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ON THE ALGEBRAIC STRUCTURE OF GLOBALLY OR LOCALLY
SU(2) INVARIANT LATTICE FIELD THEORIES

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A B S T R A C T

The local contribution to the action of the $O(3)$ σ model in $D = 2$ or pure $SU(2)$ gauge models in $D \geq 3$ dimensions are expanded and integrated on the group. There results a field of variables j , a $3nj$ coefficient $W(\{j\})$ with $n \rightarrow \infty$ and dynamical factors $f(j, \beta)$. We prove that for the gauge models a local decomposition of $W(\{j\})$ into a product of $3nj$ coefficients with $n = 2D(D-2)$ exists. We study generating functions for $W(\{j\})$ or the $3nj$ coefficients and develop an algorithm for their computation. Some of these generating functions are explicitly calculated.

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1. - INTRODUCTION

We consider lattice field theories on hypercubic lattices with field variables ranging over the group manifold $SU(2)$ itself or one of its coset spaces. The action may be either globally or locally $SU(2)$ invariant. After local harmonic analysis, i.e., expanding each local contribution of the action into a series of $SU(2)$ characters or representation functions, the integration over the field variables can be performed. Instead there arise expressions that can be formulated as contractions of $3j$ coefficients.

The group invariants j can be regarded as a field supported by a certain geometrical structure which is constructed from the original (real space) lattice (from now on "RS lattice") or its dual. For a given field $\{j\}$ the contraction of the $3j$ coefficients over the whole RS lattice yields a functional $W(\{j\})$ of the j field. It possesses the structure of a $3nj$ coefficient with n very big for large lattices. It is known that such $3nj$ coefficients can be reduced into sums of products of $3n'j$ coefficients, $n' < n$ (e.g., $6j$ coefficients) by the introduction of and summation over further variables j (see Section 2). Therefore the domains of j fields are not uniquely defined.

On the other hand, the W functional defines only part of the whole theory. The other parts are dynamical factors $f(j)$ which are local functions of the j field. Correspondingly we split the domain of the j field into a kinematic part where $f(j) = 1$ and a dynamical part where $f(j) \neq 1$. Only the kinematic domain is ambiguous. For example we get the generalized Villain model if we set ¹⁾:

$$f(j) = \begin{cases} e^{-\frac{2j(j+1)}{\beta}} & \text{on the dynamic domain} \\ 1 & \text{on the kinematic}^{(1)} \text{ domain} \end{cases}$$

After an appropriate normalization the W functional and the dynamic factors times $2j+1$ define a real measure on the space C^N , where C is the space of group invariants

$$C = \left\{ 0, \frac{1}{2}, 1, \frac{3}{2}, \dots \right\} = \frac{1}{2} \mathbb{Z}_+ \quad (2)$$

and N is the cardinal of the domain of the j field. The dynamical factors render this measure finite but this is not their only purpose, of course.

The aim of this investigation is to develop methods that make dual versions of $SU(N)$ invariant lattice field theories analytically tractable. The value of the dual theories lies mainly in the invariance of its field variables. In the case of locally invariant theories, this amounts to an elimination of a large set of superfluous field degrees of freedom. With this approach the low coupling expansion of the original field theory must be obtainable in an invariant way, that means, without gauge fixing. The strong coupling expansion on the other hand necessitates the calculation of invariant coupling coefficients which in our language are W functionals for j fields of small support.

In this article we show how to use generating functions ^{2),3)} for the calculation of W functionals. So far they have been derived for $6j$ and $9j$ coefficients and used to study their symmetries. In our context it makes sense to use generating functions only if they can be calculated easily. For the strong coupling expansions the generating functions need to be known only locally. We present an algorithm which gives us either a local expansion or any $6nj$ coefficient of the $SU(2)$ gauge theory if n is not too big.

Technically the results are based on the Bargmann space realization of the representations of $SU(2)$ ³⁾. This formalism can be generalized to groups $SU(N)$, $N > 2$, though not much is explicitly known yet ⁴⁾.

Finally we return to the issue of low coupling expansions without gauge fixing whose feasibility depends on the solution of technical problems which will be reserved to another work. It is connected with the large j expansion of the W functional, involving the classical angular momentum limit as first approximation. A superficial glance at Regge's work ⁵⁾ may convince the reader how difficult this task might be. On the other hand this problem seems very interesting. Its solution presents a non-Abelian gauge field theory by an equivalent perturbative field theory of only physical observables. Moreover, it enables us to distinguish and study two different types of topological excitations on the lattice: the instantons and the Coulomb gas resulting from the discreteness of the field variables j ⁶⁾.

2. - GRAPHS

For the contraction of $3j$ coefficients we use graphical methods. We follow the notations of Ref. 7). For completeness we list the most important notations and rules.

A graph consists of oriented links, oriented legs, and oriented vertices :

- 1) each vertex connects three lines which are either links or legs ;
- 2) each link connects two vertices ;
- 3) to each link we ascribe a $j \in \frac{1}{2} \mathbb{Z}_+$;
- 4) to each leg we ascribe a $j \in \frac{1}{2} \mathbb{Z}_+$ and an m , $-j \leq m \leq +j$, $j-m \in \mathbb{Z}_+$;
- 5) to each vertex we ascribe a $3j$ coefficient, e.g. :

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & -m_2 & m_3 \end{pmatrix} \quad (3)$$

The direction of lines teaches us how to put signs on the m 's in the $3j$ coefficient, and the orientation $\pm$ of the vertex $[+ \text{ in } (3)]$ fixes the order of the columns in the $3j$ coefficient, namely as even respectively odd permutation of the counterclockwise reading in the graph. For this prescription to make sense, the graph must be drawn in planar form.

- 6) Graphs can be contracted as follows :

$$\sum_{m=-j}^{+j} (-1)^{j-m} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & -m_2 & m \end{pmatrix} \begin{pmatrix} j_3 & j_4 & j \\ m_3 & -m_4 & -m \end{pmatrix} \quad (4)$$

Changing the orientation of the link causes a sign factor $(-1)^{2j}$, whereas change of orientation of a $j_1 j_2 j_3$ vertex causes a factor $(-1)^{j_1+j_2+j_3}$.

Graphs composed in this fashion may have legs, in this case they describe " j_m coefficients" ⁷⁾. If they have no legs, they describe $3nj$ coefficients. The graph corresponding to the W functional of a j field over an RS lattice has a certain periodic structure resembling that of the RS lattice. We may therefore consider such graphs as lattices themselves, " J lattices". We may formulate one issue of this article by "computing generating functions for J lattices exploiting the lattice structure".

Notice that only the connectedness and the orientations of a graph, not length and angles, have an algebraic meaning. Lines are allowed to intersect

each other without forming vertices. This freedom of distorting graphs and J lattices makes it possible to ascribe two-dimensional J lattices to multi-dimensional RS lattices.

The basic contraction of $3j$ coefficients is presented in (4). It leads to general composition and decomposition rules of graphs for $3nj$ coefficients. In obvious graphical notation we have the following three rules :

$$\begin{array}{|c|} \hline \begin{array}{c} \xrightarrow{j_1} \\ \xleftarrow{j_2} \end{array} \\ \hline \end{array} \begin{array}{|c|} \hline \alpha \\ \hline \end{array} \begin{array}{|c|} \hline \beta \\ \hline \end{array} = \delta_{j_1 j_2} \frac{1}{2j_1+1} \cdot \begin{array}{|c|} \hline \alpha \\ \hline \end{array} \begin{array}{|c|} \hline \beta \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline \xrightarrow{j_1} \\ \xleftarrow{j_2} \\ \xleftarrow{j_3} \\ \hline \end{array} \begin{array}{|c|} \hline \alpha \\ \hline \end{array} \begin{array}{|c|} \hline \beta \\ \hline \end{array} = \begin{array}{|c|} \hline \alpha \\ \hline \end{array} \begin{array}{|c|} \hline \beta \\ \hline \end{array}$$

$$\sum_{a \in \frac{1}{2} \mathbb{Z}_+} (2a+1) \begin{array}{|c|} \hline \alpha \\ \hline \end{array} \begin{array}{|c|} \hline \beta \\ \hline \end{array} = \begin{array}{|c|} \hline \alpha \\ \hline \end{array} \begin{array}{|c|} \hline \beta \\ \hline \end{array}$$

(7)

For more detailed discussions, see Ref. 7).

Instead of using J lattices one can study their dual forms which, if carefully defined, lead to an equivalent formalism. In the transition to the dual lattice, vertices go into triangles and links into sides of triangles. Note that each side of a triangle plays the rôle of an "oriented borderline"

(8)

Such dual graphs or J lattices are difficult to draw. In simple cases they can be mapped on two-dimensional triangular manifolds giving up the orientation of triangles and sides. Ponzano and Regge have shown ⁵⁾ that one can geometrize these graphs and fix the length of a triangle side j to $[j(j+1)]^{\frac{1}{2}}$. For the topics of this article this is irrelevant.

3. - THE $O(3)$ σ MODEL IN $D = 2$ DIMENSIONS

We consider the following two-dimensional lattice model. Ascribe to each side x of a quadratic two-dimensional RS lattice Λ a vector

$$\vec{n} \in S^2$$

and introduce an action by

$$-S = \sum_{x,v} \vec{n}_x \cdot \vec{n}_{x+v} \quad (9)$$

where $v = 1, 2$ labels the two lattice unit vectors $\underline{v}$. We assume periodic boundary conditions. The partition function is defined to be

$$Z = \int \prod_{x \in \Lambda} d\omega_x e^{-\beta S} \quad (10)$$

with $d\omega_x$ the normalized uniform measure on S^2 .

