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THE SUPERCURRENT IN FIVE DIMENSIONS

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ABSTRACT

The supercurrent multiplet is constructed for $N = 2$ and 4 extended supersymmetries in five dimensions. The corresponding supergravity multiplets are also exhibited and have $40 + 40$ and $128 + 128$ components respectively.

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The supercurrent - the supermultiplet which contains the x space spinor current and the energy-momentum tensor - has proven to be of use in the construction of off-shell representations of supersymmetry for supergravity theories. In particular the supercurrent derived from the $N = 2$ massive hypermultiplet has been used to find the minimal $32 + 32$ component multiplet of $N = 2$ Poincaré supergravity¹⁾ and the supercurrent for $N = 4$ Yang-Mills has been similarly employed in the context of $N = 4$ supergravity. In this note we construct the supercurrents for $N = 2$ and $N = 4$ supersymmetry in five dimensions using Yang-Mills as our basic building block. Our motivation for doing this is that by working in five dimensions we avoid conformal invariance and hence expect to find multiplets that are relevant for Poincaré, rather than conformal, supergravity. For $N = 2$ this is indeed the case; we find a multiplet with $40 + 40$ components whose corresponding supergravity multiplet reduces to the minimal $32 + 32$ Poincaré set together with an additional Yang-Mills. For $N = 4$, however, the supercurrent turns out to have the same number of components as the four dimensional superconformal one of Ref. 2) (B.d.d.) i.e., $128 + 128$; furthermore, the corresponding five dimensional supergravity multiplet remains irreducible under dimensional reduction, as indeed it should as $128 + 128$ is the minimal size for an off-shell multiplet in $N = 4$ supersymmetry (without central charges).

Nevertheless, we feel that our results are worth presenting for two reasons: firstly, the reduction of the $N = 4$ stress tensor multiplet may be performed in two ways yielding either the conformal multiplet or the multiplet relevant for massive theories with a central charge; secondly, the absence of low dimension fields enables us to follow the approach of Ref. 3) and construct an "almost simple" supergravity^{*)} in superspace which for $N = 4$ is much simpler than the full non-linear conformal superspace theory⁴⁾. For $N = 2$, it is then straightforward to compensate for the local $SU(2)$ invariance by introducing an $8 + 8$ component multiplet; on the other hand, it is not so obvious how to remove the local $Sp(4)$ invariance in the $N = 4$ case and we make some comments on this problem at the end of the paper. Throughout the paper we shall focus on the $N = 4$ case, the $N = 2$ results being obtainable by straightforward truncation.

We recall that $N = 4$ Yang-Mills in five dimensions is described on-shell by a scalar superfield W_{ij} which transforms as a $\underline{5}$ under $Sp(4)$ and satisfies^{**)}

*) By "almost simple" supergravity we mean a formulation with a minimal Poincaré multiplet.

**) $a, b, \dots$ are vector indices; $\alpha, \beta, \dots$ are spinor indices; $i, j, \dots$ $Sp(N)$ indices. The γ matrices are $\eta_{\alpha\beta}$, (charge conjugation matrix), $(\gamma^a)_{\alpha\beta}$ antisymmetric and $(\gamma^{ab})_{\alpha\beta} = 1/2(\gamma^a\gamma^b - \gamma^b\gamma^a)_{\alpha\beta}$ symmetric. Spinors are $Sp(N)$ Majorana: $\bar{\psi}_{\alpha i} = \eta_{\alpha\beta} \eta_{ij} \psi_{\beta j}$.

$$W_{ij} = -W_{ji} ; \quad \eta^{ij} W_{ij} = 0$$

$$\bar{W}^{ij} = \eta^{ik} \eta^{jl} W_{kl} \quad (1)$$

where η_{ij} is the antisymmetric $S_p(4)$ metric. One has

$$\begin{aligned} D_{\alpha i} W_{jk} &= \eta_{ij} \lambda_{\alpha k} - \eta_{ik} \lambda_{\alpha j} + \frac{1}{2} \eta_{jk} \lambda_{\alpha i} \\ D_{\alpha i} \lambda_{\beta j} &= -i (\gamma^a)_{\alpha\beta} \partial_a W_{ij} + \frac{i}{2} \eta_{ij} (\gamma^{ab})_{\alpha\beta} F_{ab} \quad (2) \\ D_{\alpha i} F_{ab} &= -(\gamma_a)_{\alpha}{}^{\beta} \partial_{b j} \lambda_{\beta i} \end{aligned}$$

where $\{D_{\alpha i}, D_{\beta j}\} = i \eta_{ij} (\gamma^a)_{\alpha\beta} \partial_a$.

