KRONECKER PRODUCTS FOR $SO(2p)$ REPRESENTATIONS

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ABSTRACT

A relatively simple algorithm for the decomposition of the product of two $SO(2p)$ representations is presented. For this purpose, generalized Young tableaux are introduced and their product defined.

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1. INTRODUCTION

There is today a renewal of interest by orthogonal groups in particle physics in the context of Grand Unified Theories\(^1\). Among the appealing features that the SO(n) groups present, let us mention the absence of anomaly in their representations, and the property of SO(4n+2) groups to possess complex conjugate representations. A popular model is given by the SO(10) group which contains a 16 dimensional spinorial representation fitting exactly 16 left-handed elementary fermions of one family. Moreover, models built from SO(n), n > 10 groups have also been proposed in order to encompass more than one family of fermions\(^2\). For the construction of Grand Unified Models, one needs to know very carefully the mathematical properties of SO(n) Lie algebras as well as of their representations. In particular, the knowledge of the decomposition into irreducible representations of the Kronecker product of two SO(n) representations, which is in itself of mathematical interest, is very useful for the gauge model builder. The reduction of the Kronecker product of representations of O(n) groups has been studied by King\(^3\) who simplified and generalized the pioneering work of Murnagan\(^4\) and Littlewood\(^5\) based on the character theory and Schur functions. The formulae so obtained\(^3\) are simple and amenable for practical calculations. Unfortunately, they cannot be used for the Kronecker product of SO(2n) representations since an irreducible representation (IR) of O(2n) may split into two irreducible ones under the restriction to SO(2n). This is not the case for orthogonal groups of odd order, the IR of O(2n+1) being the IR of SO(2n+1).

With the introduction of difference characters\(^4\), Butler and Wybourne\(^6\) were able to make an extension of Littlewood's results to the SO(2n) case, at the price of rather complicated algorithms. During the completion of our work we received a preprint by Dehuai, Wybourne and King\(^7\) in which the results obtained in Refs. 6 and 3) are presented in a systematic way. Let us mention another attempt to solve this problem recently proposed by Fischler\(^8\) based on Young tableau methods for classical groups: unlike for SU(n) groups, this method is not straightforward and generally leads to ambiguous results. In any case, the solutions proposed up to now for analyzing the Kronecker product of SO(n) representations require either the introduction of unusual objects (for the physicist) like Schur functions or complicated ways of handling usual objects like Young tableaux.

The method we propose hereafter is obtained in a rather different way and seems to us simpler to use. Actually our method reduces to calculate, given two SO(2n) IR, a finite number of products of generalized Young tableaux (GYT). As will be defined, a GYT is a tableau which can include "negative" boxes. The product
of two GYT's can be seen as a natural extension of the usual product of two SU(n) Young tableaux. Another nice feature of this technique lies in the fact that the rules for products involving vector or spinor representations are essentially the same.

2. REPRESENTATIONS OF THE O(n) AND SO(n) GROUPS

The Lie algebra of the \( O(n) \) group can be realized with the help of \( n(n-1)/2 \) infinitesimal generators \( J_{ij} = J_{ji} \), \( J_{ij} = \epsilon_{ij} \) which satisfy the commutation relations:

\[
\left[ J_{ij}, J_{k\ell} \right] = i \left( \delta_{\ell j} J_{ik} - \delta_{k j} J_{i\ell} + \delta_{i k} J_{\ell j} - \delta_{\ell k} J_{i j} \right)
\]  (2.1)

The Cartan (maximal Abelian) subalgebra can be chosen as generated by the \( p \) commuting generators \( J_{12} , J_{2j-1,2j} , \ldots , J_{2p-1,2p} \) if \( n = 2p \) or \( n = 2p + 1 \), the eigenvalues of which, in a given irreducible representation, will yield the weight components.

Any irreducible representation (IR) of the \( O(n) \) covering group can be labelled by the components of its greatest weight, i.e., by a set of \( p \) numbers \( \{ m_1, m_2, \ldots, m_p \} \), satisfying \( m_1 \geq m_2 \geq \ldots \geq m_p \geq 0 \). For a given representation, these numbers are integers (true representations) or all half integers (spin representations).

