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GEOMETRIC ANALYSIS OF INTEGRABLE NON-LINEAR SIGMA MODELS

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ABSTRACT

The classical, two dimensional non-linear σ models on symmetric spaces are reviewed. As main results, the Lax representation, non-local charges, reduced systems and local conservation laws are discussed.

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INTRODUCTION

In recent years, efforts to solve models of quantum field theory exactly have had some beautiful successes. The most systematic, and most ambitious, way to deal with exactly integrable quantum field theories is the quantum inverse scattering method. In a rather direct way, this formalism uses the fact that the corresponding classical models are completely integrable. This property in turn is expressed most systematically in the classical inverse scattering method.

The interplay of classical and quantum integrability motivated most of the investigations of integrable classical systems carried out in recent years. By now, there exists a large variety of two dimensional models sharing properties such as the existence of solitons, of higher conservation laws, or, presumably most important, of a linear Lax representation. In these lectures I shall confine my attention to a particular class of models: the generalized non-linear σ models (NLSM). This class, containing several models of special interest like the S^N , CP^N and principal (chiral) models and showing all types of inner symmetry groups, admits a unified, "streamlined" treatment of integrability questions. Moreover, it is related to various other classes of models like the generalized sine-Gordon systems, the generalized Toda lattices and the Gross-Neveu models. The NLSM have been investigated using notions and methods of differential geometry¹⁾⁻⁴⁾. In fact, the structure of the NLSM mainly depends on the geometrical properties of groups entering, for example, as isometry or stabilizer groups of certain manifolds. Thus the NLSM are particularly suitable for looking systematically at the interplay of classical dynamics, geometric structures and integrability properties.

These notes review this development, starting in Chapter 1 with a brief summary of well-known properties of the simplest example, the S^2 σ model. In Chapter 2, a sufficiently general formulation of the NLSM on a Riemannian homogeneous space is given. In Chapter 3, I explain the central statement that all NLSM on symmetric spaces possess a linear Lax representation (yielding higher non-local conservation laws). After some mathematical considerations concerning symmetric spaces, the general definition of the reduced system (the "generalized sine-Gordon equation") corresponding to the NLSM on a symmetric space is formulated in Chapter 5, while Chapter 6 finally deals with the subtle question of the existence of higher local conservation laws.

I have tried to restrict myself to the main arguments, leaving out most of the mathematical and technical details; the general theory is illustrated throughout by rederiving the respective statements for the simplest non-trivial example, the S^2 model. For both mathematical rigor and more complicated (sometimes also more significant) examples, the reader is referred to the original articles¹⁾⁻⁴⁾ and the reviews^{5),6)}.

1. INTEGRABILITY PROPERTIES OF THE CLASSICAL S^2 NON-LINEAR σ MODEL

Our prototype and main example⁷⁾ is defined in terms of a real field vector $\vec{q}(x^0, x^1)$ taking values on the two dimensional sphere S^2 :

$$\vec{q}^2(x^0, x^1) = \sum_{i=1}^3 |q^i(x^0, x^1)|^2 = 1 \quad (1.1)$$

Incorporating this constraint by introducing a Lagrange multiplier, the dynamics is given by the action

$$S = \frac{1}{2} \int d^2x \left\{ \partial_\mu \vec{q} \partial^\mu \vec{q} + \lambda (\vec{q}^2 - 1) \right\} \quad (1.2)$$

and the field equations

$$\square \vec{q} + (\partial_\mu \vec{q} \partial^\mu \vec{q}) \vec{q} = 0, \quad \vec{q}^2 = 1. \quad (1.3)$$

Action and field equations show that this classical field theory is invariant under Lorentz and conformal transformations and under global $O(3)$ rotations of $\vec{q}$. The rotation invariance implies the conservation of the Noether current ($\vec{q}$ denoting a column vector)

$$\partial_\mu j^\mu = 0, \quad j_\mu = \vec{q} (\partial_\mu \vec{q})^T - (\partial_\mu \vec{q}) \vec{q}^T. \quad (1.4)$$

The following system of differential equations⁷⁾ plays a central role in the classical S^2 model:

$$\begin{aligned} \partial_\xi U^{(\gamma)} &= (1 - \gamma^{-1}) U^{(\gamma)} j_\xi \\ \partial_\eta U^{(\gamma)} &= (1 - \gamma) U^{(\gamma)} j_\eta \end{aligned} \quad \gamma \in \mathbb{R}. \quad (1.5)$$

Here, ξ and η denote light cone co-ordinates:

$$\begin{aligned} \xi &= \frac{1}{2} (x^0 + x^1), & \eta &= \frac{1}{2} (x^0 - x^1) \\ j_\xi &= j_0 + j_1, & j_\eta &= j_0 - j_1. \end{aligned}$$

(1.5) is a system of differential equations for an $O(3)$ valued function $U^{(\gamma)}(\xi, \eta)$. Given the components j_ξ and j_η of the conserved Noether current [taking values in the Lie algebra $\mathfrak{o}(3)$ of $O(3)$], (1.5) determines the $U^{(\gamma)}$ up to normalization provided both equations are compatible. But the compatibility condition, obtained by cross differentiation, is exactly the equation of motion, hence we may consider (1.5) as a Lax representation of the $S^2 \sigma$ model.

Unfortunately, it has not yet been possible to solve (1.3) via (1.5) in the sense of the inverse scattering method. For reasonable boundary conditions, i.e., $j_\mu \rightarrow 0$ as $|x^1| \rightarrow \infty$, the $U^{(\gamma)}$ in (1.5) are asymptotically constant and not plane waves. So it is unclear in which way an inverse scattering formalism as a generalized Fourier transformation⁸⁾ can be constructed around (1.5).

Nevertheless, (1.5) can be used to derive an infinite set of higher continuity equations⁹⁾ starting with

$$\partial_\eta j_\xi + \partial_\xi j_\eta = 0 \quad (1.6.1)$$

$$\begin{aligned} & \partial_\eta \left\{ \frac{1}{2} \left[\int_{-\infty}^{\xi} d\xi' j_\xi(\xi', \eta), j_\xi(\xi, \eta) \right] - j_\xi(\xi, \eta) \right\} \\ & + \partial_\xi \left\{ \frac{1}{2} \left[\int_{-\infty}^{\eta} d\eta' j_\eta(\xi, \eta'), j_\eta(\xi, \eta) \right] \right\} = 0. \end{aligned} \quad (1.6.2)$$

(1.6.1) is the (local) isospin conservation law (1.4), while (1.6.2) and all the following ones are non-local conservation laws involving more and more integrations.