Here, $\lambda_{\alpha i}$ represents the four spinors in the theory and F_{ab} is the spin one field strength and there are no further independent components of W_{ij} . Furthermore, the theory is on-shell:

$$\not{\partial} \lambda_i = \partial^a F_{ab} = \square W_{ij} = 0 \quad (3)$$

As is well known, these results may be obtained from the spin one superspace Bianchi identities⁵⁾ with

$$F_{\alpha i \beta j} = i \eta_{\alpha\beta} W_{ij} \quad (4)$$

There are evidently just two bi-linear superfields one can construct from W_{ij} (without using derivatives) and the supercurrent transforms as a 14 under $Sp(4)$. Explicitly we have

$$C_{ijkl} = W_{ij} W_{kl} - \frac{1}{20} (\eta_{ij} \eta_{kl} - 2\epsilon_{ijkl}) W^{mn} W_{mn} \quad (5)$$

It is now straightforward to obtain the components of C and their super-symmetry transformations by differentiating repeatedly with the spinor covariant derivative. The components are the following (in four dimensional dimensions):

Dim 2

$$C_{ijhl}$$

Dim 5/2

$$\chi_{\alpha i, jh} = W_{ij} \lambda_{\alpha h} - \frac{1}{5} (\eta_{ij} W_{hl} \lambda_{\alpha}^l + 2 \eta_{hi} W_{jl} \lambda_{\alpha}^l + 2 \eta_{jh} W_{il} \lambda_{\alpha}^l) \quad (6)$$

Dim 3

$$\begin{aligned} S_{ij} &= \frac{i}{4} \lambda_i^{\alpha} \lambda_{\alpha j} \\ V_{ij}^a &= \frac{i}{4} \lambda_i^{\alpha} (\gamma^a)_{\alpha}{}^{\beta} \lambda_{\beta j} - \frac{1}{4} W_{ih} \overleftrightarrow{\partial_a} W_{hj} \quad (7) \\ N_{ij}^{ab} &= \frac{i}{4} (\lambda_i^{\alpha} (\gamma^{ab})_{\alpha}{}^{\beta} \lambda_{\beta j} - \frac{1}{4} \eta_{ij} \lambda_h^{\alpha} (\gamma^{ab})_{\alpha}{}^{\beta} \lambda_{\beta}^h) \\ &\quad - W_{ij} F^{ab} \end{aligned}$$

Dim 7/2

$$\begin{aligned} \phi_{\alpha i}^a &= (\gamma^{bc} \gamma^a)_{\alpha}{}^{\beta} \lambda_{\beta i} F_{bc} + 2 \lambda_{\alpha l} \overleftrightarrow{\partial_a} W_{li} \\ &\quad - \frac{2}{5} (\gamma^{ab})_{\alpha}{}^{\beta} \partial_b (W_{ij} \lambda_{\beta}^j) \quad (8) \end{aligned}$$

Dim 4

$$\begin{aligned} T_{ab} &= F_{ac} F_b{}^c - \frac{1}{4} \eta_{ab} F_{cd} F^{cd} \\ &\quad - \frac{1}{4} (\partial_a W_{ij} \partial_b W^{ij} - \frac{1}{2} \eta_{ab} \partial_c W^{ij} \partial_c W_{ij}) \\ &\quad - \frac{i}{2} \lambda^{*i} (\gamma_a \partial_b)_{\alpha}{}^{\beta} \lambda_{\beta i} \quad (9) \end{aligned}$$

$$X_a = \epsilon_{abcde} F^{bc} F^{de} - \frac{i}{2} \partial^b (\lambda^i \gamma_{ab} \lambda_i)$$

S_{ij} and V_{ij}^a are symmetric, N_{ij}^{ab} is skew on a, b and is a $\bar{5}$ under $Sp(4)$ and $\chi_{i,jk}$ is a $\underline{16}$ under $Sp(4)$. V_{ij}^a are the currents corresponding to $Sp(4)$ invariance, $\psi_{\alpha i}^a$ is the spinor current and T_{ab} is the energy momentum tensor, all three being conserved. X_a is also conserved, although identically so.