Let us now remind ourselves that any irreducible \( O(2p+1) \) representation is irreducible under \( SO(2p+1) \). This is also the case for an \( O(2p) \) representation with the last component of its greatest weight vanishing \( (m_p = 0) \). The situation is rather different if the last index \( m_p \) is non-vanishing. Indeed, in this last case, the \( O(2p) \) representation labelled by \( \{ m_1, \ldots, m_p \} \) splits into two IR of \( SO(2p) \) with the corresponding greatest weight \( \{ m_1, \ldots, m_p \} \) and \( \{ m_1, \ldots, -m_p \} \): two such representations are called conjugate.

Another way of labelling an \( SO(n) \) representation, which appears naturally in Cartan's construction, is often used. An \( SO(n) \) representation is then characterized by \( p \) non-negative integers related to the \( m_i \)'s by the relations:

\[
\begin{align*}
q_i &= m_j - m_{j+1} \\
q_p &= m_{p-1} + m_p
\end{align*}
\]  \( \{ j = 1, 2, \ldots, p-1 \} \)  \( q_p = \ell p \)}}
and

\[
\begin{aligned}
q_j' &= m_j - m_{j+1}, \\
q_p &= \ell m_p
\end{aligned}
\]  

(\(j = 1, 2, \ldots, p-1\)) \quad \text{if} \quad n = 2p+1

Let us conclude this short section by recalling that the SO(4\(v\)+2) representations \((m_1, \ldots, m_{2\nu+1})\) with \(m_{2\nu+1} \neq 0\) are the only SO(n) representations which are complex - actually \((m_1, \ldots, m_{2\nu+1})\) is complex conjugate to \((m_1, \ldots, -m_{2\nu+1})\). All the SO(4\(v\)+2) representations with \(m_{2\nu+1} = 0\), as well as all the representations of SO(4\(v\)) or SO(2p+1), being real or quaternionic real.

3. GENERALIZED YOUNG TABLEAUX AND THEIR PRODUCT

With the O(n) IR \((m_1, \ldots, m_p)\) \(- n = 2p \) or \( n = 2p + 1 \) - it is possible to associate a Young diagram with \(m_i\) boxes in the \(i\)th row, \(i = 1, 2, \ldots, p\) if the considered IR is a true one, or with \(m_i - 1/2\) boxes in the \(i\)th row if the IR is a spinorial one.

What about an SO(2p) IR? If the last index \(m_p \geq 0\) there is of course no problem, but with its conjugate representation, if \(m_p \neq 0\), we will associate a new type of Young tableau, the last row of which being called a "negative row":

\[
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
m_1 \\
m_p \\
\end{array}
\]

Such a tableau will be called the generalized Young tableau or GYT associated with the SO(2p) representation \((m_1, \ldots, m_p)\). In the algorithm we will propose, we are also led to define a wider class of GYTs, i.e., in the context of SO(2p) groups, tableaux associated with the ordered set of positive, null and negative integers \([\alpha_1, \ldots, \alpha_p]\) satisfying \(\alpha_1 \geq \alpha_2 \geq \ldots, \geq \alpha_p\), the introduction of such tableaux being related to the study of the different weights in an SO(2p) IR. In the following, we shall denote such a GYT by \([\ldots]\). Let us insist on the fact that we can only associate an SO(2p) representation with such a GYT if \(\alpha_1 \geq \ldots, \geq \alpha_{p-1} \geq |\alpha_p| \geq 0\). As an example, in SO(8) the GYT \([1, 0, -1, -2]\) will be represented as follows:
Finally, let us point out that an SO(2p) GYT cannot have more than p rows.

Now let us concentrate on the way of defining the product of two GYTs. Actually, the general product we will have to solve in our method will be that of a completely arbitrary GYT \([\alpha_1, \ldots, \alpha_p]\), \(\alpha_1 \geq \ldots \geq \alpha_p\), by a classical Young tableau \([m_1, \ldots, m_p]\), \(m_1 \geq \ldots \geq m_{p-1} \geq m_p > 0\). Therefore, we will restrict our study to this case. If the product we have to consider concerns two tableaux with only positive rows (classical Young tableaux), then the product law will be the product law of two SU(p) Young tableaux, the only difference being that an SO(2p) GYT with p "positive rows" \([m_1, \ldots, m_p]\) is not equivalent to the simplified one \([m_1 - m_p, \ldots, m_{p-1} - m_p, 0]\).

