will, in an essentially way, be founded on the concept of the center of charge-coordinates of a droplet-like particle, prepared already in the preceeding Section.

Then the fundamental new postulate of NLQM is that $\psi(x)$ governed by the linear law (2.1) determines the probability amplitude of the location of the center x of a $u(y)$ -soliton, the precise position of which can be evaluated as functional of $u(y)$ via (3.17) or alternatively from (3.18).

According to this proposal, the internal structure of a particle is determined by a (quasi-) deterministic nonlinear law such as (2.6, 2.7). On the other hand, due to the functional relation (3.18) between the $u(y)$ and $\psi(x)$ waves, part of the initial conditions of the soliton-solution $u(y)$ are subject to stochastic fluctuations, which as such follow a similar but linear differential

$$p^\alpha j^\mu x = 0 \quad (3.19)$$

in the case of vanishing spin tensor $\gamma_{\alpha\beta}$ as an implicit relation specifying such a point. Note that in this instance (which, however, is not true for Dirac fields as they carry also intrinsic spin given by (3.15)) our definition (3.18) would not be completely unrelated to (3.19).

It is also important to note that these coordinates may well disagree for certain soliton configurations $\{h_i\}$. In fact, de Broglie [29] (see also Ref. 17) advocates to employ (3.17) and (3.18) in the definition of the excentricity

$$e^\mu := p_\nu u^\nu (x^\mu - x^\mu) \quad (3.20)$$

of a spin fluid droplet.

Based on these concepts Bohm and Vigier [17] have formulated a classical theory of rotating fluid masses with the aim [13-16] to provide a causal interpretation of quantum mechanics. In later papers, Vigier and collaborators [61, 26] speculate on further consequences of the idea to regard electrons as extended fluid droplets. If one would follow their picture, a classical counterpart of the spin with a topological justification of its half-integer value could be obtained. Furthermore, it also allows to eliminate the usual "self-accelerated" solution of the classical Lorentz-Dirac electron and prevents an extended electron to radiate in Bohr's orbits around the nucleus.

Following another strand of ideas, de Broglie et al. [32, 33] developed a bilocal theory with an internal rotator structure of the particle. Accordingly, the wave function $\psi(x,z)$ depends not only on space-time coordinate x_μ but on the coordinate z of 3-dimensional complex manifold, the latter being invariant with respect to the group $SO(3)^*$ of complex rotations. (This group is known to be isomorphic to the Lorentz group.)

Following an earlier suggestion of Yukawa [168] such bilocal theories of extended particles have been considerably developed in Japan [169]. Except for a later remark (see Sec. IV) on their relation to NLQM, we will not pursue this idea here further. Rather, we will regard it as a support for our view that even in QCD being based on Yang-Mills-type gauge fields refuge has been taken [58] to a droplet model as a means to achieve the necessary confinement of the elusive quarks!

Then the set (2.1), (2.6, 2.7), (3.16), (3.17) and (4.1) is strictly speaking an intricate system of nonlinear integro-differential equations, thereby constituting a nonlocal NLQM.

Due to this complicated mathematical coupling between u and ψ waves no Lagrangian formalism seems to be available which could yield all these equations from a Hamilton's principle. This, although a Lagrangian such as (2.9) may be written down for each of the fields, if treated separately. In this respect NLQM acquires a rather different structure than QED, although in a self-consistent approach [5] the latter formally also becomes nonlinear, integro-differential equation (see also [144, 143]). In another attempt to formulate a "Unitary Quantum Mechanics" a somewhat related mathematical structure as that occurring in NLQM has been advocated by Sapogin [131].

NLQM may be also compared with an interesting suggestion of Drechsler [41]. Based on the theory of fibre bundles he has developed a bilocal theory of extended hadrons, which involves nonlinear equations being also of the integro-differential type. In contrast to our approach there the internal equation is linear, whereas the "external" wave equation for the spinor $\psi(x, \xi)$ becomes nonlinear and nonlocal. This theory has in common with NLQM that the hadronic matter field $\psi(x, \xi)$ after factorization [42] becomes a mixture of a semiclassical quantity $\psi(\xi)$ with ξ varying over the internal fibre and the usual $\psi(x)$. Only after field quantization does the latter become a q -number. Such "nonlocal" [71] field theories may even be formulated with resort to the concepts of Finsler geometry [72].

As a matter of elucidation it should be pointed out that the relation of the u and ψ -waves in NLQM is quite different from those devised by de Broglie [28]. In his Theory of the Double Solution it is assumed that the phases of a solution

$$\psi = a(x) e^{\frac{i}{\hbar} g(x)} \quad (4.2)$$

of (2.1) and the phases of a solution

$$u = w(x) e^{\frac{i}{\hbar} \hat{g}(x)} \quad (4.3)$$

of the corresponding nonlinear equation (2.6) (internal indices are not considered) coincide for all four spinor components:

equation. As already stated, this linearity for the $\psi(x)$ -wave is indispensable in order to obtain an (-as we will see later on-approximate) superposition principle required by the probability interpretation of $|\psi(x)|^2 d^3x$.

The epistemological relation of the u and the ψ -waves, may also be expressed as follows: The (internal) dynamical structure of a particle is essentially described by the universal wave $u(y)$ following a nonlinear law, whereas the probability amplitudes $\psi(x)$, though being themselves dynamical, are only contingent relations for the choice of initial conditions for $u(y)$. Stated otherwise, $\psi(x)$ describes only the time-evolution of our fundamental ignorance of the location of the $u(y)$ -soliton.

Opposed to Einstein's point of view that there must be a nonlinear field theory containing the quantum laws, our approach appears to be more in accordance with the following visionary hypothesis of Born: "Every theory, built up on classical foundations, requires, for the completion of its assertions, an extension by initial and boundary conditions, satisfying only statistical laws. The quantum theory of the field provides this statistical completion, not, however, in the external manner of the classical theory, but through an inner fusion of the statistical and causal laws" [21, footnote p.434].

Although u and ψ waves correspond to different conceptual levels a further coupling between these two waves is inherent in our scheme. For nonlinear laws such as (2.6, 2.7) not only the "bare" mass μ but also self-energy terms contribute to the field energy $P_0(u)$ -given by (3.13)-of a static u -soliton solution. In nonlinear scalar field models this effect has been explicitly studied and interpreted as a kind of classical "mass renormalization" [94, 34, 96]. Therefore, in an arbitrary reference frame the Einstein relation (3.16) in general yields a deviating mass m which may be regarded as the physical mass of $u_s(y)$. Consequently, we propose that exactly this mass should occur as a parameter in the relativistic wave equation (2.1) for $\psi(x)$ in order to retain its conventional quantum-mechanical interpretation. On the other hand, the probability function $\psi(x)$ may exert effects on the u -solitons if in (2.6, 2.7) an additional term such as

$$NL(\psi) = -\alpha_s L^\mu J_\mu(\psi) \quad (4.1)$$

is allowed. It may be regarded as a "quantum potential" [16] penetrating even to the (semi-) classical u -level.

$$g_k(x) = \hat{g}_k(x) \quad k = 1, \dots, 4 \quad (4.4)$$

Dropping here for the moment complications (see Ref. 28, Chapter XVI(4)) due to the spin, the energy-momentum vector of the particle - in the absence of external potentials A_μ - may then be expressed by

$$p_x = -\partial_x g(x) \quad (4.5)$$

As well-known, this formula extends the classical formula $\vec{p} = -\text{grad } S$ of Jacobi's theory beyond the limits of geometrical optics. This has the consequence, that the motion of the particle at each point of its trajectory in the wave is determined by the "guidance formula" [31]

$$\vec{v} := c \frac{\vec{p}}{p_0} = -c^2 \frac{\partial \text{grad } g}{\partial g / \partial t} \quad (4.6)$$

Instead of (4.4) it has also been proposed [128] to employ rather (3.13) in the deduction of (4.5).

