INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS

GEOMETRIC ORIGIN OF CENTRAL CHARGES

J. Lukierski

and

L. Rytel

1981 MIRAMARE-TRIESTE
GEOMETRIC ORIGIN OF CENTRAL CHARGES

J. Lukierski† and L. Ryta†
International Centre for Theoretical Physics, Trieste, Italy

ABSTRACT

The complete set of $N(N-1)$ central charge generators for $D=4$, $N$-extended super-Poincaré algebra is obtained by suitable contraction of $\text{Osp}(2N; 4)$ superalgebra. The superspace realizations of the spinorial generators with central charges are derived. The conjugate set of $N(N-1)$ additional bosonic superspace coordinates is introduced in an unique and geometric way.

MIRAMARE - TRIESTE
May 1981

* To be submitted for publication.
† On leave of absence from the Institute of Theoretical Physics, University of Wroclaw, 50-205 Wroclaw, Poland.

1. The central charges were introduced originally in [1] as a mathematical possibility, but recently they appeared to be an important enlargement of the fundamental super-Poincaré algebra. Already the study of $N=2$ extended supersymmetry shows (see e.g. [2]) that in some cases only the presence of central charges allows for the existence of nontrivial (i.e. describing interesting theory) field representations. It is also believed that the difficulties with the formulation of $N \geq 3$ extended supergravities are related with the lack of proper understanding of the role of degrees of freedom generated by central charges.

The aim of this paper is to show that the central charge generators have a clear meaning as the contracted generators from the internal symmetry sector. If we assume that the linearly realized group on extended super-Poincaré generators is $\text{SL}(2, \mathbb{C}) \times U(N)$ the supergroup which provides such a contraction can be uniquely determined as $\text{Osp}(2N; 4)$. In our contraction scheme all the bosonic generators survive the contraction limit: the translation generators are obtained from the four noncompact generators describing $\text{SU}(1,1)$ and $\text{SU}(N-1)$ central charges are obtained from the compact generators in the coset $\text{O}(2N)/U(N)$. From such a construction it follows that the central charges transform under $\text{SU}(N)$ as antisymmetric second rank tensor $\mathbb{Z}^{ab} = X^{ab} + i Y^{ab}$, denoted by $(2,0)$.

We would like to mention here that central charges are scalars under the "exact" internal symmetry of the S-matrix, but can transform under spontaneously broken internal symmetry. One should assume therefore that the $\text{U}(N)$ symmetry of $N$-extended supersymmetry is spontaneously broken in the presence of central charges. The reduced "exact"symmetry group for different choices of central charge generators has been recently calculated by Ferrara, Savoy and Zumino [3].

2. Let us recall that the superalgebra $\text{Osp}(2N; 4)$ has the bosonic sector $\mathcal{g} = \mathcal{g}_1 \oplus \mathcal{g}_2$ where

$\mathcal{g}_1 = \text{Sp}(4, \mathbb{R}) \times \text{O}(3, 2)$ (space-time symmetries)

$\mathcal{g}_2 = \text{O}(2N)$ (internal symmetries)

and the fermionic sector is generated by $2N$ four-component Majorana charges.

* J. Lukierski, private communication.
† An alternative equivalent formulation uses $2N$ 2-component complex Weyl spinors (see e.g. [4]).
We introduce five $h \times h$ real matrices, describing the Majorana representation of $O(3,2)$ Clifford algebra

$$\{\gamma_\alpha, \gamma_\beta\} = 2\delta_{\alpha\beta} \quad \epsilon_{AB}^* = \text{diag} (-1,1,1,1,1,1)$$

The supercharges $Q^i_\alpha$, extending the Lie algebra $Sp(4, \mathbb{R}) \otimes O(2N)$ to the superalgebra $OSp(2N; \mathbb{R})$ satisfy the following relations:

$$\{Q^i_\alpha, Q^j_\beta\} = \delta^{ij} \left(\sigma^{\mu\nu} \gamma_\sigma\right)_{\alpha\beta} \Gamma_\mu \bar{\psi} +$$

$$+ m \left[\delta^{ij} \left(\sigma^{\mu\nu} \gamma_\sigma\right)_{\alpha\beta} \Gamma_\mu \bar{\psi} + g \left(\Gamma_\sigma \gamma_\rho\right)_{\alpha\beta} \bar{\psi} \right]$$

(1)

where $i,j = 1 \ldots 2N$, $\sigma, \rho = 1 \ldots 4$, $\Gamma_\mu = \frac{i}{2} (\gamma_\mu, \gamma_\rho)$ and

$$\gamma^{ij} = -\gamma^{ji} = -\gamma^{ji} \Gamma_\rho,$$ describe $O(2N)$ generators expanded in the adjoint $2N \times 2N$ matrix basis, denoted by $r_{ij} (r = 1 \ldots N(2N-1))$.