In the general case we have to study, the multiplication law will be a direct generalization of the product of two SU(n) Young tableaux and could be summarized as follows:

- Call the boxes in the first line \(a\), those of the second line \(b\) and so on up to the \(p\)th line of the GYT \(\mathcal{G}[m_1, \ldots, m_p]\) satisfying \(m_1 \geq \ldots \geq m_{p-1} \geq |m_p| > 0\).

- Add to the other GYT \(\mathcal{G}'[\alpha_1, \ldots, \alpha_p]\) with \(\alpha_1 \geq \ldots \geq \alpha_p\) one box \(a\) of \(\mathcal{G}\) using all different ways so that one always gets a GYT. Note that the box \(a\) added to the negative row of \(\mathcal{G}\) will cancel the box furthest left in this row. Then add a second \(b\) to the obtained tableaux and so on using the usual SU(n) prescriptions.

As an illustration, let us consider the following product relative to SO(6):

\[
[1,1,-1] \times [3,2,1] = [4,1,0] + [3,2,0] + [3,1,1] + [2,2,1]
\]
Notice that the tableau

\[
\begin{array}{c}
\alpha \\
\beta \\
\gamma
\end{array}
\]

does not exist since, before adding the \( b \) boxes, we would get the tableau

\[
\begin{array}{c}
\alpha \\
\gamma
\end{array}
\]

i.e., \([3, -1, 0]\) which is not a GYT.

4. RULES FOR THE PRODUCT OF TWO \( \text{SO}(2p) \) REPRESENTATIONS

In the analysis of the Kronecker product of two \( \text{SO}(2p) \) IR we need to consider a subset of GYT of crucial usefulness. These are GYT with null and/or negative \( p \) labels, say

\[
\left[ o, o, \ldots, o \right] \ ; \left[ o, o, \ldots o, \ell, \ell \right] ; \left[ o, o, \ldots o, \ell, -\ell \right] ; \left[ o, o, \ldots o, \ell, -\ell, \ell \right] \ ; \left[ o, o, \ldots o, \ell, -\ell, -\ell, \ell \right] 
\]

We shall denote them in the following way:

\[
L_p^{\mathfrak{L_1}} \left( \alpha_i \right) = \left[ o, \ldots, o, \left( -1 \right)^{\nu_1}, \left( -1 \right)^{\nu_2}, \ldots \left( -1 \right)^{\nu_k} \right] \quad \text{ (4.1)}
\]

where \( \left\{ \alpha_i \right\} \) is a set of even or null integers which indicate the multiplicity of \( \left( -1 \right) \) and satisfy \( \sum_{i=1}^{k} \alpha_i = 2k \). The lower index \( p \) just reminds us of the total number of allowed labels given the rank of the group under consideration. Given the fact that the \( \alpha_i \) are even, the \( \left( -1 \right) \) are repeated an even number of times and if \( p \) is odd the first label is always \( 0 \). In any case, the number of \( 0 \) is easily deduced to be \( p - \sum_{i=1}^{k} \alpha_i \). As an example, in the \( \text{SO}(8) \) case we have:

\[
L_4^{\mathfrak{L_1}} \left( 2 \right) = \left[ 0, 0, 1, -1 \right] \ ; \ L_4^{\mathfrak{L_1}} \left( 4 \right) = \left[ -1, 1, -1, -1 \right] \ ; \ L_4^{\mathfrak{L_1}} \left( 0, 2 \right) = \left[ 0, 0, 1, 1 \right]
\]
Let us now consider the product of two IR of $SO(2p)$, $[m_1, \ldots, m_p]$ and $[n_1, \ldots, n_p]$. If $m_p$ and $n_p$ are both negative, it is simpler to make the product $[m_1, \ldots, |m_p|]$ $[n_1, \ldots, |n_p|]$ and then replace each term of the result by its conjugate.

