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ON THE LOOP SPACE FORMULATION OF GAUGE THEORIES

B. Durhuus
NORDITA - Copenhagen

and

J.M. Leinaas
CERN - Geneva

A B S T R A C T

We discuss a possible group theoretical approach to the loop description of gauge theories. In this approach classical configurations are interpreted as unitary representations of the "group of loops". The structure of this group is examined, and it is in particular shown how a metric can be introduced. Problems related to the local structure of the group are discussed, and it is suggested that a regularization of the theory should be connected to a possible generalization of Fourier transformations on groups. In this picture the description of the Yang-Mills field in terms of the Wilson loop average is dual to that of the Lagrangian formulation. We illustrate the ideas by some simple examples.

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1. INTRODUCTION

In the past few years several attempts have been made to reformulate Yang-Mills theories in terms of variables which are more suitable for handling the infra-red structure and the problem of quantization of the theories. One of the most interesting approaches is the description of Yang-Mills fields by use of path dependent variables¹⁻⁷⁾. These variables give a description which takes into account more explicitly the geometrical interpretation of the Yang-Mills field than a description in terms of the gauge dependent potentials. Some progress has been made along these lines, in particular for the evaluation of the Wilson loop average $\langle W(C) \rangle = \langle \text{Tr} P \exp \oint_{\mu} A_{\mu} dx^{\mu} \rangle$. Both on the lattice and in the continuum string equations for this quantity have been found, and it is possible that by suitable approximations one can obtain useful information about the quantum gauge theories through these equations.

However, the general approach along these lines is hindered by difficult obstacles. Thus, even if a reformulation of the Yang-Mills theories in terms of functions on the space of paths, at the more formal level, looks very attractive, the mathematical problems of handling this space are extremely difficult. It is indeed very likely that the problems one meets in giving a well-defined meaning to the quantities and operations used in the "loop space" formulations are comparable to the problems one meets in other approaches to the quantization of Yang-Mills theories.

Nevertheless, it seems of interest to make an attempt to clarify the ideas involved to see how it might be possible to give them meaning beyond the formal level. In this paper we examine some ideas which are based on the fact that the loop space has a natural group structure^{8),9)} and that the classical configurations of the Yang-Mills field can be identified as unitary representations of this "loop group" (see also Ref. 5)). We study the structure of this group and, in particular, show how a metric can be introduced, thereby giving the group a topological structure.

We further discuss some of the problems related to the local structure of the loop group. To handle these problems a regularization is needed and we suggest that such a regularization should be connected to a possible generalization of Fourier transformations on groups. This is partly motivated by the fact that, at the formal level, the Wilson loop average can be related to the action by a Fourier transformation, and the description in terms of the non-local variables $\langle W(C) \rangle$ in this sense is dual to the Lagrangian formulation.

We would like to stress the point that this paper is meant primarily to discuss some of the ideas involved in the loop space formulation and to suggest a possible approach to give more definite meaning to these ideas. On the other hand, we do not attempt to make a real attack on the serious mathematical problems encountered in this formulation. At the end we therefore also include some examples where these ideas take on a particularly simple form and where the more serious mathematical problems are completely avoided. We furthermore choose to present the discussion in a rather detailed way, even if that means that in the first part of the paper we include things which are also known from other publications.

2. YANG-MILLS THEORY ON LOOP SPACE

The basic quantity in a path dependent formulation of Yang-Mills theory is the unitary matrix

$$U(C) = P \exp i \int A_\mu dx^\mu \quad (1)$$

This matrix, which takes on values in the gauge group G , has a direct geometrical interpretation as it represents the parallel transport of the theory. Due to this interpretation it seems natural to regard $U(C)$ as a more fundamental object than the gauge potential A_μ , since A_μ depends on the choice of gauge. In fact, the matrices $U(C)$ contain all information about the classical field configuration and this is the case even if we restrict C to be a closed curve with a fixed base point¹⁰⁾. In the following we will make such a restriction. $U(C)$ then represents a unitary operator acting within the internal (colour) space associated with the base point ($x = 0$) of the loop C .

When using as the fundamental variable of the theory the field of operators $U(C)$, defined on the space of all loops C , one is faced with the fact that there are many constraints imposed on this field. Thus, if we identify a loop C as a continuous mapping from the unit interval in $\mathbb{R}$ into $\mathbb{R}^4$

$$C : x(\tau) \in \mathbb{R}^4, \quad 0 \leq \tau \leq 1, \quad x(0) = x(1) = 0 \quad (2)$$

then the operators $U(C)$, for any field configuration, satisfy the following identities:

- 1) Reparametrization

$$U(C') = U(C) \quad \text{when} \quad x'(\tau) = x(f(\tau)) \quad (3)$$

for all continuous $f(\tau)$, which satisfy the boundary conditions $f(0) = 0$, $f(1) = 1$.