In the framework of NLQM such a guidance principle could only be accepted for the probability amplitude $\psi(x)$ but not for the $u(y)$ function. Our argument against a identification (4.4) of the phases of u and ψ spinors can be deduced from the fact that the nonlinear equations (2.6, 2.7) admits the following formal solutions $u(y)$:

- A) Assume that the components $v^{(i)}$ of $v = v_L(y)$ satisfy (2.6) corresponding linear Dirac equation. Then a solution of (2.6, 2.7) is of the form (4.3) with all the phase functions given by

$$\hat{g}_k = \bar{g}(y) = -\frac{3}{8} \hbar \ell^*{}^2 \int \bar{w} L^5 L_\mu w dx^\mu \quad (4.7)$$

(compare with Ref. 166).

- B) If the phases are specified by

$$g_k = \hat{g}(y) = -\mu c \int |w_0|^2 \frac{\bar{w}_0 L_\mu w_0}{\bar{w}_0 L^\nu \bar{w}_0 L_\nu w_0} dx^\mu \quad (4.8)$$

a solution $u(y)$ of the form (4.3) can be obtained for (2.6) provided that $v = v_0(y)$ satisfies the corresponding massless, nonlinear equation.

In both cases the phase of a solution depends crucially on the path of integration. In the case A the formal relation $v^{(i)}(x) = C\psi(x)$ holds by construction. Then already (4.6) shows that the phases of u and ψ -waves in general cannot be identical.

Destouches [35] in a recent paper discussed an alternative possibility of recovering a probability amplitude function from the more fundamental universal wave $u(y)$. Under certain conditions a functional mean

$$\psi := \langle u \rangle_{\vec{x}} / \| \langle u \rangle_{\vec{x}} \| \quad (4.9)$$

can be constructed from a given nonlinear u -wave which obeys a corresponding linear equation. As an example, the representation (4.3) together with (4.7) may be formally inverted. Since, by construction, the functions $v(y)$ satisfy

a linear Dirac equation, we may put

$$\psi(x) := \frac{w(u(y))}{C |w(u(y))|} \quad (4.10)$$

where v is via (4.3, 4.7) implicitly determined from a solution $u(y)$. This ψ -wave may now be given the probabilistic interpretation of QM, possibly along the lines of thought proposed by Rybakov [128].

At this stage several critical remarks may come to the mind of an attentive reader. We have outlined a theory which accounts in a rational way for the observed extension of particles and have taken care not to spoil the well-tested features of conventional quantum mechanics. However, the crucial question remains: Should it not be possible to dismiss QM with its interpretatory difficulties (see Sec. IX) completely and try to construct a so-called causal theory?

Bohm and Vigier [16] in an interesting paper on the fluid aspects of Schrödinger's theory have suggested to postulate a "subquantum medium" which affects the particle's trajectory by its random fluctuations. Also in NLQM it should be promising to adopt a related point of view.

"But, in reality, small random external actions and small random fluctuations in the boundary conditions always do enter into the picture, and these actions which, by producing a sort of Brownian movement that causes the particle (or representative point) to jump constantly from one unperturbed motion to another, assures realization of the density of presence as $|\psi|^2$. In this way, while still retaining the average physical significance of the trajectories predicted by the causal theory, one succeeds in superimposing upon them a kind of Brownian movement. It is curious to note that in this way there would be achieved a synthesis of the concepts of the causal theory and of Einstein's frequently reiterated affirmation that the successes of the statistical interpretation of Wave Mechanics imply underlying particle-movements of a Brownian character" [28, p.173]. Later, de Broglie has expanded this idea and speculated on a "hidden thermodynamics of particles" [31].

The most detailed attempt into such a direction has been undertaken by Nelson [110, 111] (see also [57] in this context). He developed a close formal analogy between the time evolution of a Schrödinger wave packet and a stochastic process of a classical, nonrelativistic point particle. By considering the two possibly physical interpretations of this formal construction more closely Meink and Tengstrand [104] have to question such a reinterpretation of QM (compare also with Ref. 88). Although their arguments - in particular against a field interpretation of Nelson's derived ψ -function - appears to be correct, it is interesting to note that their reasoning could be easily modified in such a way that it is in support of the general view underlying this NLQM!

V. CORRESPONDENCE PRINCIPLES

There are several levels with respect to which NLQM contains other conventional or by now "classical" theories as limiting cases.

The first correspondence will be revealed we consider the limiting case of vanishing "fundamental" length ℓ^* characterizing the "strength" of the nonlinear self-interaction (2.7). To this end it is instructive to see how such a procedure affects the size or mean soliton radius

$$r_s \equiv \langle |\vec{Y}|^2 \rangle_u^{1/2} := \left(\frac{\int_0^{\infty} |\vec{Y}|^2 j_{\nu} \sqrt{19} d\Sigma^{\nu}}{\int_0^{\infty} j_{\sigma} \sqrt{19} d\Sigma^{\sigma}} \right)^{1/2} \quad (5.1)$$

of a localized u-wave. Our definition is a transfer of that considered by the author [95] in the scalar case (compare with Ref. 148). Despite the covariant notion (5.1) has a good physical meaning only in the rest frame of the center of the soliton. (A notion of an effective particle radius harmonizing more with the theory of general relativity has been discussed by Tredner [142]).

For definiteness let us choose a representative, radially localized soliton

$$|\mu_s(y)|^2 = \frac{4}{\pi^3 \ell^{*3} \cosh(1\vec{Y}-\vec{X}/\ell^*)} \quad (5.2)$$

being centered around x in a rest frame. Note that (5.2) is not a solution to (2.6, 2.7) but simply serves as an convenient example in order to illustrate our point. Using the normalization 3.523.5 and the definite integral 3.523.7 of Ref. 59 we obtain (5.1) the result

$$r_s = \frac{\pi}{2} \sqrt{\frac{5}{2}} \ell^* \quad (5.3)$$

Since this is larger than the half-width of the exponentially decreasing function, it indicates that the definition (5.1) provides a sufficiently good measure for the mean size of a soliton.

In the limit of zero characteristic length ℓ^* we may deduce that

$$|\mu_s(y)|^2 | \vec{Y} - \vec{X} | \rightarrow \delta^{(3)}(\vec{Y} - \vec{X}) \quad (5.4)$$

is a delta-convergent sequence [55] for $\ell^* \rightarrow 0$; i.e., the above function converges to a δ -function in the sense of distributions. For (3.17) this has

$$X^0(y) = Y^0, \quad \vec{X}(y) = \vec{X} \quad (5.5)$$

as an immediate consequence. Furthermore (2.6, 2.7) reduces to (2.1) component-wise. Consequently for $\psi(x)$ the usual point-particle picture of relativistic quantum mechanics has been retained.

A similar point of view has been envisioned by Born [21]. In an attempt to quantize a nonlinear theory of the electromagnetic field, "the usual quantum mechanics is [considered as] the limiting case in which the self-field is regarded as rigidly bound to the centre. The field quantities become, then, only functions of the motions of the centres of particles."

At this stage, we may enter the "ultraquantum" region where even the Heisenberg uncertainty relations could be violated. This curious instant can be observed in nonlinear models if the mean value of the non-relativistic kinetic energy is defined according to a more classical prescription [122]. Different results are obtained if the Weyl form [87, p.45] of the uncertainty relations are studied in nonlinear models [89, 95*, 96].

relations are studied in nonlinear models [89, 95*, 96].

In this region NLQM would presumably correspond rather to a nonlinear and nonlocal classical field theory than to a quantum theory. On the other hand, for atomic structures characterized by an extension of 10^{-8} cm an electron of a radius of the order of 10^{-13} cm could well be treated as point-like. Nevertheless, more detailed checks on the computability of NLQM with the conventional results would be instructive.