Because of rotational invariance we have

$$\begin{aligned} e^{\beta \vec{n}_x \cdot \vec{n}_{x'}} &= \sum_{\ell \in \mathbb{Z}_+} (2\ell+1) C_\ell(\beta) P_\ell(\vec{n}_x \cdot \vec{n}_{x'}) \\ &= 4\pi \sum_{\ell \in \mathbb{Z}_+} C_\ell(\beta) \sum_{m=-\ell}^{+\ell} Y_\ell^m(\vec{n}_x) \overline{Y_\ell^m(\vec{n}_{x'})} \end{aligned} \quad (11)$$

where $C_\ell(\beta)$ is a cylinder function

$$C_\ell(\beta) = \left(\frac{\pi}{2\beta} \right)^{\frac{1}{2}} I_{\ell+\frac{1}{2}}(\beta) \quad (12)$$

At each site x of Λ we have to integrate over s^2

$$(4\pi)^2 \int d\omega_x \overline{Y_{m_{(x,1)}}^{l_{(x,1)}}(\vec{n}_x)} Y_{m_{(x,2)}}^{l_{(x,2)}}(\vec{n}_x) \overline{Y_{m_{(x-1,1)}}^{l_{(x-1,1)}}(\vec{n}_x)} \\ \times \overline{Y_{m_{(x-2,2)}}^{l_{(x-2,2)}}(\vec{n}_x)} \quad (13)$$

This integration is performed by means of standard formulae [Ref. 8), Eqs. (4.6.3) and (4.6.5)] and yields

$$+ [(2l_{(x,1)}+1)(2l_{(x,2)}+1)(2l_{(x-1,1)}+1)(2l_{(x-2,2)}+1)]^{\frac{1}{2}} \\ \times \sum_{l_x \in \mathbb{Z}_+} (2l_x+1) \sum_{m=-l_x}^{+l_x} \begin{pmatrix} l_{(x,1)} & l_{(x,2)} & l_x \\ 0 & 0 & 0 \end{pmatrix} \\ \times \begin{pmatrix} l_{(x-1,1)} & l_{(x-2,2)} & l_x \\ 0 & 0 & 0 \end{pmatrix} (-1)^{l_{(x-1,1)}-m_{(x-1,1)}+l_{(x-2,2)}-m_{(x-2,2)}} \\ \times (-1)^{l_x-m} \begin{pmatrix} l_{(x,1)} & l_{(x,2)} & l_x \\ m_{(x,1)} & m_{(x,2)} & m \end{pmatrix} \begin{pmatrix} l_{(x-1,1)} & l_{(x-2,2)} & l_x \\ -m_{(x-1,1)}-m_{(x-2,2)}-m \end{pmatrix} \quad (14)$$

The first two $3j$ coefficients in (14) restrict the j field to integers l and to

$$l_{(x,1)} + l_{(x,2)} \triangleq l_x \pmod{2} \quad (15)$$

$$l_{(x,1)} + l_{(x,2)} \triangleq l_{(x-1,1)} + l_{(x-2,2)} \pmod{2} \quad (16)$$

The j field has as dynamical domain the links of the RS lattice Λ and as kinematical domain the sites of Λ .

The second pair of $3j$ coefficients of (14) after contraction over m yield a graph

The orientation of the vertices is irrelevant due to (15) and (16) and therefore dropped. Contracting all these graphs gives the J lattice of Fig. 1, which has been distorted so as to obtain regular hexagonal form. The co-ordinate axes of the underlying RS lattice appear then to enclose an angle $\pi/3$.

If we call the $3nj$ coefficient of this J lattice $w(\{l\})$ then the W functional is

$$W(\{l\}) = \prod_{x \in \Lambda} \begin{pmatrix} l_x & l_{x,1} & l_{x,2} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_x & l_{x-1,1} & l_{x-2,2} \\ 0 & 0 & 0 \end{pmatrix} w(\{l\}) \quad (17)$$

and the partition function is

$$Z = \sum_{\{l_x, l_{x,1}, l_{x,2}, x \in \Lambda\}} \prod_{x \in \Lambda} [(2l_x+1)(2l_{x,1}+1) \times (2l_{x,2}+1) C_{l_{x,1}}(\beta) C_{l_{x,2}}(\beta)] W(\{l\}) \quad (18)$$

We are aware of the fact that we deviate from the terminology introduced in Section 1 slightly (e.g., the difference between w and W) but we hope that this does not cause any confusion.

4. - SU(2) GAUGE MODELS

We consider a hypercubical RS lattice Λ in $D \geq 3$ dimensions and use Wilson's action ⁹⁾

$$-S = \frac{1}{2} \sum_P \text{Tr } u_P, \quad P = (x, \mu, \nu), \quad \mu < \nu \quad (19)$$

$$u_p = u_{(x,\mu)} u_{(x+\mu,\nu)}^{-1} u_{(x+\nu,\mu)}^{-1} u_{(x,\nu)}^{-1} \quad (20)$$

The partition function is defined by

$$Z = \int \prod_{(x,\mu) \in \Lambda} du_{(x,\mu)} e^{-\beta S} \quad (21)$$

x denotes sites, (x,μ) the links, (x,μ,ν) plaquettes of Λ , and du is the normalized Haar measure on $SU(2)$ *).

We integrate after expanding the exponent into characters. With

$$\text{Tr } u = 2 \cos \frac{\varphi}{2} = \chi_{\frac{1}{2}}(u) \quad (22)$$

$$\chi_j(u) = \frac{\sin(j+\frac{1}{2})\varphi}{\sin \frac{1}{2}\varphi}, \quad j \in \frac{1}{2}\mathbb{Z}_+ \quad (23)$$

$$e^{\beta \frac{1}{2} \text{Tr } u} = \sum_{j=0}^{\infty} (2j+1) C_j(\beta) \chi_j(u) \quad (24)$$

We easily find

$$C_j(\beta) = \frac{2}{\beta} I_{2j+1}(\beta) \quad (25)$$

Each link variable occurs in $2D-2$ plaquettes and to each plaquette belongs a representation j of $SU(2)$ due to (24).

Consider a link $(x,\mu) \in \Lambda$. The plaquettes attached to this link are

$$(x,\mu,\nu), \mu < \nu; (x,\nu,\mu), \nu < \mu \quad \text{all } \nu$$

and

$$(x-\nu,\mu,\nu), \mu < \nu; (x-\nu,\nu,\mu), \nu < \mu \quad \text{all } \nu$$

*) For the notations see the Appendix.

Each character can be expanded $[P = (x, u, v)]$

$$\begin{aligned} \chi_j(u_P) &= \sum_{m_1, \dots, m_4} D_{m_1 m_2}^j(u_{(x, \mu)}) D_{m_2 m_3}^j(u_{(x+\mu, v)}) \\ &\quad \times D_{m_3 m_4}^j(u_{(x+\nu, \mu)^{-1}}) D_{m_4 m_1}^j(u_{(x, \nu)^{-1}}) \\ j &= j(x, \mu, \nu) \end{aligned} \quad (26)$$

We will later replace u^{-1} by u through

$$D_{m_1 m_2}^j(u^{-1}) = (-1)^{j-m_1+j-m_2} D_{-m_2, -m_1}^j(u) \quad (27)$$

[see Ref. 8), Eq. (4.2.7)]. At each link of Λ we have the integral

$$\begin{aligned} \int du_{(x, \mu)} \prod_{\nu > \mu} D_{m_1 m_2}^{j(x, \mu, \nu)}(u_{(x, \mu)}) \prod_{\nu' < \mu} D_{m_3 m_4}^{j(x, \nu', \mu)}(u_{(x, \mu)}) \\ \times \prod_{\lambda > \mu} D_{m_5 m_6}^{j(x-\lambda, \mu, \lambda)}(u_{(x, \mu)}) \prod_{\lambda' < \mu} D_{m_7 m_8}^{j(x-\lambda', \mu, \lambda')}(u_{(x, \mu)}) \end{aligned} \quad (28)$$

We start our algebraic analysis with a symmetric form of the integral

(28)

$$\int du \prod_{i=1}^N D_{m_i m'_i}^{j_i}(u) = \left(\begin{matrix} j_1 & j_2 & \dots & j_N \\ (m_1 m'_1) & (m_2 m'_2) & \dots & (m_N m'_N) \end{matrix} \right) \quad (29)$$

$[N = 2D-2$ in (28)]. It can be expressed by sums of products of $3j$ coefficients in different ways. A kernel for these expressions will be given in Section 5. For simplicity we start with the case $N = 4$ ($D = 3$). Then

$$\begin{aligned}
 & \left(\begin{array}{ccc} j_1 & j_2 & j_4 \\ (m_1, m'_1) & (m_2, m'_2) & (m_4, m'_4) \end{array} \right) = \\
 & = \sum_{j=0}^{\infty} (2j+1) \sum_{m=-j}^{+j} (-1)^{j-m} \left(\begin{array}{ccc} j_1 & j_2 & j \\ m_1 & m_2 & m \end{array} \right) \left(\begin{array}{ccc} j_3 & j_4 & j \\ m_3 & m_4 & -m \end{array} \right) \\
 & \quad \times \sum_{m'=-j}^{+j} (-1)^{j-m'} \left(\begin{array}{ccc} j_1 & j_2 & j \\ m'_1 & m'_2 & m' \end{array} \right) \left(\begin{array}{ccc} j_3 & j_4 & j \\ m'_3 & m'_4 & -m' \end{array} \right) \quad (30)
 \end{aligned}$$

The internal sums can be represented by the graphs

Note that a change of the orientation of any line or vertex does not matter as long as we do it in both graphs simultaneously. Keeping this in mind we may replace these two graphs by any pair

with the contractions inside the bubbles unspecified. In a similar fashion we may express the integral (29) for arbitrary N by a sum of products of two factors

$$\begin{aligned}
 & \sum_{\{j_i\} \in (\frac{1}{2} \mathbb{Z}_+)^{N-3}} \prod_{i=1}^{N-3} (2j_i+1) \left[\begin{array}{ccc} j_1 & j_2 & \dots & j_N \\ m_1 & m_2 & & m_N \end{array} \middle| j_1, j_2, \dots, j_{N-3} \right] \\
 & \quad \times \left[\begin{array}{ccc} j_1 & j_2 & \dots & j_N \\ m'_1 & m'_2 & & m'_N \end{array} \middle| j_1, j_2, \dots, j_{N-3} \right] \quad (31)
 \end{aligned}$$

where we represent both j_m coefficients by a pair of graphs of the same internal structure

The internal structure can be fixed later by convenience. The derivation of (31) is based on repeated use of the equation [Ref. 8), Eq. (4.3.2)]

$$D_{m_1 m'_1}^{j_1}(\omega) D_{m_2 m'_2}^{j_2}(\omega) = \sum_j (2j+1) \sum_{m, m'=-j}^{+j} (-1)^{j-m+j-m'} \\ \times \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix} D_{mm'}^j(\omega) \begin{pmatrix} j_1 & j_2 & j \\ m'_1 & m'_2 & -m' \end{pmatrix} \quad (32)$$