The $S^2 \sigma$ model is closely related to the sine-Gordon theory⁷⁾. Defining

$$\cos \alpha = \frac{(\partial_\xi \vec{q} \cdot \partial_\eta \vec{q})}{\|\partial_\xi \vec{q}\| \|\partial_\eta \vec{q}\|}, \quad (1.7)$$

(1.3) implies the equations

$$\square \alpha + \|\partial_\xi \vec{q}\| \|\partial_\eta \vec{q}\| \sin \alpha = 0 \quad (1.8)$$

$$\partial_\eta \|\partial_\xi \vec{q}\| = 0$$

$$\partial_\xi \|\partial_\eta \vec{q}\| = 0.$$

For reasons to become clear later (cf. Chapter 5), (1.8) has been called the "reduced S^2 system". Equations (1.8) are the compatibility conditions of the Lax pair

$$\partial_{\xi} g^{(\gamma)} = g^{(\gamma)} \begin{pmatrix} 0 & -\gamma^{-1} \|\partial_{\xi} \vec{q}\| & 0 \\ \gamma^{-1} \|\partial_{\xi} \vec{q}\| & 0 & \partial_{\xi} \alpha \\ 0 & -\partial_{\xi} \alpha & 0 \end{pmatrix} \quad (1.9)$$

$$\partial_{\eta} g^{(\gamma)} = \gamma g^{(\gamma)} \begin{pmatrix} 0 & -\|\partial_{\eta} \vec{q}\| \cos \alpha & -\|\partial_{\eta} \vec{q}\| \sin \alpha \\ \|\partial_{\eta} \vec{q}\| \cos \alpha & 0 & 0 \\ \|\partial_{\eta} \vec{q}\| \sin \alpha & 0 & 0 \end{pmatrix}$$

Again, $g^{(\gamma)}$ is a one parameter family of $O(3)$ valued functions, which, by the principle of "gauge equivalence"¹⁰⁾, are closely related to the $U^{(\gamma)}$ of (1.5).

The invariance of (1.3) with respect to local scale transformations

$$(d\xi, d\eta) \longrightarrow (d\xi', d\eta') = (h(\xi) d\xi, k(\eta) d\eta), \quad h, k > 0,$$

allows us, choosing $h(\xi) = \|\partial_{\xi} \vec{q}\|$, $k(\eta) = \|\partial_{\eta} \vec{q}\|$, to obtain the sine-Gordon theory itself: in the new, normalized co-ordinates, we have

$$\|\partial_{\xi'} \vec{q}\| = \|\partial_{\eta'} \vec{q}\| = 1, \quad (1.10)$$

and (1.8) and (1.9) reduce to the sine-Gordon equation and to its well-known Lax pair, respectively [written in the fundamental representation of $O(3)$ rather than $SU(2)$]. The sine-Gordon theory is known to be a Hamiltonian system, completely integrable by the inverse scattering method¹¹⁾. It has an infinite series of higher local conservation laws starting with energy momentum conservation:

$$\partial_{\eta} \frac{1}{2} (\partial_{\xi} \alpha)^2 = \partial_{\xi} \cos \alpha \quad (1.11.1)$$

$$\partial_{\eta} \left\{ \frac{1}{2} (\partial_{\xi}^2 \alpha)^2 - \frac{1}{8} (\partial_{\xi} \alpha)^4 \right\} = \partial_{\xi} \left\{ -\frac{1}{2} (\partial_{\xi} \alpha)^2 \cos \alpha \right\} \quad (1.11.2)$$

etc.

The continuity equations (1.11), together with those obtained by interchanging ξ and η , provide a complete set of action variables for the sine-Gordon system. Using (1.7), they can be re-expressed in terms of the $\vec{q}$ field and its derivatives⁷⁾:

$$\partial_\eta \left\{ \frac{1}{2} (\partial_\xi^2 \vec{q})^2 \right\} = \partial_\xi (\partial_\xi \vec{q} \cdot \partial_\eta \vec{q}) \quad (1.12.1)$$

$$\partial_\eta \left\{ \frac{1}{2} (\partial_\xi^3 \vec{q})^2 - \frac{5}{8} (\partial_\xi^2 \vec{q})^4 \right\} = \partial_\xi \left\{ -\frac{1}{2} (\partial_\xi^2 \vec{q})^2 (\partial_\xi \vec{q} \cdot \partial_\eta \vec{q}) \right\} \quad (1.12.2)$$

etc.

(Of course, the restriction to the normalized co-ordinates (1.10) can be eliminated by replacing

$$\partial_\xi \rightarrow \|\partial_\xi \vec{q}\|^{-1} \partial_\xi, \quad \partial_\eta \rightarrow \|\partial_\eta \vec{q}\|^{-1} \partial_\eta \quad .)$$

Thus, the Lax pair (1.9) produces at the same time an infinite set of local conservation laws for the σ model itself. However, it is not known whether one can construct a complete set of action variables for the σ model out of both series of non-local (1.6) and local (1.12) conservation laws.

It has turned out¹⁾⁻³⁾ that the properties

(1.5) - Lax pair for the σ model

(1.6) - higher non-local conservation laws for the σ model

(1.8) - reduced system

(1.9) - Lax pair for the reduced system

(1.11) - higher local conservation laws for both the reduced system and the σ model

(1.12)

are not restricted to the S^2 model. They can be extended to a considerably large class of appropriately generalized σ models, and this will be explained in Chapters 2 - 6.

2. CONSTRUCTION OF NON-LINEAR σ MODELS ON RIEMANNIAN HOMOGENEOUS SPACES

Since the construction of generalized σ models has been treated extensively in the original papers¹⁾⁻⁴⁾ as well as in reviews^{5),6)}, I shall be rather brief. Let us start with a Lie group G with Lie algebra $\mathcal{G}$ and a closed subgroup H of G with Lie algebra $\mathcal{H}$. Denoting the complement of $\mathcal{H}$ in $\mathcal{G}$ by $\mathcal{m}$, we have

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{m} \quad (2.1)$$

$$[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}, \quad [\mathcal{H}, \mathcal{m}] \subset \mathcal{m}.$$

We require the existence of an inner product $(\cdot, \cdot)$ on $\mathfrak{g}$ which is invariant under the adjoint representation of G and positive definite on $\mathfrak{m}$ and which extends to the tangent bundle of the coset space G/H so that the latter becomes a Riemannian homogeneous space.

The non-linear σ model on G/H is defined in terms of a G valued field $g(x^0, x^1)$ with the action

$$S = \frac{1}{2} \int d^2x (D_\mu g, D^\mu g) \quad (2.2)$$

where the covariant derivative $D_\mu g$ of g with respect to the "gauge group" H is given by decomposing (cf. 2.1) the tangent vector $g^{-1} \partial_\mu g$ into its vertical part $A_\mu \in \mathfrak{h}$ (the gauge field) and its horizontal part $g^{-1} D_\mu g \in \mathfrak{m}$ (the left translated covariant derivative):

$$\underbrace{g^{-1} \partial_\mu g}_{\mathfrak{g}} = \underbrace{A_\mu}_{\mathfrak{h}} + \underbrace{g^{-1} D_\mu g}_{\mathfrak{m}} \quad (2.3)$$