Also, as noted in the preceding section, there are two types of representations of the orthogonal groups: the true representations (T) and the spin representation (S); then there are three types of products to be considered:

$$T_a \otimes T_b = \bigoplus_{i \in \mathbb{Z}} T_i \quad S_a \otimes S_b = \bigoplus_{i \in \mathbb{Z}} S_i \quad T_a \otimes S_b = \bigoplus_{i \in \mathbb{Z}} S_i$$

With an $S$ type representation say $[m_1, \ldots, m_p]$ where $m_1$ are half integers, one associates a tableau $[\mu_1, \ldots, \mu_p]$ with $\mu_1 = m_1 - 1/2$. Given this procedure, in the case $T_a \otimes S_b$ one must add $1/2$ to each label of each term in the product decomposition, whereas in the case $S_a \otimes S_b$ one adds one unit in the same way to obtain the final result.

Apart from this proviso the three types of product can be made with the same rules.

The general formula can be written in the following compact form for the product of the IR $[\mu] \otimes [\nu]$ of $SO(2p)$ with $(m_p > 0, n_p > 0)$; (the case $n_p < 0$ will also be deduced from this formula; (see further on):

$$[\gamma] \otimes [\delta] = [\gamma_1, \ldots, \gamma_p] \otimes [\delta_1, \ldots, \delta_p] =$$

$$\sum_{Q} \sum_{i \in \mathbb{Z}} \left\{ \left( L_{\mu}^\delta (\nu_i) \otimes [\gamma] \right)_A \otimes [\gamma] - \left( L_{\mu}^\delta (\nu_i) \otimes [\gamma] \right)_{AM} \otimes [\gamma] \right\}$$

(4.2)

$$Q = \sum_{i \in \mathbb{Z}} n_i : \text{for SO}(4q) \text{ and } SO(4q+2).$$

$$L_{\mu}^\delta = [\cdots, - (n_1 + n_2), - (n_1 + n_2), \cdots]$$

$$\sum_{i \in \mathbb{Z}} n_i \leq \min (\text{number of non zero labels in } [\mu] \text{ and } [\nu]) + 1$$

$\nu_1$ and $\nu_1$ are integers defined by $\mu_1 = m_1$, $\nu_1 = n_1$ (if $[\mu]$, $[\nu]$ are of T type), and $\mu_1 = m_1 - 1/2$, $\nu_1 = n_1 - 1/2$ (if $[\mu]$, $[\nu]$ are of S type).
In the above formulae we assume that $p_i$ is such that $\sum_{i=1}^p n_i = N \leq M = \sum_{i=1}^p n_i$ and that if $M = N$ we have $n_p = m_p$; if $m_p = n_p$ then $m_{p-1} = m_{p-1}$ and so on. The subscripts $A$ and $\Lambda$ on the brackets mean that in the product of the GYT's one only keeps the terms which fulfill some conditions fixed by $[n]$ and $[m]$. In $\{\}_{A}$ one keeps only the "allowed" terms $[\lambda]$ defined in the following way:

i) $\sum_{i=1}^p |\ell_i| \leq N$ ($\ell_i = \lambda_i + 1/2$ S type), and $|\ell_i| \leq n_i$.

ii) If $\ell_1$ or $|\ell_p|$ is equal to $n_1$, $[\lambda]$ must not contain any label $\lambda_j$ such that $|\ell_j| \geq n_2$. If one of the $\ell_j$ satisfies $|\ell_j| = n_2$, then $[\lambda]$ must not contain any label $\lambda_k$ such that $|\ell_k| \geq n_3$, and so on.

iii) if $n_1 = \ell_1$, or $n_1 = |\ell_{p-i+1}|$, $i = 1, 2, \ldots, p$ one must have $n_{p-i} = n_1 = \ell_{p-i+1}$.

Note that an allowed GYT $[\lambda]$ such that for each $\lambda_j$ there is a $n_j = |\ell_j|$ has to be considered only once for a fixed value of $k$ in $\ell_p(\{n_j\})$.

Finally, in the product $\{\}_{\Lambda}$ one keeps only the "non-allowed" GYT's.

The last prescriptions are:

i) in the final result one will keep only the GYT's $[\rho]$ which can be associated with an $SO(2p)$ IR which is such that $\rho_1 \geq \rho_2 \geq \ldots \geq |\rho_p|$;

ii) IR which appear in the second term and not in the first term of the right-hand side of Eq. (4.2) have to be omitted.