However, if we were to employ the dual formulation of the free Maxwell theory with an antisymmetric tensor instead of a vector ⁶⁾, X_a would correspond to a central charge current.

Corresponding to the above current multiplet, there is a multiplet of supergravity fields with the following components:

Gauge bosons

$$h_{ab}, A_a \text{ (dim. 1) ; } V_a^{ij} \text{ (dim. 2)}$$

Other bosons

$$S_{ij}, N_{ij}^{ab} \text{ (dim. 2) ; } C_{ijkl} \text{ (dim. 3)} \quad (10)$$

Gauge fermions

$$\psi_{a\alpha i} \text{ (dim. } \frac{3}{2} \text{)}$$

Other fermions

$$\chi_{ijkl} \text{ (dim. } \frac{5}{2} \text{)}$$

where with the exceptions of the graviton (h_{ab}), the gravitino ($\psi_{a\alpha i}$) and A_a we have used the same letters for the fields as for the current to which they couple. We remark that for $N = 4$ (10) does not contain all the physical fields of five dimensional Poincaré supergravity⁷⁾ (there are five vectors, one scalar and four spinors missing) whilst for $N = 2$ it does^{*)}. Reducing (10) to four dimensions, we find the following multiplet:

Gauge bosons

$$h_{ab}, A_a, B_a \text{ (dim. 1) ; } V_a^{ij} \text{ (dim. 2)}$$

Other bosons

$$A, B \text{ (dim 1) ; } C_{ij}, \dots \text{ (dim. 3) ;} \quad (11)$$

$$S_{ij}, N_{ij}^{ab}, N_{ij}^a, V_{ij} \text{ (dim. 2)}$$

Gauge fermions

$$\psi_{a\alpha i} \text{ (dim. } \frac{3}{2} \text{)}$$

^{*)} The $N = 4 \rightarrow N = 2$ truncation is as follows: $\underline{5}$ and $\underline{14}$ in $Sp(4)$ become $SU(2)$ singlets; the $\underline{10}$ (e.g., S_{ij}) becomes a $\underline{3}$ and the $\underline{16}$ and $\underline{4}$ become $\underline{2}$'s.

Other fermions

$$\lambda_{\alpha i} \text{ (dim. } \frac{3}{2} \text{)} ; \chi_{\alpha i, jh} \text{ (dim. } \frac{5}{2} \text{)} \quad (11) \text{ cont.}$$

Here B_a and B are the additional components of h_{ab} in five dimensions; $A, N_{ij}^a, V_{ij}, \lambda_i$ are the extra components of $A_a, N_{ij}^{ab}, V_{ij}^a, \psi_{\alpha i}$ respectively. For $N = 2$ this multiplet is reducible and one may separate off a Yang-Mills multiplet $(A_a, A, B, \lambda_i, V_{ij})$ leaving the $32 + 32$ minimal Poincaré multiplet. In $N = 4$ however, it is clear that this cannot be done, although the multiplet we have is not identical in form to that of B.d.d. In particular, we have two dimension 0 gauge bosons replacing one of their 6 non-gauge dimension one antisymmetric tensors; $Sp(4)$ instead of $SU(4)$ gauge fields and $h_{ab}(\psi_i)$ undergoes no Weyl (S supersymmetry) transformation. The question as to whether our multiplet is different from B.d.d.'s or not may be answered by returning to the supercurrent. If we reduce the underlying Yang-Mills to four dimensions conventionally (i.e., $\partial/\partial x^5 = 0$), then from (7) one finds