The action (2.2) is invariant under Lorentz and conformal transformations, under local H valued gauge transformations

$$g(x) \rightarrow g(x) h(x) \quad , \quad h(x) \in H \quad ,$$

and under global G valued transformations

$$g(x) \rightarrow g_0 g(x) \quad , \quad g_0 \in G \text{ constant.}$$

The continuity equation corresponding to the global G invariance is

$$\partial_\mu j^\mu = 0 \quad , \quad j_\mu = -D_\mu g g^{-1} . \quad (2.4)$$

At the same time, (2.4) is the field equation corresponding to (2.2); usually it is written in the form

$$D_\mu D^\mu g - D_\mu g g^{-1} D^\mu g = 0 . \quad (2.5)$$

The simplest non-trivial example for this general prescription is given by

$$G = O(3), \quad H = O(2), \quad G/H = S^2 . \quad (2.6)$$

$$\begin{aligned} \text{Then } \mathcal{G} &= \{ \text{real antisymmetric } 3 \times 3 \text{ matrix} \} \\ \mathcal{H} &= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \mid x \text{ real antisymmetric } 2 \times 2 \text{ matrix} \right\} \\ \mathcal{M} &= \left\{ \begin{pmatrix} 0 & -z^T \\ z & 0 \end{pmatrix} \mid z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{R}^2 \right\} . \end{aligned}$$

Writing $g = (\vec{q}, Y) = (\vec{q}, \vec{Y}^{(1)}, \vec{Y}^{(2)})$ for the field, we require $g \in O(3)$, i.e.,

$$\vec{q}^T \vec{q} = 1, \quad \vec{q}^2 = 1, \quad Y^T Y = 1_2, \quad \vec{q}^T Y = 0. \quad (2.8)$$

The gauge group H acts on g as

$$(\vec{q}, Y) \rightarrow (\vec{q}, Yh) \quad \text{with } h(x^0, x^1) \in O(2). \quad (2.9)$$

According to (2.3),

$$A_\mu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & Y^T \partial_\mu Y \end{pmatrix}, \quad D_\mu (\vec{q}, Y) = (\partial_\mu \vec{q}, \partial_\mu Y - Y Y^T \partial_\mu Y) = (\partial_\mu \vec{q}, D_\mu Y). \quad (2.10)$$

Inserting (2.10) into (2.2), (2.5) and (2.4) [the inner product is the Cartan-Killing form of $O(3)$], we find the action

$$S = \int d^3x (\partial_\mu \vec{q} \partial^\mu \vec{q}) = \int d^3x \text{tr} (D_\mu Y (D^\mu Y)^T), \quad (1.2)$$

the field equations

$$\square \vec{q} + (\partial_\mu \vec{q} \partial^\mu \vec{q}) \vec{q} = 0 \quad \text{and} \quad D_\mu D^\mu Y + Y (D_\mu Y)^T D^\mu Y = 0, \quad (1.3)$$

and the current

$$j_\mu = \vec{q} (\partial_\mu \vec{q})^T - (\partial_\mu \vec{q}) \vec{q}^T = Y (D_\mu Y)^T - (D_\mu Y) Y^T. \quad (1.4)$$

Thus we have rederived the S^2 non-linear σ model; the generalization to $S^{N-1} = O(N)/O(N-1)$ is obvious. More complicated examples, like the complex Grassmannian $SU(p+q)/S[U(p) \times U(q)]$, can be found in the literature cited above.

As the reader will have noticed, the main ingredient in the definition of the NLSM is not the global symmetry group G , but the homogeneous space G/H . Therefore, we prefer the terminology " S^{N-1}_σ model" rather than " $O(N)_\sigma$ model". For $N > 3$, there are various different NLSM with global $O(N)$ symmetry corresponding to the various closed subgroups of $O(N)$.

As a side remark, observe that we obtain two different formulations of the S^{N-1} model: the first (the usual one) involves the field vector $\vec{q}$ and is gauge invariant, whereas the second one involves the Y field and has a (for $N > 3$) non-Abelian $O(N-1)$ gauge symmetry. Classically, this simply amounts to different parametrizations of S^{N-1} . For the corresponding quantum field theories, however, the consequences are not clear: for example, the $1/N$ expansion, working smoothly in terms of $\vec{q}$, in terms of Y immediately leads to the problem of summing planar Feynman graphs.

3. LAX REPRESENTATION, NON-LOCAL CONSERVATION LAWS AND SYMMETRIC SPACES

At present, there exists no systematic way of deciding whether a given two dimensional model has a Lax representation or not. We can, however, take Eq. (1.5) as a guide and try to generalize it. So let us look at the ansatz

$$\begin{aligned} \partial_{\xi} U^{(\gamma)} &= (1-\gamma^{-1}) U^{(\gamma)} j_{\xi} \\ \partial_{\eta} U^{(\gamma)} &= (1-\gamma) U^{(\gamma)} j_{\eta} \end{aligned} \quad (3.1)$$

where $j_{\mu} = -D_{\mu} g g^{-1} \in \mathfrak{g}$, $U^{(\gamma)} \in G$, $\gamma \in \mathbb{R}$. (3.1) is a system of linear differential equations on the group G . Given the components j_{ξ} and j_{η} of the conserved G current, (3.1) determines a one parameter family of G valued transformations $U^{(\gamma)}$ (up to normalization) provided both equations are compatible. Using current conservation, i.e., the field equation, the compatibility condition for (3.1)

$$\partial_{\xi} (1-\gamma) j_{\eta} - \partial_{\eta} (1-\gamma^{-1}) j_{\xi} + [(1-\gamma^{-1}) j_{\xi}, (1-\gamma) j_{\eta}] = 0 \quad (3.2)$$

takes the form

$$\partial_{\xi} j_{\eta} - \partial_{\eta} j_{\xi} + 2 [j_{\xi}, j_{\eta}] = 0. \quad (3.3)$$

Inserting $j_{\mu} = -D_{\mu} g g^{-1}$ and observing that (2.3) implies

$$F_{\xi\eta} = \partial_{\xi} A_{\eta} - \partial_{\eta} A_{\xi} + [A_{\xi}, A_{\eta}] = [g^{-1} D_{\eta} g, g^{-1} D_{\xi} g]_{\mathcal{H}}, \quad (3.4)$$

(3.3) can be written as

$$[g^{-1} D_{\xi} g, g^{-1} D_{\eta} g]_m = 0. \quad (3.5)$$

(Subscripts $\mathcal{H}$, m , etc., always denote projection to the respective subspaces.) Because of $g^{-1} D_{\mu} g \in m$, this gives the additional condition

$$[m, m] \subset \mathcal{H}. \quad (3.6)$$

(3.6) is the (infinitesimal) defining property of a certain subclass of homogeneous spaces, the so-called symmetric spaces¹²⁾.

Before explaining this in more detail, let me continue the discussion of (3.1). Whenever we start from a homogeneous space G/H satisfying (3.6), the linear equations (3.1) have the field equation of the NLSM on G/H as their compatibility condition. Hence (3.1) provides a Lax representation for the σ model the status of which is the same as for the S^2 case: for boundary conditions like $j_\mu \rightarrow 0$ as $|x^1| \rightarrow \infty$, we do not know how to solve (2.5) via (3.1) in the sense of the inverse scattering method. Nevertheless, (3.1) turns out to be the basis of all integrability properties of the NLSM; therefore, I shall concentrate in the following on the class of NLSM on symmetric spaces.