VI. RELATIVISTIC HYDROGEN ATOM IN NONLINEAR MODELS

The bound-state problem presented by the hydrogen atom admits well-known [11] exact solutions in the case of the linear Dirac equation (2.1). Except for the Lamb-shift - conventionally accounted for via quantization of electromagnetic field - and other hyperfine structure effects, the resulting energy levels agree extremely well with the observed spectrum. Under these circumstances could we risk modifications by adopting a NLQM?

An interesting treatment of a nonlinear spinor equation being analogue to (2.6, 2.7) has been performed by Ráñada [118] in this case. In order to obtain normalizable solutions it was found that the Coulomb potential

$$A_{\bullet}(Y) = - \frac{\sum \alpha}{v} \quad r = |\vec{y} - \vec{x}| \quad (6.1)$$

has to be modified by a term corresponding to a finite sphere of uniform charge density at the origin. Under this physically justifiable assumption it is shown numerically as well as by analytic approximations [121] that the nonlinear term induces deviations of the energy levels from those of the linear theory which are of the order α_0^6 ($\alpha_0 = e^2/\hbar c$ being Sommerfeld's fine structure constant). This

* Note that in Ref. 95 the definition (17) of the mean squared momentum needs to be corrected by dividing it by a normalization integral similar to (40) of Ref.

The second correspondence is familiar from conventional QM (see, e.g., Landau & Lifshitz [87] p52). In order to facilitate our reasoning, note that every spinor component of ψ - for vanishing A_μ and Γ_μ - satisfies the Klein-Gordon equation

$$0 = \psi(x) \left[\left(\frac{\psi}{2m} \right)^2 + x e^{x\psi} \right] \quad (5.6)$$

whereas the components of $u^{(i)}(y)$ due to (2.6, 2.7) in the same case have to satisfy the nonlinear Klein-Gordon equation

$$[\partial_\mu \partial^\mu + \frac{3\mu c}{4\hbar} \bar{u} \gamma^5 \gamma_\mu \gamma^5 \gamma^\mu + \frac{9}{64} (\bar{u} \gamma_\mu \gamma^5 u)^2 + (\mu c/\hbar)^2] u^{(i)}(y) = 0 \quad (5.7)$$

(See equ. (2.5) of Ref. 34, compare also with [149]). Due to the quantum mechanical assignment

$$-i\hbar \partial_\mu \rightarrow p_\mu \quad (5.8)$$

both the linear as well as the nonlinear Klein-Gordon equation yield the Einstein relation

$$p_k^{\mu} = m^2 c^2 \quad (5.9)$$

of relativistic mechanics in the limit $\chi \rightarrow 0$. The reason simply being, that the nonlinear terms in (4.7) become multiplied by χ and χ^2 , respectively. In the deduction of (5.9) $\mu \rightarrow m$ for $\chi \rightarrow 0$ is obligatory. This follows as a consequence of the postulated functional relation (3.16) between u and ψ -waves, if transferred to scalar fields.

However, for interaction distances of particles comparable to the fundamental length ℓ^* one should expect that the dynamics of the u-wave will dominate over that of the ψ -wave. The reason being that then the inherent nonlocality of the field equations for $\psi(x)$ becomes noticeable. Furthermore, its linearity is lost and in the wake of this, at small distances, also the superposition principle. Consequently at diminutive interaction distances we may have to give up the probabilistic interpretation of $|\psi(x)|^2 dx$ being so successful in the medium range.

escapes experimental detection. Since these energy eigenvalues are so close to those of the linear theory, conventional, relativistic QM of the hydrogen atom may be regarded as the linear limit of a nonlinear theory.

In NLQM with its double solution picture, analogous arguments as the preceding ones are expected to hold for the $\psi(x)$ - wave, even after taking its nonlocal coupling - due to (3.16) - to the u-level into account. In the u equation (2.6) the assumed nonlinear self-interaction (2.7) of the electron is expected to dominate on a length scale of the order

$$\ell^* \approx 10^{-12} \text{ cm} \quad (6.2)$$

For an average distance of $\sqrt{|\vec{x} - \vec{x}_p|} \approx 1\text{\AA} = 10^{-8} \text{ cm}$ of the extended electron from the nucleon located at x_p the Coulomb potential would be negligible in (2.6,2.7), at least for the lowest energy, localized soliton-solution of the u-equation, representing the electron. Therefore, we would obtain a movable soliton solution localized at the center of (its!) charge position x which, as such, is very little affected by the "far-off" static electric potential. On the other hand, the probability distribution arising from this picture for the $\psi(x)$ -wave, would be almost identical to the linear limit discussed before.

In NLQM our general view of the hydrogen atom is therefore in contradistinction to that advocated by Rnada. From a linearization of a nonlinear spinor model he concludes that "in a Coulomb field the kink [soliton] has a radius of order $1\text{\AA}$, and this must be considered as the radius of the electron itself in this situation" (quotation from Ref. 118, p.809).

However, a difficulty arises if the nucleon itself is described by an analogous soliton solution $u_p(y)$ around x_p . This is due to the fact that the probability $|\psi(x - x_p)|^2 d^3x$ of the $S_{1/2}$ state is known to have a maximum at the origin. Then, the two "solitons", one for the electron and one for the proton, would have a chance - due to their extension - to interact on the semi-classical u-level. This in turn would cause a noticeable deviation from the known behaviour of the ψ -waves as calculated according to conventional QM. In order to prevent gross effects, we are obliged to allow (almost) only elastic scattering of the localized u_e and u_p waves, at least in the dynamical situation posed by the hydrogen atom. In fact, this is exactly the property of solitons in the restricted sense [90 p.11]. The question arises, however, to what extent these features survive in the physically more relevant

cases of non-integrable solutions we are dealing with. Let us remark here, that the Lamb-shift, in QED [11] commonly accounted for by considering the vacuum fluctuations of the quantized electromagnetic field, according to some speculations [135] may be recovered from a linear limit of a nonlinear theory [129]. According to another proposal, the Lamb-shift may be deducible via a self-consistent treatment [5] of the nonlinear equations of QED.

VII. IDENTICAL PARTICLES

In classical mechanics, the elements of a system of N identical objects do not lose their individuality, despite the identity of their physical properties. The reason being that one can imagine these "particles" to be numbered at some initial time and then follow each of their trajectories as time evolves. In quantum mechanics, this is not possible, because of the uncertainty principle.

Conventionally, in relativistic quantum mechanics, an abstract configuration space of the system is introduced, which is constituted by the aggregate of $4N$ coordinates $x_{(n)}^\mu$, $n = 1, \dots, N$ for N particles. The motion of the system is then described by a wave function $\psi(x_{(1)}, \dots, x_{(N)})$ obeying the first order partial differential equation

$$\left[\sum_{n=1}^N i \gamma^\mu \frac{\partial}{\partial x_{(n)}^\mu} - \frac{e}{\hbar c} \gamma^\mu A_\mu(x_{(1)}, \dots, x_{(N)}) - mc/\hbar \right] \psi(x_{(1)}, \dots, x_{(N)}) = 0 \quad (7.1)$$

generalizing (2.1). The quantity $\bar{\psi} d^4x_{(1)} \dots d^4x_{(N)}$ measures the probability of the presence of particle 1 in the four-dimensional volume element $d^4x_{(1)}$ of the physical space, particle 2 in volume element $d^4x_{(2)}$ of physical space and so on.

However, in the absence of particle localization in the conventional theory, the construction of a configuration space out of the physical coordinates of the N particles seems ambiguous if not "of paradoxical nature" [30, Chapter VI].

is determined by a certain, fixed fraction of say the half-width

$u^{(n)}(y_{2D}) = \frac{1}{2} u^{(n)}$ of the n^{th} - "bump" (Fig. 1.1). This prescription can only make sense as long as these domains are disjunct. Otherwise, if two such domains begin to overlap in an interaction, it is natural to regard these "two" corresponding bumps already as one object.