Equation (4.2) looks very complicated at first sight, but it simplifies in practice. Let us remark that the second term for subtraction does not generally appear when $m_p \neq 0$ and that it can appear only for

$$m_{p-1} + m_p < k \quad (m_p > 0).$$

In particular, very compact formulae can be given in the cases of two completely symmetric, or two completely antisymmetric $SO(2p)$ representations if $m \geq n$:

$$SO(2p) \left[ m, 0, \ldots, 0 \right] \otimes \left[ n, 0, \ldots, 0 \right] = \sum_{k=0}^n \sum_{k} m+n-k-2k, k, 0, \ldots, 0 \right]$$

where $k$ satisfies $m + n - 2k \geq 2k > 0$ and $(n-k) \geq k$. (4.3)
\[ SO(4q) : \left[ m, m, \ldots, m \right] \otimes \left[ n, n, \ldots, n \right] = \sum_{\{k_i\}} [m+n-k_1, m+n-k_2, m+n-k_{q-1}, \ldots, m+n-k_q, m+n-k_q] \]

with the set \( \{k_i\} \) satisfying \( 2n \geq k_q \geq k_{q-1} \geq \ldots \geq k_1 \geq 0 \).

\[ SO(4q+2) : \left[ m, m, \ldots, m \right] \otimes \left[ n, n, \ldots, n \right] = \sum_{\{k_i\}} [m+n, m+n-k_1, m+n-k_2, \ldots, m+n-k_q] \]

with \( \{k_i\} \) again satisfying \( 2n \geq k_q \geq \ldots \geq k_1 \geq 0 \).

Let us consider the case in which one IR, let us say \( [n] \), has the last label negative, \( n_p < 0 \). We can, without loss of generality, assure \( \sum_{i=1}^{2p} n_i = 2N < M \). In this case we compute the product

\[ L_p^{2k}(k_i) \otimes [\nu_1, \ldots, \nu_p]^c \]

where \([ ]^c\) is the GYT associated with the representation conjugate to the representation \([n]\), i.e., with \( n_p > 0 \). In the final result we change all the signs of the obtained GYTs, rearrange them in ordered form and then insert them into the right-hand side of Eq. (4.2). This works for \( SO(4q+2) \), whereas for \( SO(4q) \) before the insertion, one has to subtract \( 2n_{2q} \) from the last label of the rearranged GYTs obtained in (4.6). If \([n]\) is a spinorial representation, the change of sign has to be performed after adding \( 1/2 \) to each label.

In order to illustrate the method, let us calculate in \( SO(10) \) the Kronecker product: \([1111]\) \( \otimes \) \([3/2 1/2 1/2 1/2 1/2]\) of respective dimensions 126 and 144. We shall operate on the spinorial representation, since \( N = 7/2 < M = 5 \), and consider the product \([\mu] \otimes [\nu]\) with \( \mu_i = 1 \) \((i = 1, 2, \ldots, 5)\) and \( \nu_1 = 1, \nu_2 = \ldots, \nu_5 = 0 \). Since the quantity \( Q = \sum_{i=1}^{6} n_i = 3 \), we will have to consider the cases \( k = 0, 1, 2, 3 \).
$k = 0$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ \end{bmatrix}$$

$k = 1$:

$$\begin{bmatrix} 0 & 0 & 0 & -1 & -1 \\ \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ \end{bmatrix}_A + \begin{bmatrix} 0 & 0 & 0 & -1 & -1 \\ \end{bmatrix}_A$$

therefore

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \end{bmatrix} \otimes \left\{ \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -1 & -1 \\ \end{bmatrix} \right\} = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ \end{bmatrix}$$

In this case there are only allowed GYTs.