As for the S^2 case, we can derive an infinite series of non-local conservation laws from (3.1), proceeding as follows. Write

$$U^{(\gamma)} = \exp v^{(\gamma)} \quad (3.7)$$

where $v^{(\gamma)} \in \mathcal{G}$. The Lie algebra $\mathcal{G}$ being a linear space, it makes sense to perform a power series expansion of $v^{(\gamma)}$ around $\gamma = 1$:

$$v^{(\gamma)} = \sum_{k=1}^{\infty} \varepsilon^k v^{(k)} \quad (3.8)$$

where $\varepsilon = 1 - \gamma$ and the normalization $U^{(1)} = \mathbf{1}$ has been chosen. Insert (3.7) and (3.8) into (3.1):

$$\begin{aligned} (\varepsilon-1) \exp \left\{ - \sum_{k=1}^{\infty} \varepsilon^k v^{(k)} \right\} \partial_\xi \exp \left\{ \sum_{k=1}^{\infty} \varepsilon^k v^{(k)} \right\} &= \varepsilon j_\xi \\ \exp \left\{ - \sum_{k=1}^{\infty} \varepsilon^k v^{(k)} \right\} \partial_\eta \exp \left\{ \sum_{k=1}^{\infty} \varepsilon^k v^{(k)} \right\} &= \varepsilon j_\eta. \end{aligned} \quad (3.9)$$

Using the formula for the derivative of the exponential map¹²⁾

$$\exp(-X) \partial_\mu (\exp X) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} [X, \dots, \underbrace{[X, \partial_\mu X]}_{n \text{ brackets}}] \quad (3.10)$$

and collecting the terms with the same power of ε in (3.9), one can recursively determine the $\partial_\xi v^{(k)}$, $\partial_\eta v^{(k)}$, $v^{(k)}$ (non-local!) to obtain the $\mathcal{G}$ valued currents entering the conservation laws

$$\partial_\eta (\partial_\xi v^{(k)}) - \partial_\xi (\partial_\eta v^{(k)}) = 0 \quad k = 1 \dots \infty . \quad (3.11)$$

Explicitly:

$$k=1 \quad \partial_\eta j_\xi + \partial_\xi j_\eta = 0 \quad (3.12.1)$$

$$k=2 \quad \partial_\eta \left\{ \frac{1}{2} \left[\int_{-\infty}^{\xi} d\xi' j_\xi(\xi', \eta), j_\xi(\xi, \eta) \right] - j_\xi(\xi, \eta) \right\} \\ + \partial_\xi \left\{ \frac{1}{2} \left[\int_{-\infty}^{\eta} d\eta' j_\eta(\xi, \eta'), j_\eta(\xi, \eta) \right] \right\} = 0 \quad (3.12.2)$$

etc.

Let us now look closer at condition (3.6). Not all homogeneous spaces fulfill it: counterexamples are the real and complex Stiefel manifolds

$$\left. \begin{array}{l} SO(N)/SO(M) \\ SU(N)/SU(M) \end{array} \right\} 2 \leq M \leq N-2$$

or flag manifolds like

$$U(3)/U(1) \times U(1) \times U(1) .$$

However, decomposing the respective Lie algebras, as done in (2.7), the reader can easily convince himself that the following homogeneous spaces match with (3.6):
The real and complex Grassmann manifolds

$$SO(p+q)/SO(p) \times SO(q)$$

$$SU(p+q)/S(U(p) \times U(q))$$

containing the spheres S^{N-1} discussed in Chapter 2 as special cases, or the spaces

$$SU(N)/SO(N)$$

$$SO(2N)/U(N)$$

etc.

Actually, E. Cartan was able to give a complete list of the symmetric spaces. This list can be found in Ref. 12) p. 516 ff; it contains

- several series of compact spaces based on classical groups
- several compact spaces based on exceptional groups
- non-compact ("dual") versions of both the classical and exceptional spaces.

Cartan's classification exhibits yet another important type of symmetric space, namely, the Lie groups themselves. In fact, observe that any Lie group G with Lie algebra $\mathfrak{g}$ can be represented as $G \simeq G \times G / \Delta G$ where $\Delta G = \{(g, g) | g \in G\}$ is the diagonal of $G \times G$, the diffeomorphism being induced by the map

$$\Gamma: G \times G \rightarrow G, \quad \Gamma(g_1, g_2) = g_1 g_2^{-1}$$

with

$$\Gamma((g_1, g_2) \cdot (g, g)) = \Gamma(g_1 g, g_2 g) = \Gamma(g_1, g_2).$$

Then

$$\mathfrak{g} \oplus \mathfrak{g} = \mathcal{H} \oplus \mathfrak{m}$$

where

$$\mathcal{H} = \{(a, a) | a \in \mathfrak{g}\}$$

$$\mathfrak{m} = \{(a, -a) | a \in \mathfrak{g}\}$$

and obviously

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathcal{H}.$$

The corresponding NLSM can be formulated either in terms of a $G \times G$ valued field with gauge group ΔG or in terms of a G valued field without gauge group^{1), 2)}. In both cases, the model has a global $G \times G$ symmetry motivating the name "chiral field model". In the Russian literature¹³⁾, the chiral models are called "principal models". The reason for this is the following: one can show²⁾ that the solution space of the principal model on a Lie group G contains the solution spaces of the NLSM on all spaces G/H , H being any subgroup of G so that G/H is symmetric, as subspaces. Thus a complete description of the principal field dynamics implies a complete description of the dynamics of all G/H models where G/H is symmetric.

4. SOME MATHEMATICAL TOOLS

In the first lecture, dealing with the general construction of σ models, the determination of the Lax representation and the computation of the non-local conservation laws, the main role was played by the $\mathfrak{g}$ valued currents

$$j_\mu = -D_\mu g g^{-1}.$$

In the second lecture, dealing with the generalization of the reduction procedure (1.8) and (1.9) and the computation of local conservation laws for NLSM on symmetric spaces, this role will be taken over by the quantities

$$A_\mu \in \mathcal{H} \quad \text{and} \quad k_\mu = g^{-1} D_\mu g \in \mathfrak{m}$$

defined by (2.3) because these directly reveal the decomposition of $\mathfrak{g}$ into its vertical and horizontal part.

For symmetric spaces, the field equation (2.5) takes the form

$$D_\eta D_\xi g - D_\eta g g^{-1} D_\xi g = 0 \quad , \quad \text{in terms of } k_\mu: \quad D_\eta k_\xi = 0 \quad (4.1)$$

or, equivalently,

$$D_\xi D_\eta g - D_\xi g g^{-1} D_\eta g = 0 \quad , \quad \text{in terms of } k_\mu: \quad D_\xi k_\eta = 0 \quad (4.2)$$

Here,

$$D_\mu k_\nu = \partial_\mu k_\nu - [k_\nu, A_\mu] \in \mathfrak{m} \quad .$$

Furthermore, (3.4) and (3.6) give the equation

$$F_{\xi\eta} = \partial_\xi A_\eta - \partial_\eta A_\xi + [A_\xi, A_\eta] = [k_\eta, k_\xi] \quad (4.3)$$

The fields A_μ and k_μ will enter Chapters 5 and 6 mainly through Eqs. (4.1) - (4.3).

In the following, we shall need a refinement of the decomposition $\mathfrak{g} = \mathcal{H} \oplus \mathfrak{m}$ ¹²⁾.

