Definitions of the gauge-theory coupling in lattice and continuum quantum chromodynamics: Implications of change in the lattice action

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(Received 29 July 1981)

We investigate various effects of a change in action for lattice QCD. The lattice scale parameter Λ_L depends on the action used and $\Lambda_L/\Lambda_{\text{MOM}}$ is computed by the background-field method in the continuum limit for Manton's and the generalized Villain's (the heat-kernel) action. When comparing with results from Monte Carlo simulations we find evidence for sizable three-loop corrections to the lattice β function. Creutz's renormalization-group analysis is extended to Manton's and the heat-kernel action. The method produces good results with the actions we study. The static quark-antiquark potential we obtain, which is close in form to a phenomenological charmonium potential, exhibits an independence of the lattice action used. Finally, we present a discussion of the effects of changing the form of the action, relating these to a line of critical points that the theory may possess.

I. INTRODUCTION

In Wilson's¹ approach to lattice gauge theory² the expectation value of a gauge-invariant observable $\mathcal{O}$ is given by

$$\langle \mathcal{O} \rangle = Z^{-1} \int_G \prod dU(L) \mathcal{O}(L) \exp[-S_W(\{U_p\})], \quad (1.1)$$

where $U(L)$ are elements of the group G associated with the links L of the lattice and $\{U_p\}$ are the plaquette variables formed from the $U(L)$'s in a standard fashion. The gauge group we shall consider is $SU(2)$. In terms of the link variables U_p Wilson's lattice action S_W is given by

$$\begin{aligned} S_W &= \sum_P S_{W,P} \\ &\equiv \frac{1}{g_0^2} \sum_P \text{Tr}(2I - U_P - U_P^\dagger), \end{aligned} \quad (1.2)$$

where g_0 is the lattice coupling constant. In (1.1), Z is the partition function defined in such a way that the expectation value of the unit operator is one. Eventually, a continuum limit must be taken corresponding to the weak-coupling limit $g_0 \rightarrow 0$

where the theory is supposed to approach a critical point.

For sufficiently small g_0 a renormalization-group calculation leads to the relation

$$\begin{aligned} \Lambda_L^2 &= \lim_{a \rightarrow 0} \left[\frac{1}{a^2} [\beta_0 g^2(a)]^{-\beta_1/\beta_0^2} \right. \\ &\quad \left. \times \exp \left[-\frac{1}{\beta_0 g_0^2(a)} \right] \right], \end{aligned} \quad (1.3)$$

where a is the lattice spacing and Λ_L the lattice scale parameter. The change of the lattice coupling constant with respect to the lattice spacing is determined by the perturbative expansion of the β function³

$$a \frac{d}{da} g_0(a) = \beta_0 g_0^3(a) + \beta_1 g_0^5(a) + O(g_0^7), \quad (1.4)$$

where, for $SU(2)$, $\beta_0 = 11/24\pi^2$ and $\beta_1 = 17/96\pi^4$. β_0 and β_1 are renormalization-scheme-independent numbers.

Recently Monte Carlo simulations have been used to obtain a variety of numerical information

on lattice gauge theories.^{4–11} (The list of Refs. 4–11 is not exhaustive.) In particular, a sharp crossover between the strong- and weak-coupling regions has been revealed⁵ and the relation between the physical string tension σ and the scale parameter Λ_L has been estimated^{5,10}:

$$\Lambda_L = (0.012 \pm 0.002) \sqrt{\sigma}. \quad (1.5)$$

It is important to remember that this relation depends on the regularization scheme used. Furthermore, the numerical computation leading to (1.5) is necessarily done at a finite value of the lattice coupling, hence the limit in (1.3) is never approached in practice. We shall return to these questions later.

With a lengthy calculation Hasenfratz and Hasenfratz¹² have determined the connection between the SU(2) momentum-subtraction scale parameter (in the Feynman gauge), Λ_{MOM} , and Λ_L to be

$$\Lambda_{\text{MOM}} = 57.5 \Lambda_L. \quad (1.6)$$

The rather large number in (1.6) cancels the small number in (1.5) yielding a “reasonable” value for Λ_{MOM} even for SU(2). Creutz¹¹ has recently performed an SU(2) Monte Carlo calculation checking the large factor entering in Eq. (1.6). Wilson’s action (1.2) has the necessary property of reducing to the continuum action when the lattice size a goes to zero. This criterion does not, however, uniquely determine the lattice action. In fact, several alternative actions have been suggested in the literature. Manton¹³ recently proposed as an example the use of the shortest distance $d(U_P, I)$ from U_P to the unit element I in group space, i.e.,

$$S_M = \sum_P S_{M,P} \equiv \frac{1}{g_0^2} \sum_P [d(U_P, I)]^2. \quad (1.7)$$