For dimensional reasons we introduced the mass parameter $m$ and dimensionless coupling $g$ characterizing respectively de Sitter radius and the Yang-Mills coupling in $O(2N)$ sector. The covariance relations look as follows (compare with (5)):

$$\frac{4}{i} \left[ P_\mu, Q^i_\alpha \right] = \frac{m}{2} \left(\Gamma_\mu \gamma_5\right)_{\alpha\beta} Q^i_\beta$$

$$\frac{4}{i} \left[ M_{\mu\nu}, Q^i_\alpha \right] = \left(\sigma^{\mu\nu}\right)_{\alpha\beta} Q^i_\beta$$

$$\frac{4}{i} \left[ T_\sigma, Q^i_\alpha \right] = g \gamma^{i\alpha\beta} Q^i_\beta$$

(2a)

(2b)

(2c)

The generator of "curved translations" $P_\mu$ satisfies $Z_2$-graded de Sitter algebra

$$\frac{4}{i} \left[ P_\mu, P_\nu \right] = -m^2 M_{\mu\nu}$$

(3)

3. Let us introduce the $Z_2$-grading in the internal symmetry sector by the following split of the $O(2N)$ Lie algebra

$$\begin{pmatrix} T^{ij} \end{pmatrix} \in O(2N): \quad \begin{pmatrix} A & S \\ -S & A \end{pmatrix} \oplus \begin{pmatrix} X & Y \\ -Y & -X \end{pmatrix}$$

(4)

where the $N \times N$ real matrices satisfy the relations:

$$A = -A^T \quad X = -X^T \quad S = S^T \quad Y = -Y^T$$

(5)

Further we introduce generalized Weyl-Majorana spinors

$$Q^\pm = \frac{1}{\sqrt{2}} \left(1 \pm \Gamma_5 \Omega\right) Q$$

$$\bar{Q}^{\pm} = -\bar{Q}^\pm$$

(6)

satisfying the condition $Q^\pm = \sigma^{\mu}_{\alpha\beta} Q^\pm$. Choosing $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ one can write

$$Q^\pm = \frac{1}{\sqrt{2}} \left(\begin{pmatrix} Q^\alpha & \mp \Gamma_5 Q^\alpha \pm N \\ \mp \Gamma_5 Q^\alpha \mp N & Q^\alpha \pm N \end{pmatrix}\right)$$

(7)

The unconstrained $N$-component Majorana spinor describing $Q^\pm$ we denote by

$$Q^\pm_\alpha (\alpha = 1 \ldots N):$$

$$Q^\pm_\alpha = \frac{1}{\sqrt{2}} \left(Q^\alpha \pm \Gamma_5 Q^\alpha \pm N\right)$$

(8)

The commutation relations (1) split as follows ($\alpha' = \alpha m$):

$$h \times h$$

-1-
\[
\{ Q_{\alpha}^{a}, Q_{\beta}^{b} \} = \delta^{ab}(\Gamma R_{\alpha} \Gamma_{\beta})_{d} P_{\mu} + m\left[\Gamma_{d}^{\alpha} \Gamma_{\beta} \chi^{ab} + (\Gamma_{d}^{\alpha} \Gamma_{\beta} \Gamma_{5} \Gamma_{d} \gamma^{ab})_{\rho\sigma} \right]_{\rho\sigma}
\]
\[
\{ Q_{\alpha}^{-a}, Q_{\beta}^{-b} \} = m\delta^{ab}(\Sigma_{U} \Gamma_{1} \Gamma_{5} \Gamma_{d} \rho \mu \nu \lambda + m[\Gamma_{d}^{\alpha} \rho \mu \nu \lambda a b + (\Gamma_{d}^{\alpha} \Gamma_{5} \Gamma_{d} \gamma^{ab})_{\rho\sigma} ]_{\rho\sigma}
\]
\[
\{ Q_{\alpha}^{a}, Q_{\beta}^{-b} \} = \delta^{ab}(\Gamma R_{\alpha} \Gamma_{\beta})_{d} P_{\mu} + m\left[\Gamma_{d}^{\alpha} \Gamma_{\beta} \chi^{ab} - (\Gamma_{d}^{\alpha} \Gamma_{\beta} \Gamma_{5} \Gamma_{d} \gamma^{ab})_{\rho\sigma} \right]_{\rho\sigma}
\]