- let $\mathcal{A}$ denote any maximal Abelian subspace of $\mathfrak{m}$,
- define $\mathcal{l}$ to be that subalgebra of $\mathcal{H}$ the elements of which commute with all elements of $\mathcal{A}$ (the centralizer of $\mathcal{A}$ in $\mathcal{H}$),
- define $\mathfrak{n}$ to be the orthogonal complement of $\mathcal{A}$ in $\mathfrak{m}$,
- define $\mathfrak{p}$ to be the orthogonal complement of $\mathcal{l}$ in $\mathcal{H}$

Then we have the orthogonal decompositions

$$\mathcal{H} = \mathcal{l} \oplus \mathfrak{p} \quad , \quad \mathfrak{m} = \mathcal{A} \oplus \mathfrak{n} \quad , \quad \mathfrak{g} = \mathcal{l} \oplus \mathfrak{p} \oplus \mathcal{A} \oplus \mathfrak{n} \quad (4.4)$$

Frequently, we shall use the corresponding decompositions of A_μ and k_μ :

$$A_\mu = (A_\mu)_\ell + (A_\mu)_p, \quad K_\mu = (K_\mu)_\mathcal{A} + (K_\mu)_n. \quad (4.5)$$

The various subspaces obey the commutation relations

$$[\mathcal{A}, \mathcal{A}] = \{0\} \quad [\mathcal{A}, \ell] = \{0\} \quad [\ell, \ell] \subset \ell \quad [p, p] \subset \mathcal{H} \quad (4.6)$$

$$[\mathcal{A}, p] \subset n \quad [\ell, p] \subset p \quad [p, n] \subset m$$

$$[\mathcal{A}, n] \subset p \quad [\ell, n] \subset n \quad [n, n] \subset \mathcal{H}.$$

The dimension of $\mathcal{A}$ is called the rank of the symmetric space G/H . Remember that Lie groups themselves are symmetric spaces: in that case, the above notions of maximal Abelian subspace and rank of the symmetric space coincide with the usual notions of Cartan subalgebra and rank of the group.

For the example $G/H = S^2$, we have

$$\mathcal{A} = \left\{ \begin{pmatrix} 0 & -x & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x \in \mathbb{R} \right\}, \quad n = \left\{ \begin{pmatrix} 0 & 0 & -x \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix} \mid x \in \mathbb{R} \right\}, \quad (4.7)$$

$$\ell = \{0\}, \quad p = \mathcal{H},$$

$$K_\mu = \begin{pmatrix} 0 & \vec{q}^T D_\mu Y \\ Y^T D_\mu \vec{q} & 0 & 0 \\ & 0 & 0 \end{pmatrix}, \quad \text{rank}(S^2) = 1.$$

For more complicated examples consult Ref. 3).

As a second tool we shall use the polar decomposition theorem^{12),3)} for symmetric spaces. In its infinitesimal version, it states that each element $Z \in \mathfrak{m}$ can be written in the form

$$Z = h x h^{-1} \quad \text{where } h \in H, \quad x \in \mathcal{A}. \quad (4.8)$$

Essentially [i.e., up to discrete Weyl group transformations, cf., Ref. 3)], x is unique and h is determined up to transformations taking values in the centralizer L of $\mathcal{A}$ in H :

$$L = \{h \in H \mid h x h^{-1} = x \text{ for all } x \in \mathcal{A}\} \quad (4.9)$$

Observe that the Lie algebra of L coincides with $\mathfrak{l}$ defined above.

For the S^2 example, let

$$Z = \begin{pmatrix} 0 & -z_1 - z_2 \\ z_1 & 0 & 0 \\ z_2 & 0 & 0 \end{pmatrix} \in \mathfrak{m}, \quad h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix} \in H, \quad X = \begin{pmatrix} 0 & -r & 0 \\ r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{A}, \quad r > 0. \quad (4.10)$$

Then (4.8) reduces to the usual polar co-ordinates of $\mathbb{R}^2$:

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} r \cos \alpha \\ r \sin \alpha \end{pmatrix} \quad (4.11)$$

(The global version of the polar decomposition theorem yields the usual polar co-ordinates on $\exp \mathfrak{m} \simeq S^2$.)

5. REDUCTION GAUGE AND REDUCED SYSTEMS

To construct the analogue of (1.8) and (1.9), the reduced system, for the general situation, we shall use the fact that for the NLSM on a symmetric space there exists a distinguished gauge which essentially completely removes the freedom to perform H valued gauge transformations. This gauge is constructed in two steps.

First, applying the polar decomposition (4.8) to $Z = k_\xi \in \mathfrak{m}$ and taking its gauge transformation behaviour

$$k_\xi = g^{-1} D_\xi g \longrightarrow h^{-1} k_\xi h$$

into account, we see that we can always choose a gauge so that

$$k_\xi \in \mathcal{A}. \quad (5.1)$$

This is a local gauge still leaving us with the freedom to perform residual gauge transformations with values in L (4.9). Inserting the decomposition $A_\eta = (A_\eta)_\mathfrak{l} + (A_\eta)_\mathfrak{p}$ into Eq. (4.1) and using $[\mathcal{A}, \mathfrak{l}] = \{0\}$ and $[\mathcal{A}, \mathfrak{p}] \subset \mathfrak{n}$, we find the equations

$$\begin{aligned} 0 &= \partial_\eta k_\xi \in \mathcal{A} \\ 0 &= [k_\xi, (A_\eta)_\mathfrak{p}] \in \mathfrak{n}. \end{aligned} \quad (5.2)$$

Assuming that k_ξ fulfills certain regularity conditions [cf., Ref. 3)], then we can invert the commutator in the second equation to get

$$(A_\eta)_\rho = 0. \quad (5.3)$$

Second, we may use the residual gauge transformations to gauge away the ℓ part of A_ξ and A_η . In fact, the differential equations

$$\begin{aligned} \partial_\xi h^R + (A_\xi)_\ell h^R &= 0 \\ \partial_\eta h^R + (A_\eta)_\ell h^R &= 0 \end{aligned} \quad (5.4)$$

have the equation

$$\partial_\xi (A_\eta)_\ell - \partial_\eta (A_\xi)_\ell + [(A_\xi)_\ell, (A_\eta)_\ell] = 0 \quad (5.5)$$

as their compatibility condition. However, (5.5) is precisely the ℓ part of Eq. (4.3):

$$\begin{aligned} 0 &= [k_\eta, k_\xi]_\ell \\ &= (\partial_\xi A_\eta)_\ell - (\partial_\eta A_\xi)_\ell + [(A_\xi)_\ell + (A_\xi)_\rho, (A_\eta)_\ell]_\ell \\ &= \partial_\xi (A_\eta)_\ell - \partial_\eta (A_\xi)_\ell + [(A_\xi)_\ell, (A_\eta)_\ell] \end{aligned}$$

due to $[A, n]_{cp}$, (5.3), $[\ell, p]_{cp}$, $[\ell, \ell]_{cl}$. Hence, (5.4) defines an L valued gauge transformation h^R which exactly cancels $(A_\xi)_\ell$ and $(A_\eta)_\ell$. We call this gauge, in which

$$K_\xi^R \in \mathcal{A}, \quad (A_\xi^R)_\ell = (A_\eta^R)_\ell = (A_\eta^R)_\rho = 0 \quad (5.6)$$

the reduction gauge. All quantities in the reduction gauge will be distinguished by an upper index R . For L non-trivial, it is a non-local gauge due to (5.4) leaving us finally with the freedom to perform constant, L valued gauge transformations.