Change of lattice action can have a dramatic effect on the lattice gauge theory. For instance, it is known¹⁴ that Wilson’s action Eq. (1.2) in two dimensions leads to a third-order phase transition in the large- N limit. This phase transition disappears when Manton’s action S_M is used.¹⁵ Moreover it has recently been shown¹⁶ that the $1/N$ expansion converges in this case very rapidly, at least in two dimensions.

Another construction of a lattice-gauge-theory action is based on a non-Abelian generalization^{17,18} of Villain’s¹⁹ action. In terms of the heat kernel $K(U, g_0^2)$ on the group under consideration the action has the form¹⁸

$$\begin{aligned} S_H &= \sum_P S_{H,P} \\ &\equiv - \sum_P \ln \frac{K(U_P, \frac{1}{2} g_0^2)}{K(I, \frac{1}{2} g_0^2)} \end{aligned} \quad (1.8)$$

and we shall refer to it as “the heat-kernel” (HK) action. With the HK action one can easily show that in two dimensions there is no Gross-Witten phase transition in the large- N limit.

Recently²⁰ we studied the consequences of using Manton’s action or the heat-kernel action in four-dimensional systems by Monte Carlo computations. We found that Manton’s action Eq. (1.7) and the HK action Eq. (1.8) lead to earlier approaches to scaling. By making use of Eq. (1.3) the corresponding lattice scale parameters $\Lambda_L^{(M)}$ and $\Lambda_L^{(HK)}$ were determined to be

$$\Lambda_L^{(M)} = (0.0616 \pm 0.0020) \sqrt{\sigma} \quad (1.9)$$

and

$$\Lambda_L^{(HK)} = (0.0206 \pm 0.0011) \sqrt{\sigma} \quad (1.10)$$

versus $\Lambda_L^{(W)} = (0.012 \pm 0.002) \sqrt{\sigma}$ for Wilson’s action Eq. (1.5).

In this paper we intend to investigate further the implications of changing the lattice action. The ratios between the scales of Eqs. (1.9), (1.10), and (1.5) are numbers which can be computed exactly, by renormalization-group methods. It then becomes possible to compare the numerical and analytical determinations of these quantities. This will be done in Sec. II. The background-field technique, which can be used very conveniently to determine the ratio between Λ_L and other scale parameters, such as Λ_{MS} or Λ_{MOM} ,^{21–24} will be adapted to a calculation of the theoretical ratios $\Lambda_L^{(M)}/\Lambda_L^{(W)}$ and $\Lambda_L^{(V)}/\Lambda_L^{(W)}$. The agreement with the numerical results is only partial and we shall argue that the discrepancy may be attributed, through higher-loop effects, to the fact that the Monte Carlo (MC) determinations are done at finite values of the coupling parameter.

In Sec. III we shall extend to our calculations based on Manton’s and the heat-kernel action a renormalization-group analysis, originally introduced by Creutz,^{11,25} which makes use of the comparison of Wilson-loop factors on different length scales. Although the general criticism may be raised against this method that one is in practice limited to very small separations, its application to an independent determination of the ratio of scales $\Lambda_L^{(W)}/\Lambda_{\text{MOM}}$ has been remarkably successful¹¹ and

we find, indeed, that it produces good results also with the different actions we study.

Finally, in Sec. IV, we shall present a discussion of the effects of changing the form of the action, relating these to a line of critical points that the theory may possess.