The relations (2a-2b) are valid also for \( Q_{\alpha}^{a} \), but the relation (2c) remains only valid for the generators belonging to \( O(8) \) subgroup of \( O(2N) \) (see (4)).

Let us observe that

\[
\begin{align*}
\{ \tau_{\alpha}, \Omega \} &= 0 \quad \text{iff} \quad \tau_{\alpha} \in U(N) \\
\{ \tau_{\alpha}, \Omega \} &= 0 \quad \text{iff} \quad \tau_{\alpha} < \frac{O(2N)}{U(N)}
\end{align*}
\]

(10)

Writing \( x^{ab}, y^{ab} = \frac{1}{R} (x^{ab} + y^{ab}) \) \( (K = 1 \ldots \frac{N(N+1)}{2}) \) one gets

\[
\begin{align*}
\frac{1}{\mu} \left[ x^{ab}, Q_{\alpha}^{a} \right] &= \frac{1}{\mu} \tau_{\alpha} \chi^{ab} Q_{\alpha}^{b} \\
\frac{1}{\nu} \left[ y^{ab}, Q_{\alpha}^{a} \right] &= \frac{1}{\nu} \tau_{\alpha} \chi^{ab} (\Gamma_{5} \gamma^{ab})_{\rho\sigma}
\end{align*}
\]

(11)

It is easy to check that the whole superalgebra \( \text{OSP}(2N, \gamma) \) can be decomposed into the following four sectors

\[
\begin{align*}
L_{0} & \quad L_{4} & \quad L_{2} & \quad L_{3} \\
M_{\mu\nu} & \quad A^{ab}_{\gamma} & \quad Q_{\alpha}^{a} & \quad S^{ab}_{\gamma}
\end{align*}
\]

(12)

and satisfies the \( \mathbb{Z}_{4} \) grading relations\(^{*}\)

\[< L_{i}, L_{j} > < L_{i+j} \quad (i, j, i+j \mod 4)\]

(13)

where \( L_{0}, L_{2} \) are bosonic and \( L_{1}, L_{3} \) fermionic sectors. The grading (13) implies that one can perform the rescaling

\[
L_{0} \rightarrow L_{0} \quad L_{1} \rightarrow \frac{4}{\sqrt{R}} L_{1} \quad L_{2} \rightarrow \frac{4}{R} L_{2} \quad L_{3} \rightarrow \frac{4}{\sqrt{R}} L_{3}
\]

(14)

and consider the contraction limit \( R \rightarrow \infty \) in consistency with the Jacobi identities. The rescaled generators satisfy the relations (9a), (9c), but we obtain \( (x^{ab}, y^{ab}) \sim \frac{1}{R} \) and also \( (X_{K}, Q_{\alpha}^{a}) \sim \frac{1}{R} \), \( (X_{K}, Q_{\alpha}^{a}) \sim \frac{1}{R} \). The \( O(2N) \) superalgebra is split by the grading (13) into \( \mathbb{Z}_{4} \)-graded symmetric Riemannian pair \( (U(N), \Omega^{(N)}) \) with the generators from \( O(2N) \) transforming under \( U(N) \) as antisymmetric 2-tensor \( (2, 0) \). One can also write the algebra of de Sitter type (see (3)) describing the internal symmetry sector. One gets

\[
\left[ X_{K}, X_{L} \right] \sim \frac{4}{R^{2}} U(N) \quad \left[ X_{K}, Y_{L} \right] \sim \frac{4}{R^{2}} U(N)
\]

\[
\left[ Y_{K}, Y_{L} \right] \sim \frac{4}{R^{2}} U(N)
\]

(15)

where \( U(N) \) denotes the element of \( U(N) \) algebra.