In the reduction gauge, Eqs. (4.1) - (4.3) take the following form for the remaining non-zero degrees of freedom $k_\xi^R \in \mathcal{A}, k_\eta^R \in \mathfrak{m}, A_\xi^R \in \mathfrak{p}$:

$$\partial_\eta k_\xi^R = 0 \quad (4.1^R)$$

$$\partial_\xi k_\eta^R = [k_\eta^R, A_\xi^R] \quad (4.2^R)$$

$$\partial_\eta A_\xi^R = [k_\xi^R, (k_\eta^R)_n] \quad (4.3^R)$$

For k_ξ^R regular (cf. above), we can solve (4.3^R) for $(k_\eta^R)_n$:

$$(k_\eta^R)_n = \text{ad}(k_\xi^R)^{-1} \partial_\eta A_\xi^R. \quad (5.7)$$

Here we use the notation $\text{ad}(x)Y = [x, Y]$ for $x \in \mathcal{A}$, $Y \in \mathfrak{n}(p)$, the inverse operation $\text{ad}(x)^{-1}$ for x regular being a linear transformation $p \rightarrow \mathfrak{n}(\mathfrak{n} \rightarrow p)$. Introducing a field $\Phi \in \mathfrak{n}$ by

$$\Phi = \text{ad}(k_\xi^R)^{-1} A_\xi^R,$$

we have

$$k_\eta^R = \partial_\eta \Phi \quad (5.8)$$

by (4.1^R). Decomposing (4.2) into its $\mathcal{A}$ and its $\mathfrak{n}$ part and inserting (5.8), we finally arrive at the reduced system defined by the following set of non-linear partial differential equations

$$\partial_\eta k_\xi^R = 0 \quad (5.9.1)$$

$$\partial_\xi (k_\eta^R)_\mathcal{A} = \frac{1}{2} \partial_\eta [\Phi, [k_\xi^R, \Phi]]_\mathcal{A} \quad (5.9.2)$$

$$\square \Phi = [\partial_\eta \Phi, [k_\xi^R, \Phi]]_\mathfrak{n} + [(k_\eta^R)_\mathcal{A}, [k_\xi^R, \Phi]] \quad (5.9.3)$$

for the variables

$$k_\xi^R \in \mathcal{A}, \quad (k_\eta^R)_\mathcal{A} \in \mathcal{A}, \quad \Phi \in \mathfrak{n}. \quad (5.10)$$

Equations (5.9) are the compatibility conditions for the Lax pair

$$\begin{aligned} \partial_\xi g^{(\delta)} &= g^{(\delta)} ([k_\xi^R, \Phi] + \delta^{-1} k_\xi^R) \\ \partial_\eta g^{(\delta)} &= \delta g^{(\delta)} ((k_\eta^R)_\mathcal{A} + \partial_\eta \Phi) \end{aligned} \quad (5.11)$$

which is closely connected to the Lax pair (3.1) for the σ model itself: in fact, define

$$g^{(\delta)} = u^{(\delta)} g. \quad (5.12)$$

(3.1) then implies

$$\begin{aligned} \partial_{\xi} g^{(\delta)} &= g^{(\delta)} (A_{\xi} + \delta^{-1} k_{\xi}) \\ \partial_{\eta} g^{(\delta)} &= g^{(\delta)} (A_{\eta} + \delta k_{\eta}) , \end{aligned} \quad (5.13)$$

and using the reduction gauge conditions (5.6) and the definition of Φ , we arrive at (5.11).

Actually, this shows that in general the reduced system (5.9) and the NLSM are related by the principle of "gauge equivalence of two dimensional integrable models"^{10),13)}. First, by the (G valued) "gauge transformation" (5.12) we go over from the Lax pair (3.1) to the new Lax pair (5.13) - the compatibility is preserved because it is a flatness condition, cf., (3.2). Second, the high degree of gauge ambiguity in (5.12) is eliminated by choosing the reduction gauge, thus arriving at (5.11); this step motivates the name "reduced systems". Finally, the compatibility equations of (5.11) define the "dynamics" of the new model.

The experience with the sine-Gordon case leads us to hope that for suitable boundary conditions such as

$$\left. \begin{aligned} \Phi &\rightarrow 0 , & k_{\xi}^R &\rightarrow \text{const.}_1 \in \mathcal{A} \\ \partial_{\eta} \Phi &\rightarrow 0 , & (k_{\eta}^R)_{\mathcal{A}} &\rightarrow \text{const.}_2 \in \mathcal{A} \end{aligned} \right\} \text{ as } |x^1| \rightarrow \infty , \text{ cf. Ref. 17),}$$

the reduced systems can be solved by the inverse scattering method based on (5.11). As a by-product of this, one could also determine the original NLSM field $g = g^{(1)}$ from the "wave function" $g^{(\gamma)}$ of the "scattering problem" (5.11). However, the above boundary conditions do not match with $j_{\mu} \rightarrow 0$, so this is not a way of overcoming the difficulties with the NLSM itself.

To consider (5.9) and (5.11) as the reduced system generalizing (1.8) and (1.9) is justified by the fact that in the well-known special cases S^{N-1} ¹⁴⁾ and $\mathbb{CP}^{N-1}$ ¹⁵⁾ it reproduces the former results. Actually, up to slight modifications, the procedure described above is the direct generalization of the approach used in Ref. 15).

Let me now show that for S^2 , we rederive our starting point (1.8) and (1.9). According to (4.7) we write

$$k_{\xi}^R = \begin{pmatrix} 0 & -k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad (k_{\eta}^R)_{\mathcal{A}} = \begin{pmatrix} 0 & -\lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \Phi = \begin{pmatrix} 0 & 0 & -\varphi \\ 0 & 0 & 0 \\ \varphi & 0 & 0 \end{pmatrix} . \quad (5.14)$$

Inserting (5.14) into (5.9), we get

$$\begin{aligned}\partial_\eta k &= 0 \\ \partial_\xi \lambda &= \frac{1}{2} \partial_\eta (k \varphi^2) \\ \square \varphi + k \lambda \varphi &= 0.\end{aligned}\tag{5.15}$$

Choosing now $\lambda = \delta \cos \alpha$, $\partial_\eta \phi = \delta \sin \alpha$ ($\delta > 0$), (5.15) takes the form

$$\begin{aligned}\partial_\eta k &= 0 \\ \partial_\xi \delta &= 0 \\ \square \alpha + k \delta \sin \alpha &= 0,\end{aligned}\tag{1.8}$$

and (5.11) yields the Lax pair (1.9). In normalized co-ordinates (1.10) we even have $k = \delta = 1$, and the reduced system is the sine-Gordon equation itself. In addition, we note that the gauge condition (5.1)

$$k_\xi = g^{-1} D_\xi g = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

implies

$$D_\xi g = (\vec{q}, \vec{y}^{(1)}, \vec{y}^{(2)}) \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (\vec{y}^{(1)}, -\vec{q}, 0),$$

i.e.,

$$g = (\vec{q}, \partial_\xi \vec{q}, \vec{y}^{(2)}) \quad \text{and} \quad \vec{y}^{(2)} = \pm \frac{1}{\sin \alpha} (\partial_\eta \vec{q} - \cos \alpha \partial_\xi \vec{q}).\tag{5.16}$$

This is the basis of $\mathbb{R}^3$ used originally in Ref. 7) to reduce the $S^2 \sigma$ model to the sine-Gordon theory. More complicated examples are discussed in Ref. 16).