2) "Back-tracking"

$$U(C') = U(C)^{-1} \quad \text{when} \quad x'(\tau) = x(1-\tau) \quad (4)$$

3) Composition

$$U(C'') = U(C')U(C) \quad \text{when} \quad x''(\tau) = \begin{cases} x(2\tau) & 0 \leq \tau \leq \frac{1}{2} \\ x'(2\tau-1) & \frac{1}{2} \leq \tau \leq 1 \end{cases} \quad (5)$$

It seems natural to make an explicit identification of loops which cannot be distinguished by the operators $U(C)$ due to the identities above. This means that we introduce a loop space, which we will denote by Γ , where loops differing by a reparametrization have been identified and where one has also identified loops which differ by the insertion of curves which are traversed back and forth. One can readily show that the loop space Γ has a group structure, with the group multiplication defined by the ordinary composition of loops (Fig. 1). Thus, in this way one finds that the "group of loops" are the natural space on which the operators $U(C)$ are defined.

After the identification of loops described above, the composition rule 3) gets a simple interpretation. A given classical field configuration, described by the operators $U(C)$, can be identified as a unitary representation π of the loop group Γ , or equivalently as a homomorphism of Γ into the gauge group G

$$\Gamma \xrightarrow{\pi} G \quad (6)$$

Due to this interpretation we will in the following use the notation

$$U(C) = \pi(C) \quad (7)$$

for the field configuration characterized by the representation π . We note that in this picture gauge equivalent configurations $\pi(C)$ and $\pi'(C)$, which are related by

$$\pi'(C) = U\pi(C)U^{-1}, \quad U \in G \quad (8)$$

are interpreted as unitarily equivalent representations of Γ . Thus a gauge field configuration is, up to a gauge transformation, completely specified by an equivalence class of unitarity representations of Γ . Hence, one may take the point of view of regarding the set of these classes, which we will denote by $\hat{\Gamma}$, as the configuration space of the pure Yang-Mills theory.

The representation space $\hat{\Gamma}$ splits into discrete parts corresponding to the dimension n of the representations

$$\hat{\Gamma} = \bigcup_n \hat{\Gamma}_n \quad (9)$$

The space $\hat{\Gamma}_n$ is then essentially identical to what is known as the orbit space associated with the $U(n)$ gauge theory. The bare action $S_0(A)$, as well as all other gauge invariant functions $O(A)$, can be regarded as functions on this space and the functional integrals

$$\langle O \rangle = Z^{-1} \int O(A) e^{-S_0(A)} [dA] \quad (10)$$

should be defined as integrals over the space $\hat{\Gamma}_n$

$$\langle O \rangle = \int_{\hat{\Gamma}_n} O(x) d\mu(x) \quad (11)$$

The essential problem in the quantization of gauge theories is to give meaning to the formal expression (10). In the standard formulation one approaches this problem from the description of the Yang-Mills field in terms of the gauge potential. In the present framework it seems natural to examine to what extent it is possible to give meaning to the expression from the group theoretical point of view.

In fact, for "reasonable" groups H (locally compact, separable and type I) there is a canonical measure on the dual space $\hat{H}$, namely what one might call the Fourier measure (see the Appendix for more details), and it seems reasonable to try to express the functional measure $d\mu(\chi)$ in terms of the Fourier measure on $\hat{\Gamma}$ (if it exists). However, Γ is not a reasonable group in the above sense, and to give meaning to the Fourier measure it is therefore necessary to generalize the standard theory of Fourier transformations. A possible way to do this may be to replace Γ by simpler groups, which satisfy the conditions for defining Fourier transformations, and which in some sense are approximations to the group Γ . Without being able to carry out this idea in detail, we shall in the following examine more carefully some of the general problems encountered in such an approach. We begin by showing how it is possible to introduce a topology on Γ . The elements of $\hat{\Gamma}$ are then restricted to correspond to continuous representations of the topological group Γ .

3. THE STRUCTURE OF Γ

To simplify the structure of Γ we first impose some regularity condition on the loops by assuming them to be piecewise smooth curves. We next turn to the question of introducing a metric on Γ . It seems natural that this metric should be defined such that the distance $d(C_1, C_2)$ between two curves C_1 and C_2 in some sense represents the area covered by the composite curve $C_1^{-1}C_2$. This would be in accordance with our previous identification of curves which differ only by pieces which cover no area; i.e., the back and forth running parts. The area dependence is also motivated by the formal operation of "area differentiation" which appears, for example, in the Mandelstam formula for the field strength¹¹⁾.