Strictly speaking, conserved quantities like charge and four-momentum will lose their usual additivity property in the case of a N soliton system, i.e.

$$P_{\mu}^N \neq \sum_{n=1}^N P_{\mu}^{(n)} \quad (7.5)$$

This is due to our definition of the integration domain $D(n)$ which will cut-off the tail of single solitons. According to (3.13) this would affect $P_{\mu}^{(n)}$ but not the momentum P_{μ}^N of the total system. Using conventional physical terms the nonlinear soliton "tail" accounts for effects induced by the "vacuum".

What is the probabilistic representation of the N -soliton solution (7.2) in NLQM? As long as their centers are mutually infinitely separated, we can disregard interactions and may analyse the linear equation (7.1) in the free case, i.e. $U_{\mu}(x_1, \dots, x(N)) = 0$.

According to the rules of probability the full system is described by

$$\psi^N(x_1, \dots, x_{(N)}) = \prod_{n=1}^N \psi^{(n)}(x_{(n)}) \quad (7.6)$$

there $\psi^{(n)}(x_{(n)})$ denotes the probability amplitude function over physical space-time of the n^{th} soliton member of (7.2) and therefore has to satisfy the free one-particle Dirac equation (2.1). Consequently, the probability function (7.6) for the N -particle system satisfies (7.1), as well-known.

It should be noted that in NLQM the so-called configuration space $(x_1, \dots, x_N)$ as the domain of the ψ^N -wave appears to be a derived concept. Via (3.5) or (3.4) it is explicitly constructed from the space-time coordinates of the center of each of the N soliton solutions.

Furthermore, in the limit of large separation, these solitons are essentially identical on the level of the u wave if they are subject to a universal nonlinear equation and the same global conditions. Therefore, the probability distributions $\psi^{(n)}(x_n)$ are modulo space-time translations the

Let us see now, how the double solution aspects of NLQM resolves these issues posed by a many-body problem.

To this end consider a soliton solution $u_g^{(n)}(y)$ of the nonlinear Dirac equation (2.6, 2.7) in the absence of external potentials. Assume that this solution is centered around $x_{(n)}^u$ according to one of the definitions (3.17) or (3.18). Then the multi-soliton (and anti-soliton) solution

$$u(y) \simeq \sum_{n=1}^N u_{S,A}^{(n)}(y) \quad (7.2)$$

is also a solution of (2.6, 2.7) provided that the separation

$$|\vec{x}_{(n_1)} - \vec{x}_{(n_2)}| \rightarrow \infty \quad (7.3)$$

of each of their centers located at $\vec{x}_{(n)}$ tends sufficiently fast (depending on the localization of $u_{S,A}^{(n)}(y)$) to infinity ("weak" superposition principle). Since we have postulated a universal nonlinear equation for the u -wave, it should be noted that each member of the N -soliton solution acquires the same functional dependence.

$$u_S^{(n)}(y - x_{(n)}) = u_S^{(n_2)}(y - x_{(n_2)}) \quad (7.4)$$

provided we impose the same localization and global conditions on its

solutions. This, e.g., may be achieved by imposing the charge quantization condition (3.2) on an N -soliton solution. Its possible link to QFT will be exhibited in the last Section.

In general, the formation of a system of N particles on the u -level deserves a more elaborate consideration. Nevertheless, the postulate of a universal nonlinear equation is essential for this construction as also pointed out by de Broglie [28, p. 161, without the underlining and omissions]: "Since the wave equations of the [two] particles are the same, it is natural to assume that, if the u wave-trains partially overlap, the waves may be superposed and form a single u wave ... the amplitude ... here having two distinct mobile [singular] regions". For such overlapping N -soliton solutions one may obtain a reasonable notion of the charge, the four momentum, the center of gravity or of charge density as well as the mean radius of a single soliton $u_{S,A}^{(n)}(y)$ within the N -particle system if the formulas (3.2), (3.13), (3.17), (3.18) and (5.1) respectively are applied only for a certain domain $D(n)$. It should contain the maximum of the n^{th} soliton such that its boundary $\partial D(n)$

the same, too. Then these identical $u^{(n)}$ solitons are also indistinguishable on the probabilistic level and the complete system should rather obey a completely (anti-)symmetric wave function

$$\begin{aligned} \hat{\psi}^N(X_{(1)}, \dots, X_{(N)}) &= \bigwedge_{n=1}^N \psi^{(n)}(X_{(n)}) \\ &:= \frac{1}{N!} \sum_{\sigma \in S_N} \psi^{(\sigma(1))}(X_{(\sigma(1))}) \dots \psi^{(\sigma(N))}(X_{(\sigma(N))}) \\ \epsilon_{\sigma} &= \pm 1 \end{aligned} \quad (7.7)$$

leading to (Bose-Einstein) Fermi-Dirac statistics in the case of (bosons) fermions.

This prescription will break down in high energy scattering experiments in which we probe the internal structure of extended particles. Then, at distances compared to the characteristic length r_s of the particles also the equation (7.1) for the ψ -wave will inherit a complicated nonlinearity and nonlocality from the u -level. This will be induced not only via the Einstein relation (3.16) but possibly also through the "external" potential" Up which may contain terms exhibiting an intricate functional relation to the u -level. Strictly speaking, even for large soliton separation a "quantum interconnectedness of distant systems" [114, and references therein] is upheld in NLQM.

VIII. IN QUEST OF A UNIVERSAL NONLINEAR EQUATION

In our construction of a new nonlinear quantum mechanics we have followed de Broglie's suggestion [28, p.221] "... that the true u wave equation is a non-linear one...". As in the theory of the double solution in NLQM it is assumed "... that, except in a very small region constituting the "particle" in the strict sense of the word, the u wave obeys the same linear equation as the ψ wave." De Broglie acknowledges the influence of the success of Einstein's general relativity which fulfills the nonlinear law (3.12) on his ideas.

Adhering to these principles, the issue of a choice of a viable nonlinear generalization of the relativistic field equations of QM, becomes more pressing than ever in NLQM. This applies in particular to the most basic equation of physics: the Dirac equation for the fundamental fermions, such as electrons, protons etc.

Although we have already built NLQM around the Heisenberg-Pauli-Weyl equation (2.6, 2.7) let us give here a more detailed exposition of its history and the virtues of our choice:

In his 1950 paper [159], Hermann Weyl has unfolded powerful geometric reasons (in germ contained already in his 1929 paper [158]!) that a Dirac equation respecting Einstein's principle of general covariance must contain a nonlinear term of the axial vector-type, which survives even after switching off gravitation. Later this point of view has been made completely transparent by Hehl and Datta [64]. Following related works of Sciama and Kibble they could elucidate the fact that in any Poincare-invariant gauge field theory Cartan's earlier given notion of torsion induces such nonlinear terms into the Dirac equation (we refer to the by-now standard survey of Hehl et al. [65]).

After establishing the isospin symmetry $SU(2)$ in nuclear physics, Heisenberg and Pauli in 1957 made (see Ref. 68) the important suggestion that all newly found hadronic resonances should follow from one fundamental nonlinear spinor equation. Only an isospin doublet representing proton and neutron would be needed to constitute the fundamental fermions. Later Finkelstein [52] has shown that this approach would give rise to "isotorsion". By employing fibre bundles techniques over curved space-time, generic $U(f)$ -torsion may be considered [97]. From the resulting $GL(2f, \mathbb{C})$ gauge model characterized by an Einstein-Cartan-Dirac-type Lagrangian, the spinor equation (2.6, 2.7) emerges rather naturally as a prototype nonlinear relativistic equation.