We go next over to the dual lattice Λ^* of Λ . A link of Λ goes into a (D-1) cell, a plaquette into a (D-2) cell. For details on the duality transformation see the Appendix. Consider a D cell $[x']$. Ascribe the primed j_m coefficient (31) to its negative and the unprimed j_m coefficient (31) to the positive boundary (D-1) cells. Then for one of these (D-1) cells $[x', \mu] \in \Lambda^*$, there is a dual link $(x-\underline{\mu}, \mu) \in \Lambda$, and the j_m coefficient (31) has as arguments $\{j_1, j_2, \dots, j_N\}$, $N = 2D-2$, according to (28)

$$\{j(x-\underline{\mu}, \mu, \nu), \text{ all } \nu > \mu\} \cup \{j(x-\underline{\mu}, \nu, \mu), \text{ all } \nu < \mu\} \\ \{j(x-\underline{\mu}-\underline{\lambda}, \mu, \lambda), \text{ all } \lambda > \mu\} \cup \{j(x-\underline{\mu}-\underline{\lambda}, \lambda, \mu), \text{ all } \lambda < \mu\}$$

whereas the arguments $\{j_1, \dots, j_{2D-5}\}$ are denoted $\{j_{[x', \mu], 1}, \dots, j_{[x', \mu], 2D-5}\}$. We can also replace the subscripts of each j by the dual (D-2) cell co-ordinate, e.g. :

$$(x-\underline{\mu}, \mu, \nu) \rightarrow [x'+\underline{\nu}, \mu, \nu]$$

On the other hand, two (D-1) cells of the boundary of $[x']$ intersect as follows (let $\mu < \nu$)

$$(a) \quad [x', \mu] \cap [x', \nu] = [x', \mu, \nu] \rightarrow (x - \underline{\mu} - \underline{\nu}, \mu, \nu)$$

$$(b) \quad [x', \mu] \cap [x' + \underline{\nu}, \nu] = [x' + \underline{\nu}, \mu, \nu] \rightarrow (x - \underline{\mu}, \mu, \nu)$$

$$(c) \quad [x' + \underline{\mu}, \mu] \cap [x', \nu] = [x' + \underline{\mu}, \mu, \nu] \rightarrow (x - \underline{\nu}, \mu, \nu)$$

$$(d) \quad [x' + \underline{\mu}, \mu] \cap [x' + \underline{\nu}, \nu] = [x' + \underline{\mu} + \underline{\nu}, \mu, \nu] \rightarrow (x, \mu, \nu)$$

where the dual images of the (D-2) cells are also given. Replace the jm coefficient (31) by its graph and draw each leg j to the centre of that (D-2) cell whose co-ordinate j carries as a subscript. There are always two such legs that meet in one centre. We make sure that this orientation is such that one leads in and the other out. This pair of j 's belongs to a pair of D functions in the trace expression (26). We claim that the summation in (26) contracts the jm coefficients of all (D-1) cells on the boundary of the D cell $[x']$ to one $3nj$ coefficient with

$$n = 2D(D-2) \quad (33)$$

We have in fact to prove that primed and unprimed jm coefficients as chosen in our manner just happen to contract in the asserted manner. Then (33) follows by an elementary counting of variables. In Figs. 2a,b,c,d, we represent cuts of the (μ, ν) plane through the D cell together with the plaquette around that (D-2) cell where the contraction has to be performed.

In Fig. 2a we have to contract

$$\begin{aligned} & \sum_m D^j(x - \underline{\mu} - \underline{\nu}, \mu, \nu) (u_{(x - \underline{\nu}, \nu)}) D^j(x - \underline{\mu} - \underline{\nu}, \mu, \nu) (u_{(x - \underline{\mu}, \mu)}^{-1}) \\ &= \sum_m (-1)^{j(x - \underline{\mu} - \underline{\nu}, \mu, \nu) - m} + \dots D^j(x - \underline{\mu} - \underline{\nu}, \mu, \nu) (u_{(x - \underline{\nu}, \nu)}) \\ & \quad \times D^j(x - \underline{\mu} - \underline{\nu}, \mu, \nu) (u_{(x - \underline{\mu}, \mu)}) \end{aligned} \quad (34a)$$

This is a contraction of the primed jm coefficient of $[x', v]$ with the primed coefficient of $[x', \mu]$. Similarly we obtain, for Fig. 2b, the sum

$$\begin{aligned} & \sum_m D_{\cdot, m}^{j(x-y, \mu, v)} (u_{(x, \mu)}^{-1}) D_{m, \cdot}^{j(x-y, \mu, v)} (u_{(x-y, v)}^{-1}) \\ &= \sum_m (-1)^{2j(x-y, \mu, v) - 2m + \dots} D_{-m, \cdot}^{j(x-y, \mu, v)} (u_{(x, \mu)}) \\ & \quad \times D_{\cdot, -m}^{j(x-y, \mu, v)} (u_{(x-y, v)}) \end{aligned} \quad (34b)$$

which leads to a contraction of the unprimed jm coefficient of $[x' + \underline{\mu}, \underline{\mu}]$ with the primed jm coefficient of $[x', v]$. The remaining cases, Fig. 2c, 2d are treated analogously.

The signs of m in (34b) and the sign factor are not correct for a contraction (4). The same happens with Fig. 2c. This can be corrected by replacing all m_i in primed jm coefficients by $-m_i$. This amounts to a reversal of the orientation of all legs and links. If we reverse all legs only but keep the orientation of the links and remember that

$$\sum_i m_i = 0$$

for any jm coefficient, we obtain an over-all sign factor $(-1)^R$ with

$$R = \sum_{\substack{\text{all legs of primed} \\ \text{jm-coefficients}}} (j_i - m_i) \quad (35)$$

After these preparations the contractions can be performed. All sign factors are used up for the contractions. The resulting orientations are as follows :

- internal lines and vertices are equally but arbitrarily oriented in each pair of jm coefficients ;
- the links obtained by contracting two jm coefficients are oriented in each $(\mu\nu)$ plane, $\mu < \nu$, as indicated in Figs. 2a-2d by the round arrows.

Consider the case $D = 3$. In Fig. 3 we have drawn the graph on a three-cell which represents an $18j$ coefficient and results from the contractions with all orientations of lines as just discussed. This graph is repeated on all three-cells. We may distort this graph to render it planar (Fig. 4). On this planar graph we also give the orientation of vertices. It can easily be proved that this orientation repeats itself if we go from the cell $[x']$ to the cells $[x', v]$. In the notations of Ref. 7) this graph belongs to an $18j$ coefficient of the second kind. It can be decomposed into a sum of products of six $6j$ coefficients [Ref. 7), Eq. (17.2)] applying the rules (6) and (7). Call the $18j$ coefficient $w_x(\{j\}_x)$. We obtain

$$w_x(\{j\}_x) = \sum_{j_{x'} \in \frac{1}{2}\mathbb{Z}_+} (2j_{x'} + 1) (-1)^{R_{x'}} \times \left\{ \begin{array}{l} j_{[x', 2]} j_{[x' + 3, 1, 3]} j_{x'} \\ j_{[x' + 1, 1]} j_{[x', 2, 3]} j_{[x' + 1, 1, 2]} \end{array} \right\} \left\{ \begin{array}{l} j_{[x', 2, 3]} j_{[x' + 1, 1]} j_{x'} \\ j_{[x' + 1 + 2, 1, 2]} j_{[x', 3]} j_{[x' + 1, 1, 3]} \end{array} \right\} \\ \times \left\{ \begin{array}{l} j_{[x', 3]} j_{[x' + 1 + 2, 1, 2]} j_{x'} \\ j_{[x' + 2, 2]} j_{[x', 1, 3]} j_{[x' + 2, 2, 3]} \end{array} \right\} \left\{ \begin{array}{l} j_{[x', 1, 3]} j_{[x' + 2, 2]} j_{x'} \\ j_{[x' + 2 + 3, 2, 3]} j_{[x', 1]} j_{[x' + 2, 1, 2]} \end{array} \right\} \\ \times \left\{ \begin{array}{l} j_{[x', 1]} j_{[x' + 2 + 3, 2, 3]} j_{x'} \\ j_{[x' + 3, 3]} j_{[x', 1, 2]} j_{[x' + 3, 1, 3]} \end{array} \right\} \left\{ \begin{array}{l} j_{[x', 1, 2]} j_{[x' + 3, 3]} j_{x'} \\ j_{[x' + 3 + 1, 1, 3]} j_{[x', 2]} j_{[x' + 3, 2, 3]} \end{array} \right\} \quad (36)$$

with

$$R_{x'} = 2(j_{x'} + j_{[x', 1, 3]} + j_{[x' + 1 + 3, 1, 3]}) \\ + \sum_{\mu} (j_{[x', \mu]} - j_{[x' + \mu, \mu]}) + \sum_{\mu, \nu} j_{[x' + \mu, \mu, \nu]} \\ + \sum_{\mu < \nu} (j_{[x', \mu, \nu]} - j_{[x' + \mu + \nu, \mu, \nu]}) \quad (37)$$

Using the rules (6) and (7), one can also compose these $18j$ coefficients to larger $3nj$ coefficients. We apply rule (6) to a pair of vertices, say $4'$ of $[x']$ and 2 of $[x'+2]$. After reorientation of the vertices and links attached to them the vertices $4'$ and 2 can be erased and we get links between vertices $3', 4, 5'$ of $[x']$ with $2', 3, 1$ of $[x'+2]$, respectively, and the variables for these links are in the same order :

$$j_{[x'+2+3, 2, 3]}, j_{[x'+2, 1, 2]}, j_{[x'+2, 2]}$$

Since the variables ending at vertex $5'$ of $[x']$ are the same as those ending at the vertex 1 of $[x'+2]$ (namely $j_{[x'+2, 2, 3]}, j_{[x'+1+2, 1, 2]}$). We can apply rule (7) after a reorientation, and sum over $j_{[x'+2, 2]}$. Again the vertices $5'$ and 1 can be erased. The graph obtained is (apart from reorientations) the one obtained by gluing the cubes $[x']$ and $[x'+2]$ together along the side $[x+2, 2]$ and erasing all lines and vertices on this side. We can go on in this fashion but it is forbidden to close a linear array of cubes. The same construction works for any D .