II. THE RELATION BETWEEN LATTICE AND CONTINUUM SCALE PARAMETERS

In order to compute the ratios $\Lambda_L^{(M)}/\Lambda_{\text{MOM}}$ and $\Lambda_L^{(V)}/\Lambda_{\text{MOM}}$ we follow the strategy and, partly, the notation of Ref. 21. When computing the Euclidean vacuum-to-vacuum amplitude (the partition function Z) we notice that the integration measure, the gauge-fixing terms, and the ghost contribution remains unchanged when we change the lattice action. The lattice link variable $U(L)$ on the link $(x, x + \mu)$ is parametrized as

$$U_{x, x+\mu} = \exp(ig_0\alpha_\mu) \exp(iA_\mu^{(0)}a), \quad (2.1)$$

where $A_\mu^{(0)}$ corresponds to the background field and $\alpha_\mu \equiv \alpha_\mu^{(a)}T(a)$ to the quantum correction. The lattice action $S(U_p)$ should now be expanded to second order in α_μ , i.e.,

$$S_p(U_p) = S(U_p^{(0)}) + S_2 + \text{cubic terms}, \quad (2.2)$$

where $S(U_p^{(0)})$ is the lattice action of the background field.

A. Manton's action

Let us write the plaquette variable U_p in (1.2) in the form e^{iH_p} and expand Wilson's action to second order in α_μ , i.e.,

$$S_{W,p} = \frac{1}{g_0^2} \text{Tr} \left[H_p^2 - \frac{1}{12} H_p^4 + O(H_p^6) \right] \quad (2.3)$$

while the corresponding expression in the case of Manton's action is

$$S_{M,p} = \frac{1}{g_0^2} \text{Tr} H_p^2. \quad (2.4)$$

By computing the ratio $\Lambda_L^{(M)}/\Lambda_L^{(W)}$ only the difference

$$\Delta S = S_M - S_W = \frac{1}{g_0^2} \sum_p \frac{1}{12} \text{Tr} H_p^4 \quad (2.5)$$

will enter in the calculation.

If P lies in the $\mu - \nu$ plane $H_P \equiv W_{\mu\nu}$ can be written in the form

$$W_{\mu\nu} = A_{\mu\nu} + B_{\mu\nu} + O(A_\mu^{(0)2}, A_\mu^{(0)}\alpha_\mu, \alpha_\mu^2), \quad (2.6)$$

where, by expansion of Eq. (2.1),

$$A_{\mu\nu} = ig_0(\Delta_\mu\alpha_\nu - \Delta_\nu\alpha_\mu) \quad (2.7)$$

and

$$B_{\mu\nu} = ia^2 F_{\mu\nu}^{(0)}. \quad (2.8)$$

Keeping terms up to the second order in α_μ and $F_{\mu\nu}^{(0)}$ we obtain

$$\frac{1}{12g_0^2} \text{Tr} H_P^4 = \frac{1}{6g_0^2} \text{Tr} [2A_{\mu\nu}^2 B_{\mu\nu}^2 + (A_{\mu\nu} B_{\mu\nu})^2]. \quad (2.9)$$

When integrating over the quantum field $\alpha_\mu \equiv \alpha_\mu^{(a)}T(a)$ the $SU(N)$ group indices (a) will be contracted in Eq. (2.9) and therefore it contributes effectively as

$$\frac{1}{12g_0^2} \text{Tr} H_P^4 \rightarrow \frac{(2N^2 - 3)}{3g_0^2 N} \text{Tr} B_{\mu\nu}^2 \text{Tr} A_{\mu\nu}^2. \quad (2.10)$$

By a careful inspection of the Dashen and Gross calculation²¹ one realizes that a term $-(1/2g_0^2) \text{Tr} A_{\mu\nu}^2 B_{\mu\nu}^2$, which effectively, as above, contributes in the form $-[(N^2 - 1)/2g_0^2 N] \text{Tr} B_{\mu\nu}^2 \text{Tr} A_{\mu\nu}^2$, gives a (multiplicative) contribution

$$\delta \left[\frac{\Lambda_L^{(W)}}{\Lambda_{\text{MOM}}} \right] = \exp \left[\frac{3\pi^2}{11} (1/N^2 - 1) \right] \quad (2.11)$$

to the $\Lambda_L^{(W)}/\Lambda_{\text{MOM}}$ ratio. The term Eq. (2.10) therefore gives a (multiplicative) contribution

$$\delta \left[\frac{\Lambda_L}{\Lambda_{\text{MOM}}} \right] = \exp \left[\frac{3\pi^2}{11} \left(\frac{2}{3} - 1/N^2 \right) \right] \quad (2.12)$$

to the $\Lambda_L/\Lambda_{\text{MOM}}$ ratio, which one also can verify by an explicit calculation. By combining Eqs. (2.11) and (2.12) we obtain