In field theory, however, the origin of nonlinear terms may fall into two categories: The nonlinearities of the vacuum-induced-type which may be called secondary should be distinguished from the more primordial, genuine nonlinearities which occur in gauge field theories such as those of Yang and Mills [167] or Einstein-Cartan [65].

According to Ivanenko [74] who very early considered nonlinear generalizations of Dirac's theory, nonlinearities in the field equations could as well be a necessary consequence of relativistic QFT (see also Ref. 70 for such an instance). In such secondary theories the fundamental length ℓ^* appearing in nonlinear terms as (2.7) may then provide [75] a natural cut-off needed to cure divergency problems of conventional QFT. From our point of view it is more natural to adopt for the u -wave a primordial nonlinearity,

i.e., one which originates from an underlying Riemann-Cartan geometry with affine connection. However, the same nonlinear equation (2.6, 2.7) may also follow - as a secondary result - from a renormalization of QCD, and vice versa [43].

In consideration of the arguments given in favor of the nonlinear Dirac equation (2.6, 2.7) it is not surprising that much effort has been made to unveil the structure of its solutions.

Very early, Finkelstein et al. [53], Soler [138] and more recently Rafelski [117], Ránada [119] and Takahashi [139] have employed numerical methods in order to obtain radially localized solutions of related nonlinear Dirac equations. The results of Ránada [119] are particularly suggestive because the obtained energy spectrum of these spinor solitons seems to correspond to some of the observed low-lying baryon states. Exact solutions of the covariant generalization [134, see also 60] of such a nonlinear Dirac equation have been obtained by the present writer [98]. The paper contains also further references on this subject. It may be added that the massless case of (2.6, 2.7) in flat space-time will also be referred to as the Nambu-Jona-Lasinio model [108, see also 44]. By relating to a completely integrable dynamical system formal solutions for this model can be constructed [48]. Although not important to the approach proposed in this paper, it is worth mentioning that two-point functions of QFT have been treated in spinor models with Wick-ordered nonlinearities [145-147].

Let us turn now to the description of particles carrying spin different from 1/2. Scalar particles, even if they cannot be regarded as fundamental, on the conventional probabilistic level are described by a scalar field $\varphi(x)$ which obeys the Klein-Gordon equation.

$$[\square + (mc/\hbar)^2] \varphi = 0 \quad (8.1)$$

In curved space-time, this equation can be generally covariant formulated by employing the Laplace-Beltrami operator

$$\square := \frac{1}{\sqrt{|g|}} \partial_\mu (g^{\mu\nu} \sqrt{|g|} \partial_\nu) \quad (8.2)$$

According to the general principles of NLQM the u-wave should be determined by a nonlinear generalization of (8.1).

A survey of the properties of scalar nonlinearities has been given by Makhankov [90]. Let us consider as an example a nonlocal (already on the u level!!) version of the Rosen model [125]. In this model the set

$$u \equiv \{u^{(i)}(y) \mid i = 1, \dots, f\} \quad (8.3)$$

of f scalar fields would have to satisfy the integro-differential equation

$$[\square - 3\ell^4 \int \frac{d^4 y}{S^4(y)} (u^* u)^2 + (mc/\hbar)^2] u = 0 \quad (8.4)$$

in four dimensions [94, 95]. The average of the nonlinear term in (8.4) over a 2-dimensional topological sphere S^2 centered at y allows in a rest frame the construction of arbitrary spherical solutions via the familiar separation Ansatz

$$u^{(i)}(y) = f_m^{(i)}(r) Y_m^i(\theta, \phi) e^{-it\mu c^2/\hbar} \quad (8.5)$$

It is a virtue of this model that the differential equation for the radial function $f_m(r)$ admits exact [94, 47] localized (see Fig.2) solutions for arbitrary quantum number l of angular momentum. Further properties of these soliton-type solutions can be found in the literature [151, 95, 47].

In the case that $f = 2l+1$ and $m = l-1$ the calculation of the field energy [95] (Mielke, 79) of such solitons with fixed, quantized charge yields the mass formula

$$M = m_0 + a \, l(l+1) \quad (8.6)$$

Although a nonlinear scalar theory as given by (8.4) can only be a "toy model" it is encouraging to note [99] that (8.6) agrees rather well with the experimental data for the rotational band of nucleon resonances.

A similar (at least for high principal quantum number) formula as (8.6) can be obtained from a simplified, nonrelativistic version [50, 3] of the MIT "bag"-model [63]. In order to obtain wave functions of finite spatial

by the Sine-Gordon equation [127]. General relativity, e.g. yields in two dimensions only flat manifolds as solutions of the vacuum Einstein equation (3.12):

In generally covariant theories, measuring devices based upon gravity would provide a means of observing non-quadratic observables conventionally prohibited by QM. Because of this surprising fact, we have to face the following alternatives: "either gravity is not classical or QM is not orthodox" [101]. Adopting the latter point of view, even the hypothesis has been pursued that a quantum nonlinearity might indirectly originate from the nonlinearity of the background space-time when coupled to Einstein's theory of general relativity [83].

Present-day models of strong interactions such as QCD [91] are based on the hypothesis that quarks carry additional "colour" degrees of freedom. This leads to a local gauge theory, in which the spin 1 gauge fields - the so-called gluons - are expected to provide the confinement or at least saturation [49]: Quarks with observed particles are always bound to produce colour-singlets, only. Consequently, QCD employs a "hidden" gauge symmetry [112]. Its dynamics are modelled after Yang-Mills theories [167] in which the field strength

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \quad (8.7)$$

are nonlinearly related to the gauge potentials

$$A_\mu = A_\mu^j \tau_j \quad (8.8)$$

Since the matrices λ_j ($j = 1, \dots, c^2$; normalized by $\text{Tr}(\lambda_j \lambda_j) = \lambda_j^2$) are the vector operators (generalized Gell-Mann matrices) of the color group $U(c)$, the commutator $[A_\mu, A_\nu]$ in (8.7) does not vanish as in the abelian case, i.e. as for $c = 1$. Opposed to electromagnetism this is the cause for the nonlinearity in the definition of the field strength (8.7). The Yang-Mills equations themselves are (quasi-) linear as in Maxwell's theory.

In order to achieve some progress in the still unsolved confinement problem in QCD, Mills [105] has suggested to consider Yang-Mills fields the dynamics of which being determined by the generalized Lagrangian density:

extent all these models need to employ discontinuous boundary conditions at the particle "radius". Such constructions appear to be rather artificial if compared with those leading to soliton-like configurations (Fig.2) in nonlinear theories.

Other polynomial nonlinearities have also been studied for Klein-Gordon fields (see e.g. Ref. 24). Due to its stability properties [1], a scalar self-interaction related to the "squared form" [149, 34] of the Heisenberg-Pauli-Weyl equation (2.6, 2.7) has attracted some attention. It has been shown by numerical means that such a scalar equation admits localized particle-like solutions [34]. More recently, geon-type solutions thereof have been constructed by exact [96] as well as by numerical methods [100].

However, it has been questioned if non-dissipative confining (for the "quarks") solutions are the only soliton solutions even if such global parameters as charge and energy are kept fixed. According to arguments given by Werle [156] it might be necessary to include fractional nonlinearities in order to achieve the uniqueness of solitons. This may also apply to spinor models [153, 154]. Other models involve even higher order derivative couplings [120].

In order to preserve the separability of non-interacting subsystems Bialynicki-Birula and Mycielski [10] (see also Ref. 9) have proposed to introduce a logarithmic nonlinearity in non-relativistic Schrödinger mechanics. This allows - as in QM - to obtain a solution of a complete, noninteracting, system by taking the product of those solutions of the nonlinear equation which applied to single elements of the system. However, in relativistic versions [126] of this model, unphysical solutions can occur [132]. Furthermore, according to the principles of NLQM advanced in this paper, the above approach could at most be useful for the probabilistic ψ -level but would not be desirable for the universal wave function representing the internal structure of particles. In connection with the above model we mention that in one-space dimensions the time-development of the "Gaussons", i.e. of the soliton-solutions of the logarithmic Schrödinger equation has been numerically determined for simple external potentials [106].