$$N_{ij}^{a5} = \frac{i}{4} \left\{ \lambda_i^a (\delta^a)_\alpha{}^\beta \lambda_{\beta j} - \frac{1}{4} \eta_{ij} \lambda_h^a (\delta^a)_\alpha{}^\beta \lambda_{\beta h} \right\} \\ - \frac{1}{2} W_{ij} \overset{\longleftrightarrow}{\partial_a} A_5 - \frac{1}{2} \partial_a (W_{ij} A_5)$$

where A_5 is the fifth component of the original Yang-Mills vector potential. Hence N_{ij}^{a5} splits into a set of 5 conserved vectors together with the derivative of 5 new scalars. This allows us to re-identify the components of the stress tensor multiplet, and the whole multiplet then turns out to be the conformal B.d.d. current with manifest $Sp(4)$ instead of $SU(4)$ covariance. On the other hand, the same five dimensional supercurrent multiplet may be obtained from two Yang-Mills multiplets W_{ij}^x ($x = 1, 2$) with

$$C_{ij,kl} = W_{ij}^x W_{kl}^x - \text{traces} \quad (13)$$

We may now use this new $O(2)$ to reduce two Yang-Mills from five dimensions to four dimensions à la Scherk and Schwarz⁸⁾ thereby obtaining an on-shell massive spin one theory with a central charge equal to the mass. In this case it is not possible to split up N_{ij}^{a5} as above, and indeed one does not expect this to be the case. Hence we see that the five dimensional stress tensor multiplet may be reduced to either the conformal B.d.d. one or the $Sp(4)$ multiplet relevant for massive theories with a central charge, both versions having $128 + 128$ components.

Since the gravitational multiplet is obtained by coupling it to the current, it follows that in the conformal case we may re-identify its components corresponding to the re-identification of the current components [as in Eq.(12)] whereas in the central charge case we get the multiplet in the form of Eq. (11).

We now turn to the formulation of "almost simple" supergravity in super-space in five dimensions. Our basic fields are the vielbein E_M^A , the connection Ω_{MA}^B and an Abelian gauge field A_M from which one constructs the field strength tensors, the torsion T_{AB}^C , the curvature R_{ABC}^D and the spin one field strength F_{AB} *) in the standard way⁹⁾. Ω is taken to be $SO(1, 4) \times Sp(N)$ Lie algebra valued, i.e.,

$$\Omega_{M \alpha i}^{\beta j} = \delta_{\alpha}^A \Omega_{M, i}^j + \delta_i^j \Omega_{M, \alpha}^{\beta} \quad (14)$$

$$\Omega_{M, \alpha \beta} = -\frac{i}{4} (\gamma^{ab})_{\alpha \beta} \Omega_{M, ab}$$

From (10) we observe that there are no gauge covariant fields of dimension less than one**) and hence we propose the following equations to describe the full non-linear theory corresponding to (10):

dim 0

$$T_{\alpha i \beta j}^c = -i \eta_{ij} (\gamma^c)_{\alpha \beta}$$

$$F_{\alpha i \beta j} = -i \eta_{ij} \eta_{\alpha \beta}$$

dim 1/2

$$T_{\alpha i \beta j}^{\gamma k} = T_{\alpha i b}^c = F_{\alpha i b} = 0 \quad (15)$$

dim 1

$$T_{ab}^c = \eta_{ij} (T_{a, \alpha k})^{\alpha i} = 0$$

where the dimension one constraints correspond to the freedom to choose the vector part of the connection. To verify that (15) are indeed the correct equations, it is necessary to grind through the Bianchi identities; this has already been done

*) Capital indices run over vector, spinor and internal indices. M, N (A, B) are curved (tangent space) indices.

**) Here we employ geometrical dimensions, i.e., canonical bosons have dimension zero, etc.

in detail for $N = 2$ in four dimensions³⁾ and the five dimensional case is very similar so we shall not present the explicit results beyond dimension one. At that level we find

$$T_{a\alpha i\beta j} = -i\gamma_{ij}(\gamma^b)_{\alpha\beta}F_{ab} + \frac{i}{12}\gamma_{ij}\epsilon_{abcde}(\gamma^{bc})_{\alpha\beta}F^{de} \\ + i(\gamma_a)_{\alpha\beta}S_{ij} + \frac{i}{6}\epsilon_{abcde}(\gamma^{bc})_{\alpha\beta}N_{ij}^{de} \quad (16)$$

where F_{ab} is the spin one field strength, $S_{ij} = S_{ji}$ and $N_{ij} = -N_{ji}$ is a traceless $[SU(2) \text{ singlet}]$. Comparing with (10), one sees that we already have all the expected bosons with the exception of C_{ij} which has dimension two.