$$\frac{\Lambda_L^{(M)}}{\Lambda_L^{(W)}} = \exp \left[\frac{3\pi^2}{11} \left(\frac{2}{3} - 1/N^2 \right) \right] \quad (2.13)$$

and the N^2 dependence cancels in the ratio $\Lambda_L^{(M)}/\Lambda_{\text{MOM}}$. The relation^{12,22}

$$\frac{\Lambda_{\text{MOM}}}{\Lambda_L^{(W)}} = 112.5 \exp \left[-\frac{3\pi^2}{11N^2} \right] \quad (2.14)$$

then leads to

$$\Lambda_{\text{MOM}} = 18.7 \Lambda_L^{(M)} \quad (2.15)$$

and

$$\Lambda_L^{(M)} = 3.07 \Lambda_L^{(W)}. \quad (2.16)$$

B. The heat kernel

For an $SU(2)$ element of the form $U = \cos\phi + i\vec{\sigma} \cdot \hat{n} \sin\phi$ the heat kernel $K(U, \frac{1}{2}g_0^2)$ can be expressed as¹⁸

$$K(U, \frac{1}{2}g_0^2) = \mathcal{N} \sum_{n=-\infty}^{\infty} \frac{\phi + 2\pi n}{\sin\phi} \times \exp \left[-\frac{2}{g_0^2} (\phi + 2\pi n)^2 \right], \quad (2.17)$$

where $\mathcal{N}$ is a normalization constant. We are interested in the weak-coupling limit. Taking the leading term in Eq. (2.17) into account and writing U in the form $U = e^{iH}$ the definition Eq. (1.8) leads to

$$S_{H,P} = \left[\frac{1}{g_0^2} - \frac{1}{12} \right] \text{Tr} H_P^2 - \frac{1}{360} \text{Tr} H_P^4 + O(H_P^6). \quad (2.18)$$

Let us now consider the ratio $\Lambda_L^{(V)}/\Lambda_L^{(M)}$ following the strategy of calculation in Ref. 21, i.e., let us consider

$$\Delta S = S_M - S_H = \sum_P \frac{1}{12} \text{Tr} H_P^2 + O(H_P^4). \quad (2.19)$$

It is clear, from Eq. (2.18) or (2.19), that going from Manton's action to the HK action we have, to one-loop order, only to decrease the running

lattice coupling according to $1/g_0^2 \rightarrow (1/g_0^2 - \frac{1}{12})$.

In the case of $SU(N)$ one obtains

$1/g_0^2 \rightarrow (1/g_0^2 - N/24)$. The effect of such a redefinition on the $\Lambda_L^{(V)}/\Lambda_L^{(M)}$ ratio is easy to extract from the results of Ref. 21 with the result that

$$\frac{\Lambda_L^{(HK)}}{\Lambda_L^{(M)}} = \exp \left[-\frac{\pi^2}{11} \right] \quad (2.20)$$

which is independent of N and reads

$$\Lambda_L^{(M)} = 2.45 \Lambda_L^{(HK)}. \quad (2.21)$$

The results of the considerations in subsections A and B are summarized in Table I together with data from the Monte Carlo simulation.²⁰ In the table we use $\Lambda_L^{(W)} = 0.0140\sqrt{\sigma}$ which is consistent with Eq. (1.5). In fact, when applying the Monte Carlo program as described in Ref. 20 to the Wilson action we have noticed a trend of the ratio $\Lambda_L^{(W)}/\sqrt{\sigma}$ to increase with higher statistics.

C. Possible effect of three-loop β -function contributions in Monte Carlo calculations

As is clear from Table I there are substantial deviations between the theoretical and experimental values of $\Lambda_L^{(V)}/\Lambda_L^{(W)}$, $\Lambda_L^{(M)}/\Lambda_L^{(V)}$ and, especially, $\Lambda_L^{(M)}/\Lambda_L^{(W)}$. We conjecture here that this discrepancy is due to the effect of the three- (and possibly higher) loop term in the β function.

In Ref. 20 we measured

$$K(a, m) = -\ln \frac{W(m, m)W(m-1, m-1)}{W(m, m-1)^2} \quad (2.22)$$

where $W(m, n)$ denotes the expectation value of a Wilson loop of size m by n lattice units. In the continuum limit we have

$$\frac{\sigma}{\Lambda_L^2} = \lim_{a \rightarrow 0} \left[\frac{\sigma}{\Lambda_L^2} \right]_{\text{measured}}, \quad (2.23)$$

where

TABLE I. Comparison of scale parameter ratios (for different lattice actions) as obtained from a theoretical continuum limit and Monte Carlo experiments.