In general, however, we do not share the view that one may learn something of physical and not "merely" mathematical interest by considering models in one space and one-time dimensions, A prominent example being given

$$\mathcal{L}_{Y-M} = \sqrt{|g|} \{ L(\phi, \phi^*) - \frac{1}{2} \text{Tr} (A_\mu S^\mu) \} \quad (8.9)$$

Here

$$\phi := -\frac{1}{4\alpha} \text{Tr} (F^{\mu\nu} F_{\mu\nu}) \quad (8.10)$$

$$*\phi := -\frac{1}{4\alpha} \text{Tr} (F^{\mu\nu} *F_{\mu\nu}) \quad (8.11)$$

$$*F_{\mu\nu} = \frac{1}{2} \sqrt{|g|} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} \quad (8.12)$$

are the two quadratic forms which in four dimensions are the only ones permitted by gauge invariance. ($\alpha = g^2/\hbar c$ denotes here the generalized "fine structure constant" constructed from the gauge coupling constant $g (= e$ in the electromagnetic case)). Then, the Euler-Lagrange equations take the form

$$\partial_\mu F^{\mu\nu} + i [A_\mu, F^{\mu\nu}] = \frac{1}{c} S^\nu \quad (8.13)$$

whereas the modified field strength (i.e., the field momenta canonically conjugated to $-4\alpha \phi$) are given by

$$F^{\mu\nu} = \sqrt{|g|} \left(F^{\mu\nu} \frac{\partial L}{\partial \phi} + *F^{\mu\nu} \frac{\partial L}{\partial \phi^*} \right) \quad (8.14)$$

On the classical level the confinement of quarks as well as those of gluons can be achieved by choosing [105]

$$L_M(\phi, \phi^*) = \phi \left[1 - \left(1 + \frac{1}{b^2} \phi^2 \right)^{-1} \right] \quad (8.15)$$

i.e. for a model with a Yang-Mills-type, but essentially nonlinear dynamics.

There are other reasons which may necessitate the inclusion of such nonlinearities already in electrodynamics. In the classical, linear Maxwell-
Faraday theory the self-energy of a point charge is known to be infinite. Therefore, Born and Infeld [22] have expressed the view that this could be resolvable (at least partly) for the divergence problems in quantum electrodynamics. With the motivation to cure these they proposed a nonlinear electrodynamics built upon the Lagrangian

$$L_{B-I}(\phi, \phi^*) = \sqrt{1 - 2\phi - (\phi^*)^2} - 1 \quad (8.16)$$

constructed from abelian gauge fields. (In an earlier version of this model Born [21] considered the choice (8.16) without the $*\phi$ -term).

As already mentioned a related nonlinear Lagrangian is also the outcome of a quantum-field-theoretical treatment of the vacuum polarization in Dirac's theory of the positron [70].

It is instructive to consider electrostatic vacuum solutions of (8.16) in the case of a flat space-time more in detail. Since $*F_{0b} = *F_{0b} = 0$ and all other fields components are independent of time, the field equations (8.13) reduce to the familiar set of equations

$$\text{rot } \vec{E} = 0 \quad (8.17)$$

$$\text{div } \vec{D} = 0 \quad (8.18)$$

where

$$\vec{E} = \{ F_{b0} \mid b = 1, 2, 3 \} \quad (8.19)$$

$$\vec{D} = \{ F_{b0}^* \mid b = 1, 2, 3 \} \quad (8.20)$$

are the vectors of the electrical and dielectrical field strength, respectively. In the case of spherical symmetry, (8.18) has the well-known solution

$$D_r = \frac{Q r^2}{r^2} \quad (8.21)$$

The dielectrical field, on account of (8.14), is related to the electrical field

$$E_r = -\phi'(r) \quad (8.22)$$

via

$$\vec{D} = \vec{E} \frac{\partial L}{\partial \phi} \mid \phi = -\frac{1}{2\alpha} |\vec{E}|^2, \phi^* = 0 = -\alpha \frac{\partial L}{\partial \phi^*} \quad (8.23)$$

For the Born-Infeld Lagrangian (8.16) we therefore obtain

IX. REMARKS ON EPISTEMOLOGICAL PROBLEMS OF THE MEASUREMENT PROCESS

The invention of QM has created epistemological issues which have presented a challenge to the greatest scientific minds of our time. Most famous is the controversial discussion of Einstein and Bohr on the interpretation of quantum mechanics (as described by Bohr [133, p. 199].) Although the "Copenhagen interpretation" of QM has won the battle and become a widely accepted dogma, certain reservations remain, even for Wigner [163], the founder of its relativistic extension [161].

Moreover, for a philosopher like Lakatos, engaging in a critically rational standpoint the very foundation of QM appears to be completely unacceptable: "In the new, post-1925 quantum theory the 'anarchist' position became dominant and modern quantum physics, in its 'Copenhagen interpretation', became one of the main standard bearers of philosophical obscurantism. In the new theory Bohr's notorious 'complementarity principle' enthroned [weak] inconsistency as a basic factual final feature of nature, and merged subjectivistic positivism and antilogical dialectic and even ordinary language philosophy into one unholy alliance. After 1925 Bohr and his associates introduced a new and unprecedented lowering of critical standards for scientific theories. This led to a defeat of reason within modern physics and to an anarchist cult of incomprehensible chaos. Einstein protested: 'The Heisenberg-Bohr tranquillizing philosophy - or religion? - is so delicately contrived that, for the time being, it provides a gentle pillow for the true believer'" (from Ref. 85, p.145).

All epistemological problems come particularly sharply into focus by considering the measurement process [164] of QM, i.e. the process which involves interactions of the microscopical system with the macroscopic apparatus. In several critical studies de Broglie [28,30] has discussed the most prominent "Gedankenexperimente" - also known as paradoxes - which are devised to test the implications of quantum mechanics. In addition he contrasted it with the corresponding explanation emerging from his theory of the double solution (see also Ref. 2).

Although the present author feels quite unqualified to take up these controversial discussions in this paper again, may he be forgiven some very preliminary and humble remarks on the changes that the proposed NLQM could make with regard to these fundamental problems.

Let us consider a very simple experiment proposed in 1927 by Einstein

$$D_r = \frac{E_r}{\sqrt{1 - \frac{1}{2} E_r^2}} = -\frac{\phi'}{\sqrt{1 - \frac{1}{2} \phi'^2}} \quad (8.24)$$

The solution of (8.24) together with (8.21) can be expressed in closed form via elliptic integrals:

$$\phi(r) = \frac{\sqrt{2}}{r_0} \int_{\frac{r}{r_0}}^{\infty} \frac{dy}{\sqrt{1+y^2}} \quad (8.25)$$

This potential has to replace Coulomb's $\frac{1}{r}$ -law. As a further result the electrical field strength turns out to be

$$E_r = \frac{\sqrt{2}}{r_0^2} \left[1 + \left(\frac{r}{r_0} \right)^4 \right]^{-1/2} \quad (8.26)$$

The corresponding electromagnetic field energy turns out to be finite, as desired. A similar construction would apply in the case of static solutions of Mills' Lagrangian (8.15) as well as for the expanded Born-Infeld Lagrangian considered by Vazquez [150].