$k = 2$:

$$\begin{bmatrix} 0 & -1 & -1 & -1 & -1 \\ \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & -1 & -1 \\ \end{bmatrix}_A + \begin{bmatrix} 0 & 0 & -1 & -1 & -1 \\ \end{bmatrix}_A$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & -2 \\ \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 & -2 \\ \end{bmatrix}_{NA} + \begin{bmatrix} 0 & 0 & 0 & -1 & -2 \\ \end{bmatrix}_A$$

Therefore,

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \end{bmatrix} \otimes \left\{ \begin{bmatrix} 1 & -1 & -1 & -1 & -1 \\ \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1 & -1 & -1 \\ \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -2 & -2 \\ \end{bmatrix}_{NA} + \begin{bmatrix} 0 & 0 & 0 & -1 & -2 \\ \end{bmatrix}_A \right\}$$

$$= \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ \end{bmatrix}.$$

$k = 3$:

$$\begin{bmatrix} 0 & -1 & -1 & -2 & -2 \\ \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & -2 & -2 \\ \end{bmatrix}_{NA} + \begin{bmatrix} 0 & 0 & -1 & -2 & -2 \\ \end{bmatrix}_{NA}$$

$$+ \begin{bmatrix} 0 & -1 & -1 & -2 & -2 \\ \end{bmatrix}_A$$
Therefore,

\[
\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix}_A = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}
\]

In this case there is no contribution from the NA diagrams. The final result is therefore, after adding 1/2 to each index of the obtained GYTs,

\[
\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} _{144} + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} _{1440} + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} _{560} + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} _{1280} + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} _{144}
\]

5. THE WEIGHTS OF AN $SO(2p)$ REPRESENTATION

In this section we give an outline of the proof of the rules given in Section 4. The proof comes from the use of the Gelfand-Zeitlin (GZ) basis\(^{10}\) for $SO(n)$ groups.

The infinitesimal generators $J_{2i-1,2i}$ $(i=1,2,...,p)$ of $SO(2p)$ form a set of commuting Hermitean operators which are generally denoted by $H_i$ and it is therefore possible to characterize (partially) a vector of the Hilbert space for any IR of $SO(2p)$ by its set of eigenvalues $\lambda_i$. Each possible set of $\lambda_i$ can be considered as the components of a vector, usually called the weight vector in an $n$ dimensional Euclidean space. It will be clear from the context whether
we are referring to a vector in the Hilbert representation space or to a vector in the weight space being used to label the Hilbert vectors. The GZ vector is an eigenvector of the generators $J_{z_{i-1},z_{i}}$ $(i=2,\ldots,p)$ only if it is an eigenvector with the maximum possible eigenvalue of $J_{z_{i-1},z_{i-2}}$ and in this case the eigenvalue is just $\lambda_{i}$, the $i^{\text{th}}$ label of the IR. However it is possible to diagonalize these generators and introduce vectors which are specified by a set of $p$ integers or half integers (positive, null or negative) $\lambda_{i} (i=1,2,\ldots,p)$:

$$J_{z_{i-1},z_{i}} \mid \lambda \rangle = \lambda_{i} \mid \lambda \rangle \quad \text{(5.1)}$$

$$\lambda_{i} = m_{i} - l_{i} + \sum_{j=1}^{i-1} \tilde{A}_{ij}$$

where:

a) $m_{i}$ are the integer or half integer numbers specifying the $SO(2p)$ IR;

b) $l_{i}$ are all possible non-negative integer numbers such that:

$$0 \leq l_{k}^{i} \leq m_{i} \quad (i=1,2,\ldots,n; \quad k=1,\ldots,n-i)$$

$$l_{k}^{i} > l_{k+1}^{i} \quad \text{for } i+k > n; \quad l_{0}^{n} = 0$$

$$m_{i} - l_{k}^{i} > m_{i+1} - l_{k+1}^{i}$$

c) $\tilde{A}_{ij} = \delta_{j,i} \min \left( m_{i} - l_{j}^{i}, m_{i-1} - l_{j-1}^{i} \right) + k_{j}^{i}$; $j=1,2,\ldots,i+1$

$$k_{j}^{i} = m_{i-j} - l_{j}^{i} - \max \left( m_{i-j} - l_{j}^{i}, m_{i-j+1} - l_{j}^{i+1} \right)$$

$$\tilde{A}_{0} = \tilde{A}_{1} = 0$$

d) the $\tilde{.}$ on top of $\tilde{A}_{ij}$ means that we have to take any number in the following set of values:

$$\tilde{A}_{d}^{\tilde{.},\tilde{.}}, \tilde{A}_{d}^{\tilde{.},\tilde{.}}, \ldots, \tilde{A}_{d}^{\tilde{.},\tilde{.}} - 2 k_{d}^{i}.$$
Notice that in Eq. (5.1) we have to choose all the possible combinations of \( \lambda^{-i-j} \). So actually we have not only one value \( \lambda^i \) for any \( i \) but a set of values. Our notation is slightly ambiguous, but it avoids overloading the formulae. If \( \lambda^i = m^i \), one has to omit, in the expression for \( \lambda^i_{i+1} \), the term \( m^i_{i+1} - \lambda^i_{i+1} \).