	Monte Carlo experiment	Theory	MC experiment theory
$\Lambda_L^{(M)}/\Lambda_L^{(W)}$	4.40	3.07	1.43
$\Lambda_L^{(M)}/\Lambda_L^{(V)}$	2.99	2.45	1.22
$\Lambda_L^{(V)}/\Lambda_L^{(W)}$	1.47	1.25	1.18

$$\left[\frac{\sigma}{\Lambda_L^2} \right]_{\text{measured}} = K(a, ma/\xi \rightarrow \infty) [\beta_0 g_0^2(a)]^{\beta_1/\beta_0^2} \times \exp \left[\frac{1}{\beta_0 g_0^2(a)} \right], \quad (2.24)$$

where ξ/a is the correlation length in lattice units. Owing to practical limitations such as finite lattice size and boundary conditions one considers as an estimate of $K(a, ma/\xi \rightarrow \infty)$ the values obtained for Wilson loops of the order 5×5 and smaller. Keeping in mind that $a \neq 0$ when doing Monte Carlo calculations we have to investigate the effect of higher-order terms in Eq. (1.4) to the $(\sigma/\Lambda_L^2)_{\text{measured}}$ ratio. By writing

$$a \frac{d}{da} g_0(a) = \beta_0 g_0^3(a) + \beta_1 g_0^5(a) + \beta_2^L g_0^7(a) + O(g_0^9) \quad (2.25)$$

we can easily find a correction term in Eq. (1.3) which leads to the replacement

$$\left[\frac{\sigma}{\Lambda_L^2} \right]_{\text{measured}} \rightarrow \left[\frac{\sigma}{\Lambda_L^2} \right]_{\text{measured}} (1 + x_L), \quad (2.26)$$

$$g_{\text{MS}}^2 = g_0^2(a) \left\{ 1 + \left[\beta_0 \ln \frac{\pi^2}{a^2 \mu^2} + x_0 \right] g^2(a) + \left[\beta_0^2 \ln^2 \frac{\pi^2}{a^2 \mu^2} + (\beta_1 + 2\beta_0 x_0) \ln \frac{\pi^2}{a^2 \mu^2} + x_1 \right] g_0^4(a) \right\}, \quad (2.29)$$

where x_0 can be obtained from the results of Ref. 12. It follows that

$$\beta_2^L = \beta_2^{\text{MS}} - x_0 \beta_1 + x_1 \beta_0 - x_0^2 \beta_0. \quad (2.30)$$

By a two-loop calculation x_1 can be found. Values of x_1 in the range $0 \leq x_1 \leq x_0^2$ would provide for the correction factors Eq. (2.27) which, for $4/g_0^2(a) \approx 2$, can easily explain the large correction factors of Table I. Corrections of this order of magnitude from higher-order contributions in the β function in the determination of the continuum Λ parameter have been discussed in the literature.²⁷

III. THE RENORMALIZED CHARGE AND THE STATIC QUARK-ANTIQUARK POTENTIAL

A. The renormalized charge

The quark-antiquark potential and the renormalized charge may be studied following a procedure

where

$$x_L = \left[\frac{\beta_2^L}{\beta_0^2} - \frac{\beta_1^2}{\beta_0^3} \right] g_0^2(a). \quad (2.27)$$

Since β_2^L is scheme dependent it also depends on the choice of the lattice action. x^L in Eq. (2.26) therefore depends on the lattice action. The correction exhibited in Eq. (2.26) leads to

$$\frac{\Lambda_L^{(M)}}{\Lambda_L^{(W)}} \rightarrow \frac{\Lambda_L^{(M)}}{\Lambda_L^{(W)}} \left[1 + \frac{1}{2} (x^{(W)} - x^{(M)}) \right], \quad (2.28)$$

where $x^{(W)}$ ($x^{(M)}$) denotes the three-loop correction corresponding to Wilson's (Manton's) action.

Equation (2.28) can be extended to the other ratios of Table I and one can verify that a consistent parametrization, in terms of $x^{(W)}$, $x^{(M)}$, and $x^{(V)}$, can be found that takes the deviations exhibited in Table I into account.