For $N = 2$, it is straightforward to compensate the local $SU(2)$ invariance of the above theory. One merely rewrites it in a form where only the $SO(1,4)$ invariance is manifest, so that the $SU(2)$ connection appears explicitly in the redefined torsions. One may now reduce the local $SU(2)$ parameter superfield by imposing the following constraint:

$$\mathcal{L}_{\alpha(ij)k} = 0 \quad (17)$$

The remaining $SU(2)$ invariance is then given by a linear superfield $L_{ij} = L_{ji}$,

$$D_{\alpha(i}L_{jk)} = 0 \quad (18)$$

which has $8 + 8$ components. Considering the theory obtained this way as globally $SU(2)$ invariant, one sees that the compensated multiplet has the original $40 + 40$ components together with the new $8 + 8$ (which contains a vector whose divergence is essentially C). The procedure has already been studied extensively in the literature in the four dimensional^{3),10)} case and we shall not repeat the details here, suffice it to say that the final $48 + 48$ component multiplet may be used to write a globally $SU(2)$ invariant Poincaré supergravity Lagrangian.

If we now try to repeat the above construction in $N = 4$ we encounter problems. It is interesting to observe that the $N = 2$ linear multiplet (18) may be considered as a current multiplet built from an $N = 2$ hypermultiplet described by a scalar superfield W_1 with

$$D_{\alpha(i}W_{j)} = 0 \quad (19)$$

One then has

$$L_{ij} = W_{ci} \bar{W}_j \quad (20)$$

However, in $N = 4$, a superfield of the form (19) describes a spin $3/2$ multiplet and hence if we were to try and compensate using the resulting L_{ij} we would obtain a theory with a conserved spin 3 field of high dimension, thus making the construction of a Lagrangian very difficult. On the other hand, we do not know an off-shell Yang-Mills theory without a central charge so that the possibility of using five Yang-Mills multiplets is not available to us at present. Furthermore, arguments have recently been given which suggest that an off-shell $N = 4$ Yang-Mills multiplet which could be used in a Lagrangian does not exist¹¹. Hence for $N = 4$, it seems that it is difficult to find an off-shell Poincaré supergravity which could be used for a Lagrangian formulation, although the possibility of having several multiplets that conspire in the right way is not ruled out.

It is therefore reasonable to ask whether one may use the existing central charge formulation of off-shell Yang-Mills⁶). At the linearized level five of these multiplets put together have the following field content:

dim 1

$$B, B_{ij} (10), B_{cijk} (14); A_{ab}^{ij} (5)$$

dim 3/2

(21)

$$\lambda_{\alpha i} (4), \xi_{\alpha i, jk} (16)$$

dim 2

$$H, H_{ij} (10), H_{cijk} (14)$$

The A_{ab}^{ij} 's are gauge antisymmetric tensors (spin 1 in five dimensions) and the H 's the central charge variation of the B 's. If we add (21) to (10) we see that the field content of the combined multiplet has the right structure for a Lagrangian representation. A_{ab}^{ij} , B and λ_i are the missing physical fields, B_{ij} compensates for the local $Sp(4)$, the H 's are standard auxiliary fields and B_{cijk} and $\xi_{\alpha i, jk}$ have the right dimensions to be Lagrange multipliers for $C_{ij, k\ell}$ and $\chi_{\alpha i, jk}$ respectively. It is therefore possible that this combined multiplet could yield a Lagrangian formulation of $N = 4$ Poincaré supergravity at least at the linearized level. However, it is not clear that one could obtain consistency in the non-linear version and furthermore the combined multiplet does not separate into supergravity plus Yang-Mills in four dimensions.

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