The W functional can be written as

$$W(\{j\}) = \prod_{x' \in \Lambda^*} w_{x'}(\{j\}_{x'}) \quad (38)$$

in which case the kinematic domain consists of all plaquettes of Λ^*

$$\{[x', \nu]\} \xrightarrow{j \text{ field}} j_{[x', \nu]} \quad (39)$$

Using the representation in terms of $6j$ coefficients (37), the kinematic domain is enlarged to

$$\{[x', \nu]\} \cup \{[x']\}$$

For the case $D = 4$ we will not present the results completely here. For a fixed 4 cell $[x']$ the boundary consists of the 3 cells

$$[x', 1], [x', 2], [x', 3], [x', 4] \\ [x'+1, 1], [x'+2, 2], [x'+3, 3], [x'+4, 4]$$

each with a $j m$ coefficient (31) containing three internal J variables (the blobs in Fig. 5). Contracting the $j m$ coefficients we obtain the graph of Fig. 5.

5. - THE BARGMANN SPACE FORMALISM

Consider the space $H_\mu(\mathbb{C}^2)$ of analytic functions over $\mathbb{C}^2$ with the Gaussian measure

$$d\mu(\zeta) = \pi^{-2} d^2\xi d^2\eta e^{-|\xi|^2 - |\eta|^2} \quad (40)$$

$$\zeta = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \mathbb{C}^2, \quad \xi, \eta \in \mathbb{C}^1 \quad (41)$$

$$d^2\xi = d(\operatorname{Re}\xi) d(\operatorname{Im}\xi) \quad (42)$$

$$(f, g)_\mu = \int_{\mathbb{C}^2} d\mu(\zeta) \overline{f(\zeta)} g(\zeta) \quad (43)$$

$H_\mu(\mathbb{C}^2)$ carries the unitary representation D of $SU(2)$

$$T_u f(\zeta) = f(u^T \zeta), \quad u \in SU(2) \quad (44)$$

which decomposes into the direct sum

$$D = \sum_{j=0}^{\infty} \oplus D_j \quad (45)$$

of all irreducible representations D_j of spin j . In fact, the homogeneous polynomials

$$v_m^j = \frac{\xi^{j+m} \eta^{j-m}}{[(j+m)! (j-m)!]^{\frac{1}{2}}} \quad (46)$$

span a subspace $Q_j \subset H_\mu(\mathbb{C}^2)$ of dimension $2j+1$ that carries just the representation D_j .

The space $H_\mu(\mathbb{C}^2)$ is a kernel space in the sense that the linear functional δ_ζ

$$f(\zeta) = (\delta_\zeta, f)_\mu \quad (47)$$

is bounded and therefore an element of $H_\mu(\mathbb{C}^2)$ itself, thus defining a kernel

$$\delta_\zeta(\zeta') = e^{\bar{\zeta} \cdot \zeta'} \quad (48)$$

Here $\bar{\zeta} \cdot \zeta'$ denotes the scalar product in the Euclidean space $\mathbb{C}^2$

$$\begin{aligned} \bar{\zeta} \cdot \zeta' &= \bar{\xi} \xi' + \bar{\eta} \eta' \\ \|\zeta\|^2 &= \bar{\zeta} \cdot \zeta \end{aligned} \quad (49)$$

The projection operator onto Q_j can be given in the form of an integral kernel as well

$$f|_{Q_j}(\zeta) = (K_j(\zeta, \cdot), f)_\mu \quad (50)$$

$$K_j(\zeta, \zeta') = \frac{1}{(2j)!} (\bar{\zeta} \cdot \zeta')^{2j} \quad (51)$$

which, due to (48), leads to

$$\sum_{j=0}^{\infty} K_j(\zeta, \zeta') = \delta_\zeta(\zeta') \quad (52)$$

as it should.

By the standard tensoration of Hilbert spaces of functions, define the Bargmann spaces $H_\mu(\mathbb{C}^{2n}) = [H_\mu(\mathbb{C}^2)]_\otimes^n$, endowed with the measure

$$d\mu(\zeta_1, \zeta_2, \dots, \zeta_n) = \prod_{i=1}^n d\mu(\zeta_i) \quad (53)$$

Consider then the function

$$F(k) = \frac{\delta_1^{k_1} \delta_2^{k_2} \delta_3^{k_3}}{k_1! k_2! k_3!} \in H_\mu(\mathbb{C}^6), \quad k_i \in \mathbb{Z}_+ \quad (54)$$

$$\delta_1 = \xi_2 \eta_3 - \xi_3 \eta_2 := [\zeta_2, \zeta_3] \quad (55)$$

etc. cyclically in 1, 2, 3

In $H_\mu(\mathbb{C}^6)$ we define the representation

$$f \rightarrow T_u f(\zeta_1, \zeta_2, \zeta_3) = f(u^T \zeta_1, u^T \zeta_2, u^T \zeta_3) \quad (56)$$

of $SU(2)$ analogous to (44). Each Weyl invariant δ_i is invariant under (56) and therefore so is $F(k)$. Define j_i through k_i by

$$k_i = \sum_{n=1}^3 j_n - 2j_i \quad (57)$$

$$2j_i = \sum_{n \neq i} k_n \quad (58)$$

and moreover

$$j = \sum_{i=1}^3 k_i = \sum_{i=1}^3 j_i \quad (59)$$

We can then expand $F(k)$ in the basis (46)

$$F(k) = \sum_{m_1, m_2, m_3} F(j_1^i, j_2^i, j_3^i) v_{m_1}^{j_1}(\zeta_1) v_{m_2}^{j_2}(\zeta_2) v_{m_3}^{j_3}(\zeta_3) \quad (60)$$

The coefficient $F(\dots)$ is then related to Wigner's $3j$ coefficient by ^{2), 3)} :

$$\begin{pmatrix} j_1^i & j_2^i & j_3^i \\ m_1 & m_2 & m_3 \end{pmatrix} = \Delta(j_1, j_2, j_3) F \begin{pmatrix} j_1^i & j_2^i & j_3^i \\ m_1 & m_2 & m_3 \end{pmatrix} \quad (61)$$

where $\Delta(j_1, j_2, j_3)$ is the "triangle coefficient"

$$\Delta(j_1, j_2, j_3) = \left[\frac{k_1! k_2! k_3!}{(j+1)!} \right]^{\frac{1}{2}} \quad (62)$$

The generating function of the $3j$ coefficient can be defined by ^{2),3)}

$$\Phi(\tau, \zeta) = \sum_{k \in \mathbb{Z}_+^3} \tau_1^{k_1} \tau_2^{k_2} \tau_3^{k_3} F(k) \quad (63)$$

$$= \exp \begin{vmatrix} \tau_1 & \tau_2 & \tau_3 \\ \bar{\zeta}_1 & \bar{\zeta}_2 & \bar{\zeta}_3 \\ \eta_1 & \eta_2 & \eta_3 \end{vmatrix} \quad (64)$$

In turn

$$F(k) = (2\pi i)^{-3} \int_{C^3} \prod_{i=1}^3 \left(\frac{d\tau_i}{\tau_i^{k_i+1}} \right) \Phi(\tau, \zeta) \quad (65)$$

C is a contour encircling zero once in positive orientation.

As an elementary application of this formalism we calculate the integrals (29). First we introduce a kernel for the operator T_u (44)

$$\begin{aligned} D_{mm'}^j(u) &= \int d\mu(\zeta) \overline{v_m^j(\zeta)} (T_u v_{m'}^j)(\zeta) \\ &= \int d\mu(\zeta) d\mu(\zeta') \overline{v_m^j(\zeta)} K(u; \zeta, \zeta') v_{m'}^j(\zeta') \end{aligned} \quad (66)$$

with

$$K(u; \zeta, \zeta') = \exp \bar{\zeta}' \cdot (u^T \zeta) \quad (67)$$

If we parametrize $u \in \text{SU}(2)$ by

$$u = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \quad (68)$$

then

$$\bar{\xi}' \cdot (u^T \zeta) = \bar{\xi}' (\alpha \xi - \bar{\beta} \eta) + \bar{\eta}' (\beta \xi + \bar{\alpha} \eta) \quad (69)$$

Instead of (29) we compute the kernel

$$\int du \prod_{i=1}^N K(u; \zeta_i, \zeta'_i) = \int du \exp(\alpha U + \beta X + \bar{\alpha} V + \bar{\beta} Y) \quad (70)$$

with

$$\begin{aligned} U &= \sum_{i=1}^N \bar{\xi}'_i \xi_i & V &= \sum_{i=1}^N \bar{\eta}'_i \eta_i \\ X &= \sum_{i=1}^N \bar{\eta}'_i \xi_i & Y &= - \sum_{i=1}^N \bar{\xi}'_i \eta_i \end{aligned} \quad (71)$$

With the parameters (68) we have the Haar measure

$$du = \frac{1}{\pi^2} d^2\alpha d^2\beta \delta(|\alpha|^2 + |\beta|^2 - 1) \quad (72)$$

We obtain by elementary algebra the result

$$\int du \prod_{i=1}^N K(u; \zeta_i, \zeta'_i) = \sum_{n=0}^{\infty} \frac{(UV + XY)^n}{n! (n+1)!} \quad (73)$$

which can be rewritten in terms of a modified Bessel function I_1 .

Ascribing such functions (73) to each (D-1) cell of Λ^* and a generating function for the kernels K_j (51)

$$\exp \tau \bar{\xi} \cdot \zeta' = \sum_{j \in \frac{1}{2} \mathbb{Z}_+} \tau^{2j} K_j(\zeta, \zeta') \quad (74)$$

to each (D-2) cell and doing the contractions by integrating over ζ, ζ' for each (D-2) cell yields a generating function $G(\{\tau\})$ so that

$$W(\{j\}) = \prod_{(D-2) \text{ cells } l} \left(\frac{1}{2\pi i} \int \frac{d\tau_l}{\tau_l^{2j_l+1}} \right) G(\{\tau\}) \quad (75)$$

However, we were unable to calculate G . Instead we will use generating functions of the "Schwinger type" whose advantage lies in their explicit computability.