As a final remark of this section we notice that the scheme-dependent β_2 term in Eq. (2.25) has recently been computed in 't Hooft's MS-subtraction scheme.²⁶ The calculation of Ref. 12 should then be extended to the two-loop level in order to find β_2^L . To see this one writes

recently suggested by Creutz.^{11,25} For large elongated loops $\ln W(m, n)$ becomes equal to the product of the energy $E(am)$ of a pair of static charges separated by a distance ma for a time interval na . If $n \gg m \gg 1$ we can write

$$E(am) = -\frac{1}{a} \ln \frac{W(m, n)}{W(m, n-1)}. \quad (3.1)$$

This quantity still includes the self-energy of the sources and hence contains contributions proportional to the perimeter of the loop. In order to subtract the self-energy we consider the force given by

$$F(d(m)a) = \frac{1}{a^2} [E(ma) - E((m-1)a)] \equiv \frac{1}{a^2} \chi(m, n), \quad (3.2)$$

where now

$$\chi(m, n) = -\ln \frac{W(m, n)W(m-1, n-1)}{W(m, n-1)W(m-1, n)} \quad (3.3)$$

has no contributions proportional to the perimeter. $d(m)$ is a number between $m-1$ and m , which, following Ref. 11, we set equal to $[m(m-1)]^{1/2}$.

If we define a gauge-invariant, renormalized-coupling constant $g_R(r)$ at short distances by²⁸

$$F(r) = \frac{3}{16\pi} g_R^2(r) \frac{1}{r^2} \quad (3.4)$$

as motivated by asymptotic freedom, then we obtain from (3.2)

$$g_R(a[m(m-1)]^{1/2}) = \frac{16\pi}{3} m(m-1) \chi(m, n). \quad (3.5)$$

For sufficiently small distances r , the renormalized charge will fulfill the renormalization-group equation (1.4). By keeping only the first term in the expansion (1.4) and by integration we see that

$$\beta_R(r_2) = \beta_R(r_1) - \frac{11}{3\pi^2} \ln r_2/r_1, \quad (3.6)$$

where $\beta_R(r) = 4/g_R^2(r)$.

Making use of the results from the Monte Carlo calculation of Ref. 20 we have studied to what extent the short-distance renormalization-group prediction (3.5) actually describes the measured values (3.2). The size of the lattice (8^4) effectively limits the largest side of the loop to $n \leq 5$. The requirement $n \gg m$ then allows to consider only $m=1$ or $m=2$ and, correspondingly, $r_1 = \sqrt{2}a$, $r_2 = \sqrt{6}a$. In Fig. 1 (Fig. 2) we show the results in the case of Manton's (the heat-kernel) action. The measured values are remarkably close to the renormalization-

FIG. 1. Comparison of the renormalized charge, as defined in text, at different length scales for Manton's action. The solid curve represents the renormalization-group prediction.

FIG. 2. Comparison of the renormalized charge, as defined in text, at different length scales for the heat-kernel action. The solid curve represents the renormalization-group prediction.

group prediction Eq. (3.6) for both actions. In the case of Manton's (HK) action the value of $\beta_R(\sqrt{2}a) = 1.0$ corresponds to a bare coupling $\beta = 4/g_0^2(a) = 2.1$ (2.4). By making use of Eq. (1.3) we find the corresponding lattice size in units of Λ_L^{-1} . Expressed in terms of the correlation length ξ , i.e., the inverse glueball mass^{10,29} $\xi = 1/m_G$ where $m_G \approx 3\sqrt{\sigma}$, we see that for Manton's (HK) action $a(2.1) \approx 0.5\xi(a(2.4)) \approx 0.65\xi$. We find it remarkable that the continuum SU(2) prediction Eq. (3.6) adequately describes the Monte Carlo measurement data for a lattice size comparable to the correlation length ξ .

In Fig. 3 (Fig. 4) we study $\beta_R([m(m-1)]^{1/2}a)$ for Manton's (HK) action as a function of the bare coupling. From these data we extract the relations

$$\beta_R(\sqrt{2}a) = \beta - 1.108 \pm 0.017 \quad (3.7)$$

and

$$\beta_R(\sqrt{6}a) = \beta - 1.385 \pm 0.012 \quad (3.8)$$

for Manton's action and for the heat-kernel action

$$\beta_R(\sqrt{2}a) = \beta - 1.443 \pm 0.030 \quad (3.9)$$

and

$$\beta_R(\sqrt{6}a) = \beta - 1.779 \pm 0.021. \quad (3.10)$$

Here $\beta = 4/g^2(a)$.