In conclusion, the main virtues of a nonlinear modification of Maxwell's theory of electromagnetism is the avoidance of a field singularity occurring otherwise in the electrostatic potential of a "point" charge. This entails also a modification of Coulomb's law, however, only for distances compared to or smaller than a certain radius r_0 , which may be roughly associated with "electrical extension" of a particle. If matter is also described by nonlinear equations - such as (2.6, 2.7) on the u-level if the NLQM paradigm holds - it would appear natural to make a connection of r_0 with the soliton radius r_s defined by (5.1). However, then the whole derivation should be reconsidered by employing the complete set (2.6, 2.7) and (8.13) of coupled (via A_μ and S^1) nonlinear field equations (remember that B_μ contains the $U(r)$ gauge potentials A_μ as well as the spin connection). Self-consistent soliton solutions of such coupled models are rarely [152, 116] studied. An exception are the geon-type solutions of semi-classical fields coupled to gravity [100]. The general lesson to be learnt from our quest for a "universal" nonlinearity could well be this: In order to advance a realistic soliton model of particles it might be necessary to incorporate-in a "unitary" manner (see Ref. 51) all known physical fields in the construction of such configurations.

[28] and later refined to a negative - result experiment by Renninger [123].

In this Gedankenexperiment a (macroscopically) point-like source S is placed in the center of a 4π -counter D_2 , e.g. realized by a spherical photographic film. Assume that S emits particles (photons) isotropically in all directions. As long as no photon has been detected at say the grain A , following the usual interpretation of QM, the particle must be considered as present, in a potential state, over the whole thin sphere of radius $r < r_D$, with the same probability $|\psi(r)|^2 d^3x$. Note here "that the wave function gives information about the probability of one photon being in a particular place and not the probable number of photons in that place" [37, p. 1]. After the detection at A , the probability of finding it at any other point on the (macroscopically possibly rather large!) film becomes instantaneously zero. According to the "orthodox" view, wave mechanics has to employ simultaneously two conceptions, contradictory in appearance, namely that of a homogeneous plane wave, indefinitely extended and that of a localized granule. This is the essential lesson of Heisenberg's uncertainty principle and, according to Bohr, both concepts are "complementary aspects" of reality. Consequently, a measurement process changes the state of our knowledge about the particle. Through an "irreversible act of amplification" [19] a reduction (or collapse) of the probability packet (mixture) occurs and afterwards only the eigenvalues of quantum-mechanical observables corresponding to a pure state are measured. Obviously, this can only be a sketch of the subtleties of the "conventional" interpretation. However, it may also indicate why for e.g. de Broglie such a chain of hypotheses appears to be "most mysterious" [28, p. 75].

An even more paradoxical situation occurs if a scintillator D_1 , covering the solid angle $\Omega < 4\pi$ is placed between S and D_2 (Fig. 3). Notwithstanding diffraction phenomena, the probability of detecting a particle at D_1 or at D_2 is shortly after emission $p_1 = \Omega/4\pi$ and $p_2 = 1 - \Omega/4\pi$, respectively. However, if the spherical wave train ψ of the particle has past the position of D_1 and no scintillation has been produced, one is forced to assume that the probability p_1 has become suddenly zero and in turn $p_2 = 1$: According to the "orthodox" lines of thought this is caused by a "collapse" of the wave function, being conditioned by our increase of knowledge of the quantum-mechanical system. In this case, however, this cannot be due to an "act of measurement" since nothing has been observed at D_1 : It appears to us that a somewhat similar paradox occurs in Wheeler's [160] "Delayed-Choice" version of the

Double-Slit experiment, the latter being analysed in much detail by Wootters and Zurek [165]. For a proponent of the Copenhagen interpretation the general lesson which may be drawn from such Gedankenexperimente is this: "No phenomenon is a phenomenon until it is an observed phenomenon" [160].

The central issue therefore remains: in which way could NLQM contribute to a more rational understanding of e.g. Einstein's example? First of all, in the framework of NLQM, it should not surprise us that after a measurement an event is only found at a certain grain A of the film D_2 , since this should be the (at least indirect) consequence of the localization of the u -soliton representing the particle. Before the act of measurement, the probability $|\psi|^2 d^3x$ of finding the centre of the soliton somewhere in the ball is almost the same as in conventional QM. Only close to the interaction region the linearity of the equation for the ψ waves is suspended (as tacitly also assumed in the Copenhagen picture!) and the nonlocality and non-linearity induced from the u -level are expected to be operational for a "reduction" of all (weakly) superposed states ψ . By considering nonlinear models for the microscopical particle as well as for the "macroscopic" apparatus, Bohr's "irreversible act of amplification" during a measurement process may find a rational explanation in the framework of NLQM.

Some support for such an idea comes from the study of the unharmonic (nonlinear) oscillator of classical mechanics. If a sufficiently strong external force is applied to it a quasi-irreversible abrupt transition of the amplitude near the resonance frequency occurs [86, p. 89]. It is a related suspicion [118] that the very rich and almost unknown world of the stability of the solutions of nonlinear partial differential equations may offer a suggestive formalism for the study of such (quantum?) "jumps". Here is also the area where René Thom's [140] ideas may have a profound impact on the quantum physics of the future!

In another famous Gedankenexperimente - known as the Einstein-Podolsky-Rosen (EPR) paradox - Einstein and his collaborators [46] tried to demonstrate that the quantum-theoretical description is an incomplete description of the individual system [133, p. 672]. This has been refuted by Bohr [18]. (See also Ref. 133 and Ref. 28, Chapter VII).

According to Einstein QM is not a complete theory but should be supplemented by additional "hidden" variables. If such theories (see Ref. 7

for a review) respect locality (see in this context also Ref. 54) they are incompatible with the well-tested [114] statistical predictions of QM, because they satisfy Bell's inequality [8] whereas QM does not. However, nonlocal hidden variable theories compatible with QM predictions can be constructed [13-15]. Our proposed NLQM as well as other similar models [61] based on de Broglie's idea of a Double Solution can be regarded as such nonlocal hidden variable theories [31]. The scientific discussion whether these are conceptual viable alternatives to QM is very much in a state of flux. Selleri and Tarozzi [136] argue that de Broglie's [28] and Bohm's [13] hidden variable theories satisfies Bell's inequality, hence are inadequate. However Destouches [35] by taking up de Broglie's theory of the double solution and employing a functional relation (4.8) between the u and ψ waves claims to have obtained a theory which provides a solution to the EPR paradox at the same time avoiding Bell's veto. Other speculations [115] draw further attention to the importance of nonlocal interactions to solve this paradox.

In any case for Wigner [162] "... the equations of motion of quantum mechanics cease to be linear, in fact ... they are grossly non-linear if conscious beings enter the picture".

X. PROSPECTS: QUANTUM FIELD THEORY WITHOUT QUANTUM FIELD THEORY?

In this final Section further remarks on the speculative possibility of absorbing the conventional quantum field theory (QFT) by our nonlinear, relativistic quantum mechanics will be made.

In the usual formulation of QFT, the $\psi(x)$ -function are converted into operator-valued distribution (see, e.g. Ref. 79) by the method of field quantization. In the fermion case the latter method can be only made consistent with Fermi-Dirac statistics if the equal time anti-commutation relations

$$\{ \psi_\alpha(\vec{x}, t), \psi_\beta^*(\vec{x}', t) \} = \delta^{(\alpha\beta)}(\vec{x} - \vec{x}') \delta_d \beta \quad (10.1)$$

$$\{ \psi(x), \psi(x') \} = \{ \psi^*(x), \psi^*(x') \} = 0 \quad (10.2)$$

are postulated.

The general solution of the free Dirac equation (2.1) is then expanded in terms of plane waves by means of a Fourier-integral (-series). The resulting Fourier coefficients $b^+(p,s)$ and $b(p,s)$ as well as those corresponding to antiparticles become creation and annihilation operators acting on the $40X$ -space of n particle states after imposing the field ("second") quantization (10.1, 10.2). More details can be found in the text books [79, 12].