\(^{*}\) The \( \mathbb{Z}_{4} \) gradings were introduced in \([6]\) for the description of supersymmetric generalization of a Riemannian symmetric pair \( (R, \mathbb{S}) \): so-called super-Riemannian quadruple. The \( \mathbb{Z}_{4} \) gradings of physically interesting supergroups were discussed by the authors in \([7]\), where also the grading (12) is given. Already in \([7]\) the relation of internal symmetry generators from sector \( L_{0} \) with central charges in the contraction limit \( R \rightarrow \infty \) was pointed out.
It is easy to see therefore that in the contraction limit \( R \to \infty \) the \( K_{N(N-1)} \) generators \( X_k, Y_k \) become Abelian ones, and in the extended super-Poincaré algebra they appear as the central charges. Because if \( R \to \infty \) the sectors \( Q^a_+ \) and \( Q^a_- \) decouple, one can restrict the fermionic sector to \( L_1 \) (or \( L_2 \)).

If we denote \( Q^a_\pm \equiv a^a \), one gets in the limit \( R \to \infty \) from the sectors \( (L_0, L_1, L_2) \) the \( D = 4 \) extended super-Poincaré algebra with \( K_{N(N-1)} \) real central charges.

Before contraction in the \( \text{OSp}(2N|4) \) superalgebra the generators \( X_k, Y_k \) describe nonabelian or curved "precentral" charges; in order to obtain the conventional description of central charges one has to perform two steps:

a) the contraction limit. The generators become Abelian, but still do not commute with internal symmetry sector \( U(N) \) (they can be called Abelian "precentral" charges);

b) reduction of internal symmetry group \( U(N) \) to its subgroup commuting with \( a^a \). Such a reduction for different choices of the representations with nontrivially represented \( X_{ab}, Y_{ab} \) was discussed recently in [5]; see also [8].

It is already known (see e.g. [9]) that the superspace realization of supersymmetries with central charges require additional bosonic coordinates of superspace. In particular one can obtain central charges from the translations in extra bosonic dimensions [10,11] but in such a case one introduces "big" Lorentz group \( O(D-1,1) \) (\( D \times 3 \)) and the breaking mechanism into physical space-time and internal sector is not justified. In the derivation of central charges presented here the additional bosonic variables acquire a clear geometric meaning: they describe complex "curved" translations on the coset \( O(D|2) \), and in the limit \( R \to \infty \) one obtains \( K_{N(N-1)} \) additional bosonic variables \( U_{ab}, V_{ab} \) (\( U_{ab} = -U_{ba}, V_{ab} = -V_{ba} \)), with their translations generated by the central charge generators.

The \( N \)-extended superspace variables before the contraction parameterise the following coset

\[
K = \frac{\text{OSp}(2N|4)}{\text{SL}(2;\mathbb{C}) \times U(N)}
\]

Using the exponential parameterisation one can write

\[
K = \exp \left( X^\mu P^\mu + Q^a_+ \tilde{a}^a_+ + Q^a_- \tilde{a}^a_- \right) \exp \left( X_{ab} U^{ab} + Y_{ab} V^{ab} \right)
\]

Applying well-known techniques of nonlinear realisations on coset spaces (see e.g. [5,12]) one gets the transformation formulae of the super-space coordinates \( (X_k, \tilde{a}^a_+, \tilde{a}^a_-, U^{ab}, V^{ab}) \) e.g. under the supertranslation \( \exp (Q^a_+ c^a + a^a \theta) \). Using the rescaled superalgebra, one can calculate the transformations up to order \( \frac{1}{R} \). Describing the change of superfield coordinates as the result of the action of the superspace generator \( c^a \theta^a \), one gets

\[
Q^a_+ = \frac{2}{N} \tilde{a}^a_+ - \frac{1}{2} (\tilde{\theta} a^a \Gamma^a) \chi + m^a \tilde{a}^a_+ + O \left( \frac{1}{R} \right)
\]

where we did put \( \theta^a_\theta = \delta^a_\theta \). Calculating the anticommutator of the generators (19) one gets

\[
\{ Q^a_+, Q^b_+ \} = \delta^{ab} \Gamma^a \chi_\chi + 2m^a (\chi \chi^a + \chi \chi^a) + O \left( \frac{1}{R} \right)
\]

The formula (20) in the limit \( R \to \infty \) coincides with the extension of Salam-Strathdee superspace realization of \( K \)-extended super-Poincaré algebra with \( K_{N(N-1)} \) central charges.