Following Creutz^{11,25} we define a renormalized asymptotic freedom scale Λ_R in terms of the renormalized-coupling constant $g_R(r)$ by

FIG. 3. $\beta_R([m(m-1)]^{1/2}a)$ as a function of the bare lattice coupling constant for $m=2$ (●) and $m=3$ (△) in the case of Manton's action. The straight lines correspond to the equations (3.7) and (3.8) given in the text.

$$\Lambda_R^2 = \lim_{r \rightarrow 0} \left\{ \frac{1}{r^2} [\beta_0 g_R^2(r)]^{-\beta_1/\beta_0^2} \times \exp \left[-\frac{1}{\beta_0 g_R^2(r)} \right] \right\} \quad (3.11)$$

in complete analogy with Eq. (1.3). Making use of Eqs. (3.7), (3.8), (3.9), (3.10), and the definitions of Λ_R and Λ_L we obtain an estimate of Λ_R/Λ_L for the various action considered above, namely,

$$\Lambda_R = (15.7 \pm 0.4) \Lambda_L^{(M)} \quad (3.12)$$

and

$$\Lambda_R = (41.7 \pm 2.0) \Lambda_L^{(HK)}. \quad (3.13)$$

The scale Λ_R is quite close to Λ_{MOM} (Ref. 30) and the numbers given by Eqs. (3.12) and (3.13) are in good agreement with the theoretical ratios of Eqs. (2.15) and (2.21). In effect, one must look upon the results Eqs. (3.12) and (3.13) with a critical eye: Although the equations used to define Λ_R are correct, in practice the condition $n \gg m \gg 1$

FIG. 4. $\beta_R([m(m-1)]^{1/2}a)$ as a function of the bare lattice coupling constant for $m=2$ (●) and $m=3$ (△) in the case of the heat-kernel action. The straight lines correspond to Eqs. (3.9) and (3.10) given in the text.

cannot be maintained and one is forced to rather small values of m . The validity of the extrapolation may then be questioned; the numbers become quite sensitive to the form assumed for $d(m)$. Still, the results are in remarkable agreement with the other, independent determinations of the ratios of scales. From Eqs. (3.12) and (3.13), in particular, one obtains $\Lambda_L^{(M)}/\Lambda_L^{(V)} = 2.66$, to be compared with the value 2.99 obtained from the numerical determination of the string tension and the theoretical result 2.45.

B. The static quark-antiquark potential

The static quark-antiquark potential for Manton's (HK) action can be determined from Monte Carlo experiment data by making use of Eq. (3.4) and the results exhibited in Fig. 3 (Fig. 4). One obtains a force as shown in Fig. 5, where points derived from Wilson's action are also plotted.¹¹ It is interesting to compare the results of Fig. 5 with the static potential used, e.g., in the

FIG. 5. The force between a quark-antiquark evaluated according to the procedure given in the text in the case of Wilson's action (●), Manton's action (+), and the heat-kernel action (△). The solid line is a fit to the different potentials.

charmonium system.³¹ In Fig. 5 we also exhibit force corresponding to the phenomenological potential of Eichten *et al.*,³²

$$V(r) = -\frac{k}{r} + \frac{r}{a^2}, \quad (3.14)$$

where our fit yields $1/a^2 = 0.961$ and $k = 0.153$. As seen in Fig. 5 we find that all three actions considered in this paper give the same static quark-antiquark potential. Moreover they all closely approximate the phenomenological charmonium continuum potential (3.14). To compare k in (3.14) with the fit to charmonium of *et al.*³² it is reasonable to multiply k with a factor $\frac{16}{9}$, coming from the difference in the SU(2) and SU(3) one-gluon-exchange potential.