Unfortunately, up to now we do not know how to derive (5.9) from a Lagrangian density, except for the special cases S^2 , S^3 and $\mathbb{CP}^2$ which are discussed in Refs. 7) and 15). These examples indicate that the variables (5.10) might not be the appropriate ones to find a Lagrangian density for (5.9) and to bring the reduced systems into the status of a well-behaved Lagrangian field theory. However, the following general characteristic properties should hold in any, perhaps more suitable parametrization:

- i) The number of independent variables is $\dim(G/H) + \text{rank}(G/H)$. Two more degrees of freedom could be eliminated by the co-ordinate normalization (1.10), choosing $h(\xi) = (D_{\xi}g, D_{\xi}g)^{1/2}$, $k(\eta) = (D_{\eta}g, D_{\eta}g)^{1/2}$, but it is not clear whether this is helpful if $\text{rank}(G/H) > 1$.
- ii) The definition of A_{μ} and k_{μ} shows that the variables (5.10) are globally G invariant.
- iii) The local gauge freedom has been eliminated, leaving us with a global L symmetry:

$$\Phi \rightarrow h^{-1} \Phi h \quad h \in L \text{ constant.}$$

- iv) The equations (5.9) are Lorentz invariant, the couple $(k_{\xi}^R, (k_{\eta}^R)_A)$ transforming like a vector and Φ like a scalar.

6. LOCAL CONSERVATION LAWS FOR NLSM AND REDUCED SYSTEMS

Our last topic is to find the analogues of the local conservation laws (1.11) and (1.12) in the general situation. Guided by our experience from various special models, we start from the fact that the field $g^{(\gamma)}$ (5.12) can locally be decomposed uniquely as follows:

$$g^{(\gamma)} = \exp a^{(\gamma)} \exp l^{(\gamma)} \exp \omega^{(\gamma)} \quad (6.1)$$

where $a^{(\gamma)} \in \mathcal{A}$, $l^{(\gamma)} \in \mathfrak{l}$, $\omega^{(\gamma)} \in \mathfrak{n} \oplus \mathfrak{p}$. We claim that the function $a^{(\gamma)}$ is the generating functional for higher, gauge invariant local conservation laws. First let me sketch how to compute $a^{(\gamma)}$, then the questions of gauge invariance and locality will be discussed.

In the gauge (5.1) which will be used throughout this Chapter, it turns out that the functions $a^{(\gamma)}$ and $\omega^{(\gamma)}$ have power series expansions of the form

$$a^{(\gamma)} = \sum_{k=-1}^{\infty} \gamma^k a^{(k)}, \quad \omega^{(\gamma)} = \sum_{k=1}^{\infty} \gamma^k \omega^{(k)} \quad (6.2)$$

with

$$\begin{aligned} a^{(k)} &= 0 \text{ for } k \text{ even}; & \omega^{(k)} &\in \mathfrak{n} \text{ for } k \text{ odd}, \\ & & \omega^{(k)} &\in \mathfrak{p} \text{ for } k \text{ even}. \end{aligned} \quad (6.3)$$

Actually, (6.3) follows from a certain symmetry of $g^{(\gamma)}$ with respect to $\gamma \rightarrow -\gamma$ (2),3). To compute the $a^{(k)}$ and $\omega^{(k)}$, we insert the decomposition (6.1) into the differential equations (5.13) and solve for the $\partial_\mu a^{(\gamma)}$. For the ξ equation, for example, this gives

$$\underbrace{\partial_\xi a^{(x)}}_{\in \mathcal{A}} = \underbrace{e^{\omega^{(x)}} (A_\xi + \delta^{-1} k_\xi) e^{-\omega^{(x)}}}_{\in \mathcal{A} \oplus n \oplus p \oplus \ell} - \underbrace{(\partial_\xi e^{\omega^{(x)}}) e^{-\omega^{(x)}}}_{\in \ell} + \underbrace{(\partial_\xi e^{-\ell^{(x)}}) e^{\ell^{(x)}}}_{\in \ell} \quad (6.4)$$

$\uparrow$ determines $a^{(x)}$ $\downarrow$ determines $\omega^{(x)}$ $\uparrow$ decouples completely

Inserting (6.2) into (6.4) and the analogous η equation, and using once more (3.10), we can recursively compute the $\omega^{(k)}$ and $a^{(k)}$ as indicated in the scheme (6.4):

The $p \oplus n$ part determines the $\omega^{(k)}$ in terms of the $\omega^{(j)}$ ($1 \leq j \leq k-1$), the $\mathcal{A}$ part determines the $a^{(k)}$ (or rather their derivatives) in terms of the $\omega^{(j)}$ ($1 \leq j \leq k$), whereas, due to the commutation relations (4.6), the ℓ part decouples completely and need not be considered. In this way, we find the $\mathcal{A}$ valued currents entering the conservation laws

$$\partial_\gamma (\partial_\xi a^{(2k+1)}) = \partial_\xi (\partial_\gamma a^{(2k+1)}) \quad , \quad k = -1, \dots, +\infty \quad (6.5)$$

Let us look at the first three orders of the recursion in more detail:

$$\delta^{-1}: \quad \partial_\xi a^{(-1)} = k_\xi \quad (6.6)$$

$$\delta^0: \quad 0 = (A_\xi)_p + [\omega^{(1)}, k_\xi] \quad (\text{p-part})$$

$$\delta^1: \quad \begin{cases} \partial_\xi a^{(1)} = \{ [\omega^{(1)}, (A_\xi)_p] + \frac{1}{2} [\omega^{(1)}, [\omega^{(1)}, k_\xi]] \} \mathcal{A} & (\mathcal{A}\text{-part}) \\ 0 = \{ -D_\xi \omega^{(1)} + [\omega^{(1)}, (A_\xi)_p] + [\omega^{(2)}, k_\xi] + \frac{1}{2} [\omega^{(1)}, [\omega^{(1)}, k_\xi]] \} \mathfrak{n} & (\text{n-part}) \end{cases}$$

Here we have used the L covariant derivative

$$D_{\xi} \omega^{(1)} = \partial_{\xi} \omega^{(1)} - [\omega^{(1)}, (A_{\xi})_{\mathcal{L}}] \in \mathfrak{n}.$$

From (6.6) γ^{-1} , we directly get

$$\partial_{\xi} \alpha^{(-1)} = k_{\xi} \quad (6.7)$$

The $\mathfrak{p}$ part of (6.6) γ^0 gives

$$\omega^{(1)} = \text{ad}(k_{\xi})^{-1} (A_{\xi})_{\mathfrak{p}} \quad (6.8)$$

for k_{ξ} regular, cf., Chapter 5.