We would like to add the following remarks:

(a) We learned when our work was finished that Howe, Lindström [13] and Kalosh [14] introduced additional 56 bosonic coordinates for \( N = 8 \) supergravity in order to incorporate into the curved superspace formalism the \( K_{8} \).

(b) The formula (19) in the limit \( R \to \infty \) was also presented by J.G. Taylor during the supergravity workshop at ICTP, Trieste (4-6 May 1981).
"hidden symmetry" group. In our framework these additional variables are determined uniquely, and one can show that they carry the linear realizations of earlier discovered hidden symmetry $E_7^*$ (see e.g. [15]). At present we are not able to determine which "hidden symmetries" occur at given $N$ using purely geometric arguments, however a look on the additional variables and linear realizations of hidden symmetries at any given $N$ leads to the conclusion that they are "well fitted" to each other.

(b) The additional bosonic coordinates can be used for the definition of mass operator which depends on the Calabrese internal and hidden symmetry groups. These coordinates transform only under internal symmetries, and represent a new version of "old" isotopic space concept.

(c) We discuss in this note only rigid extended supersymmetries. Our way of introducing central charges if applied to supergravity requires local gauging of $O(N; h)$ symmetry. The methods developed by McDowell (see e.g. [16]) and by supergroup approach (see e.g. [17]) should be useful in deriving the supergravity actions with central charges.

(d) Because the number of new bosonic coordinates increases quadratically with $N$, it makes reasonable to look for composite superspaces, by supermetric extension of Penrose framework with "elementary" twistors and "composite" space-time coordinates. The outline of such a generalization, with more details provided for $N=1$, has been presented recently by one of the authors [18].

ACKNOWLEDGEMENTS

The authors would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for the cordial hospitality at the International Centre for Theoretical Physics, Trieste and for making our visit possible. We would also like to acknowledge the discussions with E. Cremmer and B. Milewski.

The results of this paper were presented during the Supergravity Workshop at ICTP, Trieste (4–6 May 1981).

REFERENCES

C. MUKKU and W.A. SAYED - $G(5) \times U(1)$ electroweak theory.


H.O. DIMITROV - On the one-electron theory of crystalline solids in a homogeneous electric field.


I. CAPRII, I. GIULISI and E.K. BÂDĂSCU - Model dependent dispersion approach to proton Compton scattering.

I. GIULISI and E.K. BÂDĂSCU - Sum-rule inequalities for pion polarisabilities.

K. TAHIR SHAH - Breakdown of predictability in an investigation on the nature of singularities.

S. NARISHI - QCD sum rules for the gluon component of the U(1)$_A$ pseudoscalar meson.

Y. FUKUMOTO - SU(18) unification.

RAJ.K. GUPTA, RAM RAY and S.B. KHANNAI - Proximity potential and the surface energy part of the microscopic nucleus-nucleus interaction with Scyrne force.

MAR QI-ZHI - The finite-dimensional star and grade star irreducible representations of SU(n).


D.K. CHATTURVEE, M. ROVERE, G. SENATORE and M.P. TOSI - Liquid alkali metals and alloys as electron-ion plasmas.

P. CORSERO - Ambiguities of functional integrals for fermionic systems.

W. MECKLEMBURG - Towards a unified picture for gauge and Higgs fields.

K.S. WIELKE - Outline of a nonlinear, relativistic quantum mechanics of extended particles.


M.NARAK ARMAD - Symmetric structures of coherent states in super fluid helium-4.

H.S. JAMULAR AND A.W.M. MAHOUR - Duffin-Kemmer formulation of spin one-half particle gauge theory.

M.R. DAPAF - Study of high spin states in nuclei within the framework of a single particle model.

C.C. GHIRARDI, F. MIGLIETTA, A. RIMINI and T. WEBER - Limitations on quantum measurements III. Analysis of a model example.