IV. A PHASE-DIAGRAM SCENARIO

Let us try to understand the dependence on the form of the action on more general grounds. Any class function on a group may be written as a sum over all inequivalent, irreducible representations. Thus

$$S(U, g_0^2) = c_0(g_0^2) + \sum_r c_r(g_0^2) \chi_r(U),$$

where

$$c_r(g_0^2) = \int dU S(U, g_0^2) \chi_r(U) \quad (4.2)$$

and $\chi_r(U)$ denotes the character of U in the representation r . We specialize to SU(2) and $r = \frac{1}{2}, 1, \frac{3}{2}, \dots$. Wilson's action has contributions only from the fundamental representation:

$$c_0(g_0^2) = 4/g_0^2, \quad c_{1/2}(g_0^2) = -2/g_0^2, \\ c_{r>1/2}(g_0^2) = 0.$$

Manton's action receives contributions from all representations:

$$c_0(g_0^2) = (2\pi^2/3 - 1)/g_0^2, \\ c_r(g_0^2) = (2r+1)(-1)^{2r}/g_0^2 r^2(r+1)^2. \quad (4.3)$$

The same holds true for the heat-kernel action, although the dependence on g_0^2 is more complicated; asymptotically, for large $1/g_0^2$, the coefficients approach those of Manton's action. A common feature of these two terms of the action is that one has enlarged the space of the possible couplings considered in the Lagrangian. In particular, both actions receive a contribution from the adjoint representation ($C_1 \neq 0$).

Recent numerical investigations³³ of the pure SO(3) gauge theory ($C_1 \neq 0, C_{r \neq 1} = 0$) have produced evidence of a first-order phase transition. An analysis of theories, where only the fundamental and adjoint representations appear,^{34,35} reveals the existence of critical lines, as sketched in Fig. 6. The phase transition of the Abelian Z_2 model, which is obtained in the $C_1 \rightarrow \infty$ limit and occurs at $C_{1/2} = -2/g_0^2 = -0.22$, develops into a whole line of critical points, which descends almost parallel to the C_1 axis until it merges with a line developing from the critical point of the SO(3) theory. The line of phase transition then continues with slope close to -1 and can be followed up to $C_{1/2} = C_1 = -0.5$, where the numerical evidence for a phase transition becomes faint. This pattern of critical lines has been suggested in Ref. 35 to explain the rapid crossover and the peak in the specific heat in the theory with Wilson's action.

Although Manton's action receives contributions from all representations, neglecting the characters with $r > 1$ it projects on the line $C_1 = -\frac{27}{128} C_{1/2}$ (dashed in Fig. 6). The intersection with the continuation of the critical line, approximated by $C_1 = -C_{1/2} - 1$, occurs at $C_{1/2} = -\frac{128}{101}$, i.e., $\beta = 4/g_0^2 = 1.43$, and it is interesting that this

FIG. 6. Sketch of the phase diagram in the C_1 - $C_{1/2}$ plane; Wilson's action corresponds to $C_1 = 0$ whereas Manton's action has contributions from both fundamental and adjoint as well as all higher (not displayed) representations. The full line denotes a first-order phase transition, the dots indicate roughly the position of the inflection point in $\langle S_p \rangle$ (maximum of the slope).

value is very close to the point where one observes the crossover.

Moreover, in Ref. 20 evidence has been found for a singularity in the theory with Manton's action at approximately the same value of β . However, the singularity is of a rather weak nature, manifesting itself only in the derivative of the specific heat. This would suggest that what one observes in the enlarged space of possible actions is a critical line with varying critical exponents. In two-dimensional spin systems, which bear many analogies with four-dimensional lattice gauge theories, the occurrence of such lines is common and can be related to the existence of marginal operators. In the SU(2) model the marginal operators might be associated with the excitation of monopoles or vortices.^{34,36} The very precocious scaling observed for the string tension with all forms of the action would indicate that such a physical quantity is not much affected by the marginal operators, which might however influence the scaling behavior of other observables, such as the plaquette-plaquette correlation length. An accurate

numerical study of these quantities in the neighborhood of the critical points could provide very useful information on SU(2)-lattice gauge theory.

Note added: After the completion of our work, we learned of a paper by Gonzalez-Arroyo and Korthals Altes³⁷ on the ratios of scale parameters for different actions.

ACKNOWLEDGMENTS

We wish to thank P. Hasenfratz for useful discussions and for checking our results on the theoretical ratios of scales and L. Abbott for an enlightening conversation on higher-order contributions to the scaling formula. One of the authors (C. R.) would like to thank R. Svendsen for a very useful discussion on spin systems. B.-S. S. wishes to thank J. Prentki and the TH-division for the hospitality during his stay at CERN as a fellow. B.-S. S. was supported in part by the Swedish Research Council under Contract No. 7310-108. Finally, we wish to thank G. Münster for some helpful comments.

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