To the weight vectors specified by the set of \( \lambda^i \) of Eq. (5.1) for all possible choices of \( \lambda^i_j \) and \( \lambda^{-i-j} \), one has to add the vectors specified by a set of \( \lambda^i_j \) obtained in Eq. (5.1), changing the sign of an even number of \( \lambda^i_j \) in all the possible ways.

This pattern of eigenvalues is rather complicated, but there are general features which are worth emphasizing:

1) The \( \lambda^i \) are all integers or half integers depending on whether \( m^i \) are integers or half integers;

2) For any weight \( [\lambda^i] \), the sum \( \Sigma_{i=1}^Q \lambda^i \) differs from \( M = \Sigma_{i=1}^Q m^i \) by \( 2k(k=0,1,...,Q) \) where \( Q = \Sigma_{i=1}^Q m^i \) for SO(4q) or SO(4q+2).

3) There is, in general, a degeneracy in the eigenvectors except for the one specified by the greatest weight, (that is, \( \lambda^i = m^i \); there is no degeneracy when all the labels \( m^i \) are equal or in the fundamental one \( [10...0] \).

4) The sum over all the weights of the \( \lambda^i \) components \( \lambda^i \) is zero for any fixed \( i \).

It should be stressed that the weight \( [\lambda^i] \) is not a complete set of labels since one has to know which IR the weight \( [\lambda^i] \) belongs to and moreover there are, in general, several vectors in each IR labelled by the same set of \( [\lambda^i] \). It is convenient to define subsets, each being constituted of vectors, the sum of whose components is

\[
\Lambda^i = M - 2k \quad (k = 0,1,...,Q) \quad (5.2)
\]

If we associate the \( [m^i] \) with the subset \( \Lambda^i = M \), it is possible to obtain all the vectors belonging to a subset \( \Lambda^i \) by multiplying \( [m^i] \) by the negative GYT introduced in Section 4; the different subsets will then be different terms in the product of the GYT.

The direct product of two IR \( [m^i] \otimes [n^i] \) can be computed as follows: find the greatest weight of the direct product, i.e., \( m^i + n^i \); calculate all the weights belonging to the IR \( [m^i + n^i] \); remove these weights from the set of all weights \( (\lambda^i + \lambda^i') \) where \( \lambda^i \) (\( \lambda^i' \)) are all the weights belonging to the IR
\([\nu_1], \{\nu_1\}\); find the greatest weight in the remaining set and so on. This method, which seems very cumbersome in principle, can be expressed in the form of a simple algorithm of Section 4 by the introduction of the GYTs.

The fact that we have to remove a set of weights which belongs to the already determined IR is taken into account by acting with \(L_p^2(\nu_1)\) on the smallest IR and with the help of the subtraction term in the left-hand side of Eq. (4.2).

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1) For a review see, for example, R. Barbieri, CERN preprint TH-2935, lectures given at the International School "E. Fermi", Varenna, Italy (1980); J. Ellis, CERN preprint TH-2942, lectures given at the 21st Scottish Universities Summer School in Physics, St. Andrews, Scotland (1980); D.V. Nanopoulos in proceedings of the XV Rencontre de Moriond, ed. Tran Van Tanh (1980).


8) M. Fischler, Fermilab preprint PUB-80/49-THY.


11) Our operators $j_{ik}$ defined in Section 2 differ from the generators $l_{ik}$ of Ref. 10) by a factor $-i$. With our definition, we get Hermitian operators.

12) The complete set of weights for any IR of $SO(n)$ can be derived from the greatest weight with the help of the Dynkin diagram. See, for example, R. Slansky, Los Alamos preprint LAUR 80-3495 (1980). However, to our knowledge, there has never been previously written a compact and explicit form for the weights of any $SO(2p)$ IR.