6. - SCHWINGER TYPE GENERATING FUNCTIONS FOR GRADED J LATTICES

We consider a closed J lattice which consists of oriented vertices and oriented links. Ascribing a value $j_l \in \frac{1}{2}\mathbb{Z}_+$ to each link l we obtain a j field on the J lattice. By means of (57) and (58), we can define, for the triplet of links entering or leaving a fixed vertex, a triplet of numbers k_1, k_2, k_3 , $k_i \in \mathbb{Z}_+$, so that each link carries a pair (k, k') of such numbers, namely one number for each vertex at the ends of the link. In this fashion we define a k field on the given lattice. Consider a pair of vertices with its connecting link,

the orientations are irrelevant here. Then in order that a unique variable j belongs to the link, the constraint

$$k_1 + k_2 = k_1' + k_2' \quad (76)$$

must be fulfilled [see (58)]. A k field satisfying (76) at each link is called "admissible".

Similarly we can define τ fields and ζ fields on the given J lattice which consists of pairs (τ, τ') and (ζ, ζ') respectively, carried by the links of the lattice. They are not constrained.

Assume that a τ field and a ζ field are given on the lattice. Then ascribe a generating function (64) to each vertex and observe the orientation of the vertex by the corresponding order of the columns in the determinant (64).

Consider next a link carrying the pair ζ, ζ' with the orientation of the link $\zeta \rightarrow \zeta'$. We ascribe to this link the kernel

$$e^{[\bar{\zeta}, \bar{\zeta}']} \quad (77)$$

There then exists a generating function $H(\{\tau\})$ (of "Schwinger type") so that for a given k field [see (61), (62)]

$$\prod_{\substack{\text{links} \\ e}} \left[(2\pi i)^{-2} \int_{C_e} \frac{d\tau_e}{\tau_e^{k_e+1}} \int_{C'_e} \frac{d\tau'_e}{\tau'^{k'_e+1}} \right] H(\{\tau\}) \\ = \begin{cases} W(\{j\}) \prod_{\text{vertices } t} \Delta(j_{t,1}, j_{t,2}, j_{t,3})^{-1} & \text{if the } k \text{ field is admissible} \\ 0 & \text{else} \end{cases} \quad (78)$$

This generating function is defined by a Bargmann integral formula

$$H(\{\tau\}) = \int \prod_{\text{links } e} [d\mu(\zeta_e) d\mu(\zeta'_e) \exp [\bar{\zeta}_e, \bar{\zeta}'_e]] \\ \times \exp \sum_{\text{vertices } t} \begin{vmatrix} \tau_{t,1} & \tau_{t,2} & \tau_{t,3} \\ \bar{\zeta}_{t,1} & \bar{\zeta}_{t,2} & \bar{\zeta}_{t,3} \\ \eta_{t,1} & \eta_{t,2} & \eta_{t,3} \end{vmatrix} \quad (79)$$

where orientations are taken into account as explained above.

For the integral (79) to converge it is necessary that the parameters τ are small enough. In fact, for compact lattices it can be shown that $H(\{\tau\})$ is rational in the τ with a finite domain of holomorphy around zero, i.e., there exists an ϵ so that H is holomorphic for $|\tau_{t,i}| < \epsilon$ for all t and i . We will therefore assume from now on that the J lattice we are dealing with is compact. We then make the circles C_ℓ, C'_ℓ in (78) of radius smaller than ϵ .

For simplicity, in the sequel we will only deal with grade J lattices. We will call a J lattice "graded" if the links are oriented in such a fashion that all vertices are either "positive" with all three links going out or "negative" with all three links entering the vertex. If a reorientation of a J lattice can transform it into a graded lattice, we call it gradable. A lattice of four vertices and six links in the form of a tetrahedron is not gradable, but the lattice of Fig. 1 and the graph Fig. 4 are gradable.

Thus assume there is a graded compact J lattice. As a first step we integrate over the variables ζ' of the negative vertices. The integration over ζ' in (79) leads to a replacement of

$$(\xi', \eta') \quad \text{by} \quad (-\bar{\eta}, \bar{\xi})$$

and consequently of

$$\begin{vmatrix} \tau'_1 & \tau'_2 & \tau'_3 \\ \xi'_1 & \xi'_2 & \xi'_3 \\ \eta'_1 & \eta'_2 & \eta'_3 \end{vmatrix} \quad \text{by} \quad \begin{vmatrix} \tau'_1 & \tau'_2 & \tau'_3 \\ \bar{\xi}_1 & \bar{\xi}_2 & \bar{\xi}_3 \\ \bar{\eta}_1 & \bar{\eta}_2 & \bar{\eta}_3 \end{vmatrix}$$

After this integration the exponent of the integral (79) contains only determinants that are either holomorphic or antiholomorphic in all its arguments ζ . This is typical for graded J lattices.

We proceed then as Schwinger²⁾ and rewrite the exponent as

$$\frac{1}{2} \sum_{e, e'} \lambda_{e, e'} [\zeta_e, \zeta_{e'}] + \frac{1}{2} \sum_{e, e'} \kappa_{e, e'} [\bar{\xi}_e, \bar{\xi}_{e'}] \quad (80)$$

where e, e' run over all links, and by definition

$$\lambda_{e, e'} = -\lambda_{e', e}, \quad \kappa_{e, e'} = -\kappa_{e', e} \quad (81)$$

We enumerate the positive vertices

$$t = 1, 2, 3, \dots, N_+$$

in an arbitrary fashion and the links in a positive vertex by $a = 1, 2, 3$ in the prescribed order, but with an arbitrary point of start. Then we order all links lexicographically by

$$\{e_1, e_2, e_3, \dots\} = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), \dots\} \quad (82)$$

The matrix λ has N_+ skew-symmetric 3×3 blocks on the diagonal and zero elsewhere

$$\begin{pmatrix} \begin{array}{c|c|c} \begin{array}{ccc} 0 & +\tau_{(1,3)} & -\tau_{(1,2)} \\ -\tau_{(1,3)} & 0 & +\tau_{(1,1)} \\ +\tau_{(1,2)} & -\tau_{(1,1)} & 0 \end{array} & \begin{array}{c} \bigcirc \end{array} & \dots \\ \hline \begin{array}{c} \bigcirc \end{array} & \begin{array}{ccc} \begin{array}{ccc} 0 & +\tau_{(2,3)} & -\tau_{(2,2)} \\ -\tau_{(2,3)} & 0 & +\tau_{(2,1)} \\ +\tau_{(2,2)} & -\tau_{(2,1)} & 0 \end{array} & \dots \\ \hline \vdots & \vdots & \ddots \end{array} \end{pmatrix} \quad (83)$$

whereas the matrix κ interconnects different vertex labels. It is obvious that λ has generically N_+ eigenvalues zero.

Schwinger proved ²⁾ that

$$H(\{\tau\}) = \det (\mathbb{1} + \kappa \lambda)^{-1} \quad (84)$$

This proof can easily be transcribed into the Bargmann space formalism and will therefore not be repeated here. The main task is to compute the determinant (84). We introduce the characteristic function of the matrix $\kappa \lambda$

$$\chi(\mu) = \det (\mu \mathbb{1} + \kappa \lambda) \quad (85)$$

so that

$$H(\{\tau\}) = \chi(1)^{-1} \quad (86)$$

We will prove that

$$\chi(\mu) = \mu^{N_+} [p_{N_+}(\mu)]^2 \quad (87)$$

where $p_{N_+}(\mu)$ is a polynomial of order N_+

$$P_{N_+}(\mu) = \mu^{N_+} + \sum_{k=1}^{N_+} a_k \mu^{N_+-k} \quad (88)$$

so that

$$H(\{\tau\}) = \left(1 + \sum_{k=1}^{N_+} a_k\right)^{-2} \quad (89)$$

The proof makes use of elementary linear algebra. By an orthogonal transformation $\mathcal{O}$ we can arrange λ into the form

$$\mathcal{O}^{-1} \lambda \mathcal{O} = \begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & 0 & \lambda_2 & \\ & & -\lambda_2 & 0 & \dots \\ & & & & & 0 & \lambda_{N_+} \\ & & & & & -\lambda_{N_+} & 0 \\ & & & & & & & 0 \dots & 0 \end{pmatrix} \quad (90)$$

Call the projection on the N_+ dimensional subspace of zero eigenvalues Π . Then

$$\chi(\mu) = \mu^{N_+} \det(\mu \Pi + \kappa_0 \lambda_0) \quad (91)$$

with

$$\begin{aligned} \kappa_0 &= \Pi \mathcal{O}^{-1} \kappa \mathcal{O} \Pi \\ \lambda_0 &= \Pi \mathcal{O}^{-1} \lambda \mathcal{O} \Pi \end{aligned} \quad (92)$$

Since the matrix

$$\mu \lambda_0^{-1} + \kappa_0$$

is skew-symmetric, we can find a non-singular matrix S whose elements depend rationally on μ , so that

$$S^T(\mu\lambda_0^{-1} + \kappa_0)S = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & 0 & 1 & & \\ & -1 & 0 & & \\ & & 0 & 1 & \\ & & & -1 & 0 & \ddots \\ & & & & & 0 & 1 \\ & & & & & & -1 & 0 \end{pmatrix} \quad (93)$$

It follows

$$\det(\mu\pi + \kappa_0\lambda_0) = \det\lambda_0 \cdot (\det S^{-1})^2 \quad (94)$$

Since the left-hand side is a polynomial of μ and the right-hand side a square of a rational function of μ the assertion is proved.

There remains to calculate the coefficients a_k in (88) and (89). From (85) we have

$$\chi(\mu) = \mu^{3N_+} \exp \left\{ 2 \operatorname{Tr} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n} \left(\frac{\kappa\lambda}{\mu} \right)^n \right\} \quad (95)$$

(if the modulus of all eigenvalues of $\kappa\lambda$ is smaller than one). Expanding the exponent we obtain by comparison with (87) and (88)

$$a_k = (-1)^k \sum_{\{m_n\}} \prod_{n=1}^{\infty} \frac{1}{m_n!} \left[-\frac{1}{2n} \operatorname{Tr} (\kappa\lambda)^n \right]^{m_n} \quad (96)$$

$$\sum_n n m_n = k$$

This coefficient is homogeneous of degree k in either τ or τ' and of degree $2k$ in both τ, τ' . There remains to compute the traces. For small lattices or simple graphs this can be done directly. For big lattices we need an algorithm. We develop it in the subsequent section.