We introduce the metric in the following way. A procedure is described to measure the distance $d(\mathbb{1}, C)$ from the trivial loop $\mathbb{1}$ to a given loop C . The distance $d(C_1, C_2)$ is then defined as

$$d(C_1, C_2) = d(\mathbb{1}, C_1^{-1}C_2) \quad (12)$$

To define the distance $d(\mathbb{1}, C)$ we consider a deformation of C which is performed successively by means of small loops, which cover a well-defined minimal area, and such that the curve after the last step is deformed into the trivial loop $\mathbb{1}$. We add up the areas of all the small loops and define $d(\mathbb{1}, C)$ as the minimum of this sum when varying over allowed deformations.

To be more precise, let us first choose a class S of small curves, which are of the form

$$\delta = C_x \sigma_x C_x^{-1}, \quad \delta \in S \quad (13)$$

where C_x is some open curve which connects a given point x to the base point 0 of the loops. σ_x is a closed curve starting and ending at x , with a well-defined minimal area $|\sigma_x| = |s|$ localized in an arbitrarily small region around x (Fig. 2). We further assume that for all loops $C \in \Gamma$ it shall be possible to make a decomposition of the form

$$C = \delta_1 \delta_2 \cdots \delta_n, \quad \delta_i \in S \quad (14)$$

The distance from the loop C to the trivial loop 1 is then defined as

$$d(1, C) = \inf_{\{\delta_i\}} \left\{ \sum_{i=1}^n |\delta_i| ; C = \delta_1 \cdots \delta_n, \delta_i \in S \right\} \quad (15)$$

This definition can be regarded as an extension of the definition of minimal area to all curves contained in Γ (Figs 3,4). (Thus, it is closely related to the concept of minimal surfaces, which is known from the mathematical literature, see, e.g., Ref. 12). However, we here introduce the minimal area in a way which seems more appropriate for our purpose of defining a metric on Γ .)

From the definition above it follows that the distance $d(C_1, C_2)$ is symmetric

$$d(C_1, C_2) = d(C_2, C_1) \quad (16)$$

It is also invariant under both right and left translations on the group

$$d(C_1 C, C_2 C) = d(C C_1, C C_2) = d(C_1, C_2) \quad (17)$$

Furthermore, it follows from the definition that the triangle inequality is satisfied. As a matter of fact, this inequality is equivalent to

$$d(1, C_1 C_2) \leq d(1, C_1) + d(1, C_2) \quad (18)$$

for all $C_1, C_2 \in \Gamma$, and this is clearly satisfied as a consequence of the definition (15). We conclude from this that the distance $d(C_1, C_2)$, as introduced above, satisfies the conditions necessary to define a metric on Γ . With this metric imposed, Γ becomes a topological group.

Let us now look at some of the consequences of this way of defining the metric. We first note that Γ contains a denumerable dense subset with respect to this metric. This can be seen, for example, by choosing a sequence of lattices in $\mathbb{R}^4$, with the lattice spacing going to zero, and consider the subset defined by the union of loops on these lattices. On the other hand, Γ is not a complete space. This is realized by considering a sequence of curves $\{C_n\}$, with

$$C_n = \delta_1 \delta_2 \cdots \delta_n \quad (19)$$

where $s_k = C_{x_k} x_k C_{x_k}^{-1}$ is of the form discussed above. We assume the loops s_k to be suitably separated, and the areas of the loops to satisfy the condition

$$\sum_n d(1, \delta_n) < \infty \quad (20)$$

The sequence $\{C_n\}$ is then a Cauchy sequence since

$$d(C_n, C_m) = \sum_{i=\min\{m,n\}+1}^{\max\{m,n\}} d(1, \delta_i) \quad (21)$$

but obviously C_n does not converge to be an element in Γ . However, Γ can be completed in the standard way.

More important for the following is the fact that Γ is not locally compact even after being completed. This can be seen by considering a sequence $\{C_n\}$ where C_n is of the form

$$C_n = A^n B A^{-n} \quad (22)$$

It follows from Eq. (17) that $d(1, C_n)$ is bounded in n . On the other hand, if the loops A and B are sufficiently far apart we have

$$d(C_n, C_m) = \min \{ 2d(1, B), 2|n-m|d(1, A) \} \quad (23)$$

which shows that $\{C_n\}$ does not contain a Cauchy sequence and therefore not a convergent one either.