For a comparison with NLQM it is important to note that the normal-ordered charge operator takes the form

$$Q = e \sum_s \int d^3p [N^+(p,s) - N^-(p,s)] \quad (10.3)$$

[12, equ. (13.61)] where $N^+(p,s) = b^+(p,s)b(p,s)$ denotes the number operator for a particle with momentum p and spin s whereas $N^-(p,s)$ is that of the corresponding antiparticles. Through the introduction of Wick's normal ordering the total charge of the "Dirac sea" of negative energy states has been eliminated. This amounts to a subtraction of an infinite constant!

As stressed already on several occasions, the $u(y)$ -solitons should be regarded as classical waves determining the internal structure of particles. Still some quantum meaning can be associated to the time-dependent localized solutions. To achieve this one may impose the Bohr-Sommerfeld quantization conditions on the solutions. For u field theory with infinitely many degrees of freedom, this semiclassical quantization condition takes the form [76]

$$\frac{\pi/2\hbar}{c} \int dt \int d^3y \sum_{i=1}^f \pi_{(i)} \partial_0 u^{(i)} = N \pi \hbar \quad (10.4)$$

$$N = 0, \pm 1, \pm 2, \dots$$

the time integration being performed over the semi-period $\pi/\hbar$. The canonical conjugate field momenta can be calculated from the Lagrange density (2.9) as

$$\pi_{(1)} \equiv \frac{1}{c} \frac{\delta \mathcal{L}_{0-W}}{\delta (\partial_0 u_{(1)})} = i \hbar \sqrt{g} u^{(1)} g^{0\nu} L_\nu \quad (10.5)$$

This expression

is formally valid even in a curved space-time. Let us apply this to stationary soliton solutions of the type (2.10) in a rest frame. Then the differentiation with respect to time can be easily performed in (10.4). Note also that the semiperiod of (2.10) is determined by the parameter

$$\hat{\Omega} = \omega \mu c^2 / \hbar \quad (10.6)$$

A calculation shows that the left-hand side of (10.4) becomes proportional to the charge integral (3.2) in the absence of the gauge potential A_μ , a result which suggests that it is also valid in more generic situations.

Consequently, the Bohr-Sommerfeld condition (10.4) for a solution (2.10) is equivalent to the charge quantization condition

$$Q(u) = Ne, \quad N = 0, \pm 1, \pm 2, \quad (10.7)$$

with e denoting the physical (!) quantum of charge. This condition has first been introduced by Finkelstein et al [53] into nonlinear spinor theories and later used by Mielke in various nonlinear models [95, 96, 34, 100]. Obviously this result extends to the case of well-separated multi-soliton as well as - antisoliton solutions (7.2). For such a many-particle problem a more suggestive way of expressing the total charge is

$$Q(u) = e(N^+ - N^-), \quad N^\pm = 0, +1, +2, \quad (10.8)$$

Here $N^\pm$ would then denote the number of indistinguishable solitons or anti-solitons, respectively, in the u -configuration. (In a sense, the momentum space integration over localized quantum states in (10.3) is already incorporated in the notion of a solitonic solution.)

Since no superposition principle for soliton or antisoliton solutions with overlapping field domains is available on the u -level (see the arguments at the end of Section II) there is no need for the ad-hoc construction of a "Dirac sea" of negative energy states. Moreover, the stability arguments of Finkelstein et al [53] and Rybakov [128] for the solitons or antisolitons with quantized charge may put such artificial concepts completely out of business.

With no subtraction scheme necessary in order to derive (10.8) this opens the road for interpreting (3.2) directly as the physical charge of particles and antiparticles. A scattering theory of such "quantized" N -soliton system may be devised in this semi-classical framework [40]. Our approach seems to be remotely related to the view that soliton solutions of classical Euler equations of motions are closely related to the "extended" objects created in quantum-many body system [92].

In the relativistic quantum theory of interacting fields such as QED there always arises the need of renormalization [12, Chapter 19]. Using perturbation methods, divergencies in the Feynman integrals corresponding to the vacuum polarization, the self energy and vertex correction by a redefinition of the charge and the mass of say an electron. Although this procedure seems to work order by order in the electromagnetic coupling constant e_0 it is not known whether the renormalized perturbation series of the S -matrix converges [12, p. 374]. Furthermore, it should be noted that an infinite mass renormalization - as in classical electrodynamics - is a consequence of the point-particle concept: only the degree of the divergence is weakened in QED from a linear to logarithmic dependence on the cut-off radius.

Let us see how the situation changes in NIQM. For a soliton-type solution (2.10) of a nonlinear theory the postulated charge quantization (10.7) - in contrast to a linear theory - does not simply mean a normalization of the solution. In reality, the charge integral

$$\frac{Q(u)}{e} = \frac{32\pi}{3} \frac{\mathcal{L}_\mu^2}{\mathcal{L}_\pi^2} \int_0^\infty R_S^*(s) R_S(s) \sqrt{g(s)} ds \quad (10.9)$$

= N

obtained from a substitution of (2.10) into (3.2), merely determines the initial constant C_∞ of the asymptotic solution (3.1) of the dimensionless radial spinor $R_S(s)$ [100]. Equivalently, the charge quantization normalizes an arbitrary length scale $\mathcal{L} \neq \mathcal{L}_\mu$ of the solitons with respect to the coupling constant ("fundamental length") $\mathcal{L}^*$ of the nonlinear self-interactions [95]. In effect, this induces a change of the shape of $R_S(s)$ near the origin as if the "particle" would be the cause of a kind of polarization of the "vacuum". Therefore, nonlinear terms in NIQM may mimic quantum-field-theoretical

effects of the conventional theory, or rather *vice versa*. Similarly, the field energy P_0 - see equation (3.13) - of a u-soliton is not merely constituted by the "bare" mass μ of our model, but through (3.9) receives self-energy contributions from the nonlinear self-interaction. Via the Einstein relation (3.16) these contributions yield a different invariant mass $m \neq \mu$ of the soliton being then used in equation (2.1) which in turn governs the probabilistic level of NLQM. In various nonlinear models [95, 34, 96, 100] this "mass renormalization" of solitons with quantized charge has been explicitly observed. Therefore, it may be speculated that a NLQM of extended particles with appropriate nonlinearities always leads to a "renormalization without renormalization" (to use a Wheeler-type phrase). No need for a regularization arises because typically only finite adjustments have to be made in the field theoretical constructions as well as for the physically measurable parameters.

How do these considerations affect the probabilistic aspects of NLQM? For a well-separated "free" many soliton system (i.e. "asymptotic" soliton states) compelling arguments have been advanced in Sec. VII that the probability amplitude function ψ is given by the completely antisymmetric product (7.7) in the fermion case. One would expect that these ψ obey an algebra which is similar to the anti-commutation relations (10.1, 10.2), at least in the coordinate space representation. From the correspondence (5.4) connecting NLQM with conventional QM it may be suspected that the generalized function $\delta^{(3)}(\vec{x} - \vec{x}')$ on the right-hand side of (10.1) has to be replaced in NLQM by a nonlocal distribution resulting from a possible overlap of the high field regions of the corresponding u-solitons.

Therefore we can fully support the view - earlier expressed by Yukawa [168] - that the well-known divergency difficulties in QFT can only be solved by taking the finite size of elementary particles via a non-local theory into account.

In conclusion NLQM tries to give a first, preliminary answer to the pertinent, at the same time also visionary criticism of the late Albert Einstein on conventional QFT: "At the present time the opinion prevails that a field theory must first, by "quantization", be transformed into a statistical theory of field-probabilities, according to more or less established rules. I see in

this method only an attempt to describe relationships of an essentially nonlinear character by linear methods" (Ref. 45, p.165).

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fig 1

Many-solution solutions of the u-wave and their domains of high field values.

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