7. - THE PLANAR HEXAGONAL J LATTICE

The hexagonal lattice of Fig. 1 is already graded and we may assume that all vertices are positively oriented. For the purpose of this section we relabel the links by the introduction of a triangular co-ordinate frame (Fig. 6). It possesses three basis vectors $\underline{e} \in \{\underline{1}, \underline{2}, \underline{3}\}$ satisfying

$$\underline{1} + \underline{2} + \underline{3} = 0 \quad (97)$$

The positive vertices (in the sense of gradation here and in the sequel) form a triangular lattice, the "skeleton", with co-ordinates in the triangular frame

$$x = m_1 \underline{1} + m_2 \underline{2} \quad (98)$$

All links and all fields are labelled by the co-ordinate x (98) and a label $a \in \{1, 2, 3\}$ for the direction of the link.

The matrix λ has the form (83) of (3×3) blocks on the diagonal

$$\lambda_{x,a;x',a'} = \delta_{x,x'} \begin{pmatrix} 0 & +\tau_{x,3} & -\tau_{x,2} \\ -\tau_{x,3} & 0 & +\tau_{x,1} \\ +\tau_{x,2} & -\tau_{x,1} & 0 \end{pmatrix} \quad (99)$$

The links ending at a negative vertex carry the labels $(x,1)$, $(x-\underline{3},2)$, $(x+\underline{2},3)$ and therefore the contribution of all negative vertices is

$$\sum_x \begin{vmatrix} \tau'_{x,1} & \tau'_{x-\underline{3},2} & \tau'_{x+\underline{2},3} \\ \bar{\xi}_{x,1} & \bar{\xi}_{x-\underline{3},2} & \bar{\xi}_{x+\underline{2},3} \\ \bar{\eta}_{x,1} & \bar{\eta}_{x-\underline{3},2} & \bar{\eta}_{x+\underline{2},3} \end{vmatrix}$$

We obtain six types only of non-vanishing (3×3) blocks of κ

$$\kappa_{x,a;x+\underline{1},a'} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\tau'_{x-\underline{3},1} & 0 \end{pmatrix} aa' \quad (100)$$

$$\kappa_{x,a; x+\underline{2},a'} = \begin{pmatrix} 0 & 0 & -\tau'_{x-\underline{3},2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{aa'}$$

$$\kappa_{x,a; x+\underline{3},a'} = \begin{pmatrix} 0 & 0 & 0 \\ -\tau'_{x-\underline{1},3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{aa'}$$

(100ctd)

and

$$\kappa_{x,a; x-\underline{1},a'} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & +\tau'_{x+\underline{3},1} \\ 0 & 0 & 0 \end{pmatrix}_{aa'}$$

$$\kappa_{x,a; x-\underline{2},a'} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ +\tau'_{x+\underline{1},2} & 0 & 0 \end{pmatrix}_{aa'}$$

$$\kappa_{x,a; x-\underline{3},a'} = \begin{pmatrix} 0 & +\tau'_{x+\underline{2},3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{aa'}$$

(101)

The triple of matrices (100) is related to the triple (101) by transposition, inversion, and translation

$$\kappa_{x,a; x+\underline{n},a'} = -\kappa_{x+\underline{n},a'; x,a}$$

$$x \rightarrow x - \underline{n}$$

(102)

The product $\lambda\kappa$ contains again six types of 3×3 blocks, namely

$$M_{x,\cdot; x \pm \underline{n},\cdot} = \lambda_{x,\cdot; x,\cdot} \kappa_{x,\cdot; x \pm \underline{n},\cdot} \quad (103)$$

or explicitly

$$\begin{aligned}
 M_{x,a; x+1,a'} &= \begin{pmatrix} 0 & +\tau_{x,2} \tau'_{x-2,1} & 0 \\ 0 & -\tau_{x,1} \tau'_{x-2,1} & 0 \\ 0 & 0 & 0 \end{pmatrix}_{aa'} \\
 M_{x,a; x+2,a'} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & +\tau_{x,3} \tau'_{x-3,2} \\ 0 & 0 & -\tau_{x,2} \tau'_{x-3,2} \end{pmatrix}_{aa'} \\
 M_{x,a; x+3,a'} &= \begin{pmatrix} -\tau_{x,3} \tau'_{x-1,3} & 0 & 0 \\ 0 & 0 & 0 \\ +\tau_{x,1} \tau'_{x-1,3} & 0 & 0 \end{pmatrix}_{aa'} \quad (104)
 \end{aligned}$$

and

$$\begin{aligned}
 M_{x,a; x-1,a'} &= \begin{pmatrix} 0 & 0 & +\tau_{x,3} \tau'_{x+3,1} \\ 0 & 0 & 0 \\ 0 & 0 & -\tau_{x,1} \tau'_{x+3,1} \end{pmatrix}_{aa'} \\
 M_{x,a; x-2,a'} &= \begin{pmatrix} -\tau_{x,2} \tau'_{x+1,2} & 0 & 0 \\ +\tau_{x,1} \tau'_{x+1,2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{aa'} \\
 M_{x,a; x-3,a'} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\tau_{x,3} \tau'_{x+3,3} & 0 \\ 0 & +\tau_{x,2} \tau'_{x+3,3} & 0 \end{pmatrix}_{aa'} \quad (105)
 \end{aligned}$$

It follows immediately that

$$\text{Tr}(\kappa \lambda) = 0 \quad (106)$$

In the computation of powers $(\lambda \kappa)^m$ we can make use of the fact that certain products of the 3×3 matrices (104), (105) vanish, namely $(x, x'$ arbitrary)

$$M_{x, \cdot; x \pm n, \cdot} M_{x' \pm n, \cdot; x', \cdot} = 0 \quad (107)$$

$$M_{x, \cdot; x+\underline{n}, \cdot} M_{x', \cdot; x'+\underline{n}', \cdot} = 0 \quad (108)$$

if $n' = n-1$ (cycle.)

$$M_{x, \cdot; x-\underline{n}, \cdot} M_{x', \cdot; x'-\underline{n}', \cdot} = 0 \quad (109)$$

if $n' = n+1$ (cycle.)

We introduce a graphical method. On the skeleton we draw an oriented link from x to x' if we have a (non-vanishing) (3×3) block $M_{x,a;x'a'}$. The contraction between two such matrices is drawn as a vertex. Then the product $(\lambda \kappa)^m$ yields graphs composed of m links and $m-1$ vertices and the trace of $(\lambda \kappa)^m$ consists of closed graphs with m links and vertices. Equations (107), (108) and (109) tell us that certain subgraphs have vanishing contributions (Fig. 7). Remaining graphs form a surprisingly small set. Note that a graph may run through a link arbitrarily often.

We draw the graphs for the lowest order powers m :

$m = 2$: no graphs

$m = 3$:

$$\text{Tr} (\lambda \kappa)^3 = 3 \sum_x \left[\begin{array}{c} \text{triangle with clockwise arrows} \\ \text{triangle with counter-clockwise arrows} \end{array} \right] \quad (110)$$

$m = 4$: no graphs

$m = 5$:

$$\text{Tr} (\lambda \kappa)^5 = 5 \sum_x \left[\begin{array}{c} \text{pentagon with clockwise arrows} \\ \text{pentagon with counter-clockwise arrows} \end{array} \right] \quad (111)$$

+ oppositely oriented graphs

$m = 6$:

$$\begin{aligned} \text{Tr} (\lambda \kappa)^6 = 6 \sum_x & \left[\begin{array}{c} \text{hexagon with clockwise arrows} \\ \text{hexagon with counter-clockwise arrows} \end{array} \right] \\ & + 3 \sum_x \left[\begin{array}{c} \text{triangle with 3 internal links} \end{array} \right] \quad (112) \\ & + \text{oppositely oriented graphs} \end{aligned}$$

Evaluating the traces corresponding to these graphs, we obtain :

$$\text{Tr}(\kappa\lambda)^3 = +6 \sum_x \tau_{x,2} \tau'_{x+2,3} \tau_{x-3,1} \tau'_{x+1-3,2} \tau_{x+1,3} \times \tau'_{x-2,1} \quad (113)$$

$$\begin{aligned} \text{Tr}(\kappa\lambda)^5 = +10 \sum_x \{ & \tau_{x,2} \tau'_{x+2,3} \tau_{x-3,1} \tau'_{x+1,1} \tau_{x-3+1,1} \\ & \times \tau'_{x-3+2,1,2} \tau_{x+2,1,3} \tau'_{x+1-2,1} \tau_{x+1,1} \tau'_{x-3,1} \\ & + \tau_{x,3} \tau'_{x+3,1} \tau_{x-1,2} \tau'_{x+2,2} \tau_{x-1+2,2} \\ & \times \tau'_{x-1+2,2,3} \tau_{x+2,2,1} \tau'_{x+2-3,2} \tau_{x+2,2} \tau'_{x-3,2} \\ & + \tau_{x,1} \tau'_{x+1,2} \tau_{x-2,3} \tau'_{x+3,3} \tau_{x-2+3,3} \\ & \times \tau'_{x-2+2,3,1} \tau_{x+2,3,2} \tau'_{x+3-1,3} \tau_{x+3,3} \tau'_{x-1,3} \} \end{aligned} \quad (114)$$

$$\begin{aligned} \text{Tr}(\kappa\lambda)^6 = -12 \sum_x \{ & \tau_{x,2} \tau'_{x-2,1} \tau_{x+1,1} \tau'_{x+1-2,1} \\ & \times \tau_{x+2,1,3} \tau'_{x+2,1-3,2} \tau_{x+1-3,2} \tau'_{x+1-2,3,2} \\ & \times \tau_{x-2,3,1} \tau'_{x+2-3,3} \tau_{x-3,3} \tau'_{x+2,3} \\ & + \tau_{x,2} \tau'_{x-2,1} \tau_{x+1,3} \tau'_{x-3,3} \\ & \times \tau_{x+1-3,3} \tau'_{x+1-2,3,2} \tau_{x-2,3,1} \tau'_{x-3,1} \\ & \times \tau_{x+2-3,1} \tau'_{x+2,2,3} \tau_{x+2,2} \tau'_{x-3,2} \} \\ & + 6 \sum_x [\tau_{x,2} \tau'_{x+2,3} \tau_{x-3,1} \tau'_{x+1-3,2} \\ & \times \tau_{x+1,3} \tau'_{x-2,1}]^2 \end{aligned} \quad (115)$$

Note that oppositely oriented graphs in (110)-(112) yield the same terms because they belong to transposed matrices. For any matrix of the form (104), (105) it holds true that

$$\text{Tr}(M^n) = (\text{Tr} M)^n \quad (116)$$

If a closed graph on the skeleton lattice gives a matrix of this type, running over the same contour twice gives the square of the trace for the simple graph. This is the origin of the last term in (115).