The $\mathcal{A}$ part of (6.6) γ^1 gives, using (6.8)

$$\partial_{\xi} \alpha^{(1)} = \frac{1}{2} [\text{ad}(k_{\xi})^{-1} (A_{\xi})_{\mathfrak{p}}, (A_{\xi})_{\mathfrak{p}}]_{\mathcal{A}} \quad (6.9)$$

The $\mathfrak{n}$ part of (6.6) γ^1 serves to determine $\omega^{(2)}$:

$$\omega^{(2)} = \text{ad}(k_{\xi})^{-1} \left\{ -D_{\xi} \omega^{(1)} + [\omega^{(1)}, (A_{\xi})_{\mathfrak{p}}] + \frac{1}{2} [\omega^{(1)}, [\omega^{(1)}, k_{\xi}]] \right\}_{\mathfrak{n}}.$$

Doing the same for the η equation, we find the first two conservation laws

$$\partial_{\eta} k_{\xi} = 0 \quad (6.10)$$

$$\partial_{\eta} \frac{1}{2} [\text{ad}(k_{\xi})^{-1} (A_{\xi})_{\mathfrak{p}}, (A_{\xi})_{\mathfrak{p}}]_{\mathcal{A}} = \partial_{\xi} (k_{\eta})_{\mathcal{A}} \quad (6.11)$$

The scheme indicated above holds to all orders. We determine

- the $\omega^{(2k)}$ from the $\mathfrak{n}$ part of the $\gamma^{2k-1} \xi$ equation
- the $\omega^{(2k+1)}$ from the $\mathfrak{p}$ part of the $\gamma^{2k} \xi$ equation
- the $\partial_{\xi} \alpha^{(2k+1)}$ from the $\mathcal{A}$ part of the $\gamma^{2k+1} \xi$ equation
- the $\partial_{\eta} \alpha^{(2k+1)}$ from the $\mathcal{A}$ part of the $\gamma^{2k+1} \eta$ equation.

The fact that the generating functional takes values in $\mathcal{A}$ implies that for a symmetric space of rank p , there exist p independent series of scalar local conservation laws. For the rank one spaces S^{N-1} and $\mathbb{CP}^{N-1}$, there is one series.

For the rank p spaces $SU(p+q)/S[U(p) \times U(q)]$, $p \leq q$ or for the principal model on a group of rank p , there are p independent series. Expressed in the variables (5.10), they are at the same time local conservation laws for the reduced models.

By the way, (6.10) and (6.11) coincide with the differential equations (5.9.1) and (5.9.2) for the reduced model, and in particular the derivation of (5.9.2) gives an impression of the sophisticated interplay of σ model dynamics (4.1) and (4.2) and symmetric space structure (4.3), (4.6) allowing for the conservation law (6.11).

I shall now demonstrate gauge invariance and locality for the example of the conservation law (6.11), referring the reader to Ref. 3) for the general case. Since we have used the gauge (5.1) to derive (6.11), we have to look at the behaviour of (6.11) under the residual, L valued gauge transformations preserving (5.1). Now $g \rightarrow gh$ for $h \in L$ implies

$$(k_\mu)_A \rightarrow (h^{-1} k_\mu h)_A = (k_\mu)_A \quad (\text{observe } [L, A] = \{0\}) \quad (6.12)$$

$$(k_\mu)_n \rightarrow (h^{-1} k_\mu h)_n = h^{-1} (k_\mu)_n h \quad (\text{observe } [L, n] \subset n) \quad (6.13)$$

and

$$A_\mu = (A_\mu)_L + (A_\mu)_P \rightarrow (h^{-1} (A_\mu)_L h + h^{-1} \partial_\mu h) + h^{-1} (A_\mu)_P h$$

i.e.,

$$(\text{observe } [L, P] \subset P),$$

$$(A_\mu)_L \rightarrow h^{-1} (A_\mu)_L h + h^{-1} \partial_\mu h \quad (6.14)$$

$$(A_\mu)_P \rightarrow h^{-1} (A_\mu)_P h \quad (6.15)$$

Using (6.12) and (6.15) and again $[L, A] = \{0\}$, we find that (6.11) is in fact invariant under residual gauge transformations. On the other hand, it is obviously local because we are working in the local gauge (5.1) and all manipulations leading from (6.4) to (6.11) are local ones.

Finally, let me show that for the S^2 example, (6.11) in fact reduces to the conservation law (1.12.1). From (5.16) and (2.10) we have

$$g = (\vec{q}, \partial_3 \vec{q}, \vec{y}^{(2)}) , \quad (A_g)_P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & Y^T \partial_3 Y \\ 0 & \vec{y}^{(2)T} \partial_3^2 \vec{q} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\vec{y}^{(2)T} \partial_3^2 \vec{q} \\ 0 & \vec{y}^{(2)T} \partial_3^2 \vec{q} & 0 \end{pmatrix}$$

and from (5.14), together with the co-ordinate normalization (1.10),

$$K_g = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

Then

$$ad(K_g)^{-1}(A_g)_P = \begin{pmatrix} 0 & 0 & -\vec{y}^{(2)T} \partial_3^2 \vec{q} \\ 0 & 0 & 0 \\ \vec{y}^{(2)T} \partial_3^2 \vec{q} & 0 & 0 \end{pmatrix}$$

and

$$\frac{1}{2} [ad(K_g)^{-1}(A_g)_P, (A_g)_P]_A = \frac{1}{2} \begin{pmatrix} 0 & -(\vec{y}^{(2)T} \partial_3^2 \vec{q})^2 & 0 \\ (\vec{y}^{(2)T} \partial_3^2 \vec{q})^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

For the right-hand side of (6.11) we get from (5.16) and (2.10)

$$(K_\eta)_A = (g^T D_\eta g)_A = \left(\begin{pmatrix} \vec{q}^T \\ \partial_3 \vec{q}^T \\ \vec{y}^{(2)T} \end{pmatrix} (2_\eta \vec{q}, D_\eta Y) \right)_A = \begin{pmatrix} 0 & -\partial_3 \vec{q}^T \cdot 2_\eta \vec{q} & 0 \\ \partial_3 \vec{q}^T \cdot 2_\eta \vec{q} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

Expanding $\partial_3^2 \vec{q}$ in the basis (5.16) of $\mathbb{R}^3$, we find

$$(\vec{y}^{(2)T} \partial_3^2 \vec{q})^2 = (\partial_3^2 \vec{q})^2 - 1 ,$$

and (6.11) gives the conservation law (1.12.1)

$$\partial_\eta \frac{1}{2} (\partial_\xi^2 \vec{q})^2 = \partial_\xi (\partial_\xi \vec{q} \cdot \partial_\eta \vec{q})$$

The corresponding, much more involved, calculation for the general complex Grassmannian is explained in detail in Ref. 3).

Already the first higher conservation law (6.11) displays the characteristic feature that the currents are rational functions rather than polynomial ones. Due to the $\text{ad}(k_\xi)^{-1}$, which for the n^{th} current will occur in the n^{th} power, there are denominators which can become zero. As the S^2 example shows, these denominators can be eliminated for the rank one spaces by the co-ordinate normalization (1.10). However, for spaces of higher rank, there will necessarily remain rational singularities in the currents.

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