On the other hand, we have from (96) up to $k = 6$

$$\begin{aligned} a_1 &= a_2 = a_4 = 0 \\ a_3 &= + \frac{1}{6} \text{Tr}(\kappa\lambda)^3 \\ a_5 &= + \frac{1}{10} \text{Tr}(\kappa\lambda)^5 \\ a_6 &= - \frac{1}{12} \text{Tr}(\kappa\lambda)^6 + \frac{1}{2} \left[\frac{1}{6} \text{Tr}(\kappa\lambda)^3 \right]^2 \end{aligned} \quad (117)$$

so that this last term in (115) just cancels against the second term in a_6 . The numerical coefficients in a_k are always ± 1 .

If we draw the dual J lattice and represent in it each $\tau_{x,a}, \tau'_{x,a}$ by lines along the link (x,a) , then each term in a_k yields one or more closed graphs. This is related with the fact that non-admissible k fields give zero in (78), see also Section 9.

8. - THE 18j COEFFICIENT OF THE SECOND KIND

The 18j coefficient for the $SU(2)$ gauge model in $D = 3$ dimensions possesses a generating function that can be derived with the methods of Sections 6 and 7. In the graph of Fig. 4, we reorient links and vertices in such a fashion that the graph becomes graded. We enumerate the positive vertices only and label the links by (Fig. 8)

$$\begin{aligned} (t, a), \quad t &= 1, 2, \dots, 6 \\ a &= 1, 2, 3 \end{aligned}$$

where t stands for the positive vertices and a labels the links attached to vertex t . All vertices are understood to be oriented in the order 1,2,3 of their links.

The positive vertices contribute to the exponent of (79)

$$\sum_{t=1}^6 \begin{vmatrix} \tau_{t,1} & \tau_{t,2} & \tau_{t,3} \\ \xi_{t,1} & \xi_{t,2} & \xi_{t,3} \\ \eta_{t,1} & \eta_{t,2} & \eta_{t,3} \end{vmatrix}$$

which leads to a matrix λ (80) of the form

$$\lambda_{t,a; t',a'} = \delta_{tt'} \begin{pmatrix} 0 & +\tau_{t,3} & -\tau_{t,2} \\ -\tau_{t,3} & 0 & +\tau_{t,1} \\ +\tau_{t,2} & -\tau_{t,1} & 0 \end{pmatrix}_{aa'} \quad (118)$$

The negative vertices contribute to the same exponent

$$\sum_{t=1}^6 \begin{vmatrix} \tau'_{t+1,1} & \tau'_{t-1,2} & \tau'_{t,3} \\ \bar{\xi}_{t+1,1} & \bar{\xi}_{t-1,2} & \bar{\xi}_{t,3} \\ \bar{\eta}_{t+1,1} & \bar{\eta}_{t-1,2} & \bar{\eta}_{t,3} \end{vmatrix}$$

(addition for t is always modulo 6), which yields as non-vanishing submatrices for κ (80)

$$\begin{aligned} \kappa_{t,a; t+1,a'} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & +\tau'_{t+2,1} \\ +\tau'_{t-1,2} & 0 & 0 \end{pmatrix}_{aa'} \\ \kappa_{t,a; t-1,a'} &= \begin{pmatrix} 0 & 0 & -\tau'_{t-2,2} \\ 0 & 0 & 0 \\ 0 & -\tau'_{t+1,1} & 0 \end{pmatrix}_{aa'} \end{aligned} \quad (119)$$

and

$$\begin{aligned} \kappa_{t,a; t+2,a'} &= \begin{pmatrix} 0 & 0 & 0 \\ -\tau'_{t+1,3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{aa'} \\ \kappa_{t,a; t-2,a'} &= \begin{pmatrix} 0 & +\tau'_{t-1,3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{aa'} \end{aligned} \quad (120)$$

The κ matrix has obviously more non-vanishing entries than the κ matrix of Section 7. This makes the calculation of the coefficients a_k (89), (96) rather cumbersome. In fact, so far the expressions (118), (119), (120) can be used for any $6nj$ coefficient of the second kind ⁷⁾.

For the $18j$ coefficient we have $N_+ = 6$ and therefore to compute a_k up to $k = 6$. We give the results

$$a_1 = 0$$

$$a_2 = - \sum_{t=1}^6 \tau_{t,1} \tau'_{t-1,2} \tau_{t+1,2} \tau'_{t+2,1}$$

$$a_3 = - \tau_{1,3} \tau'_{2,3} \tau_{3,3} \tau'_{4,3} \tau_{5,3} \tau'_{6,3} \\ - \tau_{2,3} \tau'_{3,3} \tau_{4,3} \tau'_{5,3} \tau_{6,3} \tau'_{1,3}$$

$$+ \sum_{t=1}^6 \tau_{t,1} \tau'_{t-1,2} \tau_{t+1,3} \tau'_{t+3,1} \tau_{t+2,2} \tau'_{t+1,3}$$

$$a_4 = - \sum_{t=1}^3 \left[\tau_{t,1} \tau'_{t+1,3} \tau_{t+2,3} \tau'_{t+4,1} \tau_{t+3,1} \tau'_{t+4,3} \tau_{t+5,3} \tau'_{t+1,1} \right. \\ \left. + \tau_{t+1,2} \tau'_{t,2} \tau_{t+2,3} \tau'_{t+3,3} \tau_{t+4,2} \tau'_{t+3,2} \tau_{t+5,3} \tau'_{t,3} \right]$$

$$- \sum_{t=1}^6 \left[\tau_{t,3} \tau'_{t+1,3} \tau_{t+2,3} \tau'_{t+3,3} \tau_{t+4,2} \tau'_{t+3,2} \tau_{t+5,2} \tau'_{t+4,2} \right. \\ + \tau_{t,1} \tau'_{t+1,3} \tau_{t+2,3} \tau'_{t+3,3} \tau_{t+4,3} \tau'_{t,1} \tau_{t+5,1} \tau'_{t+1,1} \\ + \tau_{t,1} \tau'_{t+1,3} \tau_{t+2,3} \tau'_{t+3,3} \tau_{t+4,2} \tau'_{t+3,2} \tau_{t+5,3} \tau'_{t+1,1} \\ + \tau_{t,1} \tau'_{t+1,3} \tau_{t+2,2} \tau'_{t+1,2} \tau_{t+3,3} \tau'_{t+4,3} \tau_{t+5,3} \tau'_{t+1,1} \\ \left. + \tau_{t,1} \tau'_{t+1,3} \tau_{t+2,3} \tau'_{t+4,1} \tau_{t+3,2} \tau'_{t+2,3} \tau_{t+4,3} \tau'_{t-1,2} \right]$$

$$+ \sum_{(t,s) \in \mathcal{M}} \tau_{t,1} \tau'_{t-1,2} \tau_{t+1,2} \tau'_{t+2,1} \tau_{s,1} \tau'_{s-1,2} \tau_{s+1,2} \tau'_{s+2,1}$$

$$\mathcal{M} = \{(1,3), (1,4), (1,5), (2,4), (2,5), (2,6), (3,5), (3,6), (4,6)\}$$

$$a_5 = \sum_{t=1}^6 \left[\begin{aligned} &+ \tau_{t,1} \tau'_{t-1,2} \tau_{t+4,2} \tau'_{t+5,1} \tau_{t+1,3} \tau'_{t+1,3} \tau_{t+2,3} \tau'_{t+2,3} \tau_{t+3,3} \tau'_{t+3,3} \\ &- \tau_{t,1} \tau'_{t-1,2} \tau_{t+1,2} \tau'_{t+2,1} \tau_{t+2,1} \tau'_{t+1,2} \tau_{t+4,2} \tau'_{t+5,1} \tau_{t+3,3} \tau'_{t+3,3} \\ &- \tau_{t,1} \tau'_{t-1,2} \tau_{t+1,2} \tau'_{t+2,1} \tau_{t+3,1} \tau'_{t+2,2} \tau_{t+5,2} \tau'_{t+6,1} \tau_{t+4,3} \tau'_{t+4,3} \\ &- \tau_{t,1} \tau'_{t+1,2} \tau_{t+2,2} \tau'_{t+1,1} \tau_{t+5,1} \tau'_{t+2,2} \tau_{t+3,2} \tau'_{t+6,1} \tau_{t+4,3} \tau'_{t+4,3} \\ &- \tau_{t,1} \tau'_{t,1} \tau_{t+3,1} \tau'_{t+4,1} \tau_{t+4,1} \tau'_{t+5,1} \tau_{t+5,1} \tau'_{t+1,1} \tau_{t+2,3} \tau'_{t+1,3} \\ &- \tau_{t+2,2} \tau'_{t+1,2} \tau_{t+3,2} \tau'_{t+2,2} \tau_{t+4,2} \tau'_{t+3,2} \tau_{t+5,2} \tau'_{t+4,2} \tau_{t,3} \tau'_{t+1,3} \\ &- \tau_{t,1} \tau'_{t+1,1} \tau_{t+2,2} \tau'_{t+1,2} \tau_{t+3,2} \tau'_{t+2,2} \tau_{t+4,2} \tau'_{t+3,2} \tau_{t-1,3} \tau'_{t+1,3} \\ &- \tau_{t,1} \tau'_{t+1,1} \tau_{t+2,2} \tau'_{t+1,2} \tau_{t+4,1} \tau'_{t+5,1} \tau_{t+5,1} \tau'_{t,1} \tau_{t+3,3} \tau'_{t+1,3} \end{aligned} \right]$$

$$\begin{aligned} a_6 = & + \prod_{t=1}^6 (\tau_{t,3} \tau'_{t,3}) \\ & - \sum_{t=1}^6 \tau_{t,2} \tau'_{t,2} \tau_{t+1,1} \tau'_{t+1,1} \prod_{s \neq t, t+1}^6 (\tau_{s,3} \tau'_{s,3}) \\ & + \sum_{t=1}^6 \tau_{t,3} \tau'_{t,3} \tau_{t+1,3} \tau'_{t+1,3} \tau_{t+2,2} \tau'_{t+2,2} \\ & \quad \times \tau_{t+3,1} \tau'_{t+3,1} \tau_{t+4,2} \tau'_{t+4,2} \tau_{t+5,1} \tau'_{t+5,1} \\ & + \sum_{t=1}^3 \tau_{t,3} \tau'_{t,3} \tau_{t+3,3} \tau'_{t+3,3} \tau_{t+1,2} \tau'_{t+1,2} \\ & \quad \times \tau_{t+2,1} \tau'_{t+2,1} \tau_{t+4,2} \tau'_{t+4,2} \tau_{t+5,1} \tau'_{t+5,1} \\ & - \prod_{t=1}^6 (\tau_{t,1} \tau'_{t,1}) - \prod_{t=1}^6 (\tau_{t,2} \tau'_{t,2}) \\ & - \prod_{t=2,4,6} (\tau_{t,1} \tau'_{t,1}) \prod_{s=1,3,5} (\tau_{s,2} \tau'_{s,2}) \\ & - \prod_{t=2,4,6} (\tau_{t,2} \tau'_{t,2}) \prod_{s=1,3,5} (\tau_{s,1} \tau'_{s,1}) \end{aligned} \quad (121)$$

The coefficients a_2 to a_5 have been calculated by hand and checked by a computer, the coefficient a_6 has been calculated by computer alone. The computer calculations were done by means of REDUCE.

9. - CONCLUSION

We have treated $SU(2)$ invariant lattice field theories such as the $O(3)$ σ model for $D = 2$ and pure $SU(2)$ gauge theories for $D \geq 3$ dimensions by expanding each local contribution to the action into representation functions for spin j and then performing the integration over the group. There results a field of variables j on the lattice, a $3nj$ coefficient (n very big) $W(\{j\})$ that depends as a functional on the j field and dynamical factors $f(j, \beta)$ depending locally on the j field.

We proved that the W functional for gauge models can, after introducing some additional "kinematical" variables j , be locally decomposed

$$W(\{j\}) = \prod_{x' \in \Lambda^*} w_{x'}(\{j\}_{x'}) \quad (122)$$

where x' runs over the sites of the dual real space lattice Λ^* and $w_{x'}(\{j\}_{x'})$ is a $3nj$ coefficient with

$$n = 2D(D-2) \quad (123)$$

We applied Bargmann's formalism to the W functionals and $3nj$ coefficients. For the complete lattice of the $O(3)$ σ model ($D = 2$) or for each local factor w_x (122) we were able to prove the existence of a Schwinger-type generating function. For the $O(3)$ σ model in two dimensions we calculated it explicitly in the form of a power series (till the twelfth power), increasing powers involves variables with supports of growing distance. In this sense this expansion is "local". With computer aid it is certainly possible to push this series to much higher powers. The generating function for $w_{x'}(\{j\}_{x'})$ and $D = 3$ (122) was calculated exactly.

It is in principle possible to represent each W functional as a multiple sum over products of $6j$ coefficients. Whether the method of generating functions is more economic depends on the problem studied, the computer aid available, and can only be decided by trial.

The existence of generating functions is interesting in itself. In the computation of generating functions by hand we found that each monomial contained in any coefficient a_k (96) can be represented by one or several closed loops on the dual J lattice. This entails that we can introduce a local $U(1)$ gauge transformation on the dual lattice for the field variables τ, τ' , under which each monomial and consequently the generating function $H(\{\tau\})$ (78), (79) is invariant. For the $O(3)$ σ model this transformation can be found by inspection $[\zeta_{x,a} \in U(1)]$

$$\begin{aligned}\tau_{x,a} &\rightarrow \zeta_{x,a+1} \tau_{x,a} \zeta_{x,a-1}^{-1} \\ \tau'_{x,a} &\rightarrow \zeta_{x,a+1}^{-1} \tau'_{x,a} \zeta_{x,a-1} \\ a &= 1, 2, 3 \text{ cyclically} \\ \zeta_{x,1} &= \zeta_{x+3,2}, \zeta_{x,2} = \zeta_{x+1,3}, \zeta_{x,3} = \zeta_{x+2,1}\end{aligned}\tag{124}$$

Is it possible therefore to build the generating function into a necessarily complex measure over the field of $\tau_{x,a}, \tau'_{x,a}$?

Consider the generalized Villain model (1) for simplicity, and introduce for the kinematic domain of the j field the cut-off functions

$$f^*(j) = e^{-\frac{2j(j+1)}{\gamma}}\tag{125}$$

in which we may let γ tend to infinity at the end. We may then perform the sums

$$Z(\beta, \gamma) = \sum_{\{j, j'\}} \prod_i (2j_i + 1) f(j_i) \prod_k (2j_k + 1) f^*(j_k) W(\{j\})\tag{126}$$

by inserting the integral representation (78). However, the triangular coefficients Δ do not separate into factors depending on simple variables j or J and therefore the summation is not elementary.

We can evade this problem if we use the non-Schwinger-type generating function $G(\{\tau\})$ (75). It yields a W functional defined only on the dynamical domain. The summation over the j fields can be explicitly performed in the case of the generalized Villain model (1). It leads to a complex measure

$$G(\{\tau\}) \prod_{(D-2) \text{ cells } l} \theta(\tau_l, \beta) \quad (127)$$

where θ is a kind of Jacobi theta function. The problem is to compute G .

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APPENDIX - THE DUALITY TRANSFORMATION

The lattice Λ consists of sites

$$x = n_1 \underline{1} + n_2 \underline{2} + n_3 \underline{3} + \dots + n_D \underline{D}$$

Links starting at x and ending at $x + \underline{v}$ are denoted by $(x, \underline{v})$. Plaquettes spanned by the links $(x, \underline{\mu})$, $(x, \underline{v})$, $\underline{\mu} < \underline{v}$, are denoted by $(x, \underline{\mu}, \underline{v})$.

The dual lattice Λ^* has the same structure as Λ , denote its sites by x' . We may denote its links, plaquettes, etc., in the same fashion as for Λ . However, for our purposes another nomenclature is more practical. Denote a D cell spanned by the links

$$(x', \underline{v}), \text{ all } \underline{v}, \text{ by } [x'];$$

a $(D-1)$ cell spanned by the links

$$(x', \underline{v}), \text{ all } \underline{v} \neq \underline{\mu}, \text{ by } [x', \underline{\mu}];$$

a $(D-2)$ cell spanned by the links

$$(x', \underline{v}), \text{ all } \underline{v} \neq \underline{\mu}, \underline{v} \neq \underline{\lambda}, \underline{\mu} < \underline{\lambda}, \text{ by } [x', \underline{\mu}, \underline{\lambda}]$$

etc.

The duality transformation maps $\Lambda \leftrightarrow \Lambda^*$ so that

$$\begin{aligned} x &\longleftrightarrow [x'], \quad x = x' + \frac{1}{2} \sum_{\underline{v}} \underline{v} \\ (x, \underline{\mu}) &\longleftrightarrow [x' + \underline{\mu}, \underline{\mu}] \\ (x, \underline{\mu}, \underline{\lambda}) &\longleftrightarrow [x' + \underline{\mu} + \underline{\lambda}, \underline{\mu}, \underline{\lambda}] \end{aligned}$$

A D cell $[x']$ has as boundary negative $(D-1)$ cells $[x', \underline{\mu}]$, $\underline{\mu} = 1, 2, \dots, D$ and positive $(D-1)$ cells $[x' + \underline{\mu}, \underline{\mu}]$, $\underline{\mu} = 1, 2, \dots, D$. The outer normals of the positive (negative) $(D-1)$ cells are in the same (opposite) direction as the co-ordinate axis.

Similarly a $(D-1)$ cell $[x', \underline{\mu}]$ has as boundary negative $(D-2)$ cells $[x', \underline{\mu}, \underline{v}]$ (all $\underline{v} > \underline{\mu}$) or $[x', \underline{v}, \underline{\mu}]$ (all $\underline{v} < \underline{\mu}$); and positive $(D-2)$ cells $[x' + \underline{v}, \underline{\mu}, \underline{v}]$ (all $\underline{v} > \underline{\mu}$) or $[x' + \underline{v}, \underline{v}, \underline{\mu}]$ (all $\underline{v} < \underline{\mu}$).

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FIGURE CAPTIONS

- Fig. 1 The J lattice of the $O(3)$ σ model for $D = 2$ with the distorted co-ordinate frame of the RS lattice.
- Fig. 2 The $[x']$ D cell cut with the $(\mu\nu)$ plane, $\mu < \nu$. The plaquette of Λ is given which encircles the $(D-2)$ cell where the contraction is performed. The co-ordinates of the $(D-1)$ cells are always as in (a).
- Fig. 3 A graph of the $18j$ coefficient on $D = 3$ cell in Λ^* .
- Fig. 4 The graph of the $18j$ coefficient for $D = 3$ in planar form with co-ordinates.
- Fig. 5 The $48j$ coefficient for $D = 4$. Each blob contains a contraction of six lines by means of three internal J variables. Orientations are not drawn.
- Fig. 6 The planar hexagonal lattice with positive vertices $\oplus$ and the triangular co-ordinate frame.
- Fig. 7 The skeleton lattice with subgraphs possessing vanishing contributions.
- Fig. 8 The graded graph of the $18j$ coefficient of the second kind. Labels for positive vertices and all links are given.

- Figure 1 -

- Figure 2 -

- Figure 3 -

$12 : J[x', 2]$	$1'2' : j[x'+1+3, 13]$	$11' : j[x'+1, 12]$
$23 : j[x', 12]$	$2'3' : J[x'+3, 3]$	$22' : j[x'+3, 23]$
$34 : J[x', 1]$	$3'4' : j[x'+2+3, 23]$	$33' : j[x'+3, 13]$
$45 : j[x', 13]$	$4'5' : J[x'+2, 2]$	$44' : j[x'+2, 12]$
$56 : J[x', 3]$	$5'6' : j[x'+1+2, 12]$	$55' : j[x'+2, 23]$
$61 : j[x', 23]$	$6'1' : J[x'+1, 1]$	$66' : j[x'+1, 13]$

- Figure 4 -

- Figure 5 -

- Figure 6 -

- Figure 7 -

- Figure 8 -