The fact that Γ is not locally compact implies that Γ is a very large space even locally. This can also be seen by considering geodesics in Γ , i.e., paths of shortest length. If we, for example, consider deformations of a curve C to the trivial one 1 , there is a large class of such deformations corresponding to the shortest length $d(1, C)$. In fact, for small loops the separation between two of these paths will, in general, be of the same order as the length $d(1, C)$. This is illustrated by a simple example in Fig. 5.

As a consequence of the fact that Γ is not locally compact there is no (quasi-) invariant Borel measure on Γ (see, e.g., Ref. 13)). Since an invariant measure, separability and the type I property are needed for the present theory of the Fourier transformation to work, one might attempt to find other possible metrics on Γ . However, we think that the metric introduced above is, in many respects, the most natural one, and we believe that it is impossible to introduce a reasonable topology on Γ in which Γ is locally compact and separable. The solution to the problem facing us is, we assume, connected to the fact that in some sense one has to regularize the theory^{14),15)}.

The most straightforward way of regularizing the theory would be to choose a sequence of locally compact, separable subgroups of Γ which in some sense converge to Γ . One way to do this is to consider the groups of loops which run on a cubic lattice with lattice spacing tending to zero. Indeed, with respect to the metric we have introduced above these groups become discrete and countable.

However, if the question of regularization (and renormalization) is connected with a possible extension of the theory of Fourier transformations (harmonic analysis), as suggested above, it seems desirable that the regularized groups in addition be of type I. The lattice regularization does not seem to fulfil this requirement (see example 3 below) and thus may be inappropriate within this framework. (This can be compared to the approach announced by Singer et al.^{16),17)} where the lattice regularization also seems inappropriate.) It may therefore turn out that other ways of regularizing the theory will be more convenient for our purpose and in this context it seems natural to consider possible regularizations of Γ in terms of groups with non-discrete topology.

4. FOURIER TRANSFORMATIONS ON LOOP SPACE

To see the significance of the Fourier transformations in connection with the loop group Γ (see the Appendix for a brief review of the theory of Fourier transformations on groups), we will in the following neglect the problem of introducing a measure, and simply assume that a measure dC can be defined on Γ - or a regularized version of Γ (left invariant measure when Γ is locally compact). The Fourier transformation $f \rightarrow \mathcal{F}(f) \equiv \hat{f}$ is given by

$$\mathcal{F}(f)(\chi) = \int_{\Gamma} f(C) \pi_{\chi}(C) dC \quad (24)$$

for a complex valued function f on Γ , where $\chi \rightarrow \pi_{\chi}$ is a measurable gauge choice, that is a choice π_{χ} from each class $\chi \in \hat{\Gamma}$. (Actually, in the Fourier transform χ is restricted to the irreducible representations of Γ . However, it seems reasonable to assume that the reducible representations are a set of zero measure in $\hat{\Gamma}$.) The Fourier inversion theorem^{18),19)} tells us that there is a measure $d\chi$ on $\hat{\Gamma}$, such that

$$f(C) = \overline{\mathcal{F}}(\hat{f})(C) = \int_{\hat{\Gamma}} \text{Tr}(\hat{f}(\chi) \pi_{\chi}(C^{-1})) d\chi \quad (25)$$

where $\overline{\mathcal{F}}$ is the inverse Fourier transform. (We have here adhered to the unimodular analogy; in the non-unimodular case the formulae become slightly modified.)

The physical significance of the Fourier transformation is seen by introducing the action $S(\chi)$ which, according to our previous discussion, is defined as a function on $\hat{\Gamma}$, and looking at its relation to the Wilson loop

$$W_{\chi}(C) = \text{Tr} \pi_{\chi}(C) = \text{Tr} P \exp i \int A_{\mu} dx^{\mu} \quad (26)$$

which here defines the character of the representation π . Assuming the functional measure to be proportional to the Fourier measure, the expectation value of $W(C)$ can be written as

$$\langle W(C) \rangle = \int_{\hat{\Gamma}} \text{Tr}(\pi_{\chi}(C)) e^{-S(\chi)} d\chi \quad (27)$$

(For a $U(N)$ theory we then assume $e^{-S(\chi)} = 0$ when $\dim \chi \neq N$.) Furthermore, using the Fourier inversion theorem we obtain

$$e^{-S(x)} \mathbb{1}_x = \int_C \langle W(C) \rangle \pi_x(C) dC \quad (28)$$

or

$$e^{-S(x)} = \frac{1}{N} \int_C \langle W(C) \rangle \text{Tr}(\pi_x(C)) dC \quad (29)$$

This shows that if we know $\langle W(C) \rangle$ for all loops we can determine the action and vice-versa. In this sense the description of a gauge theory in terms of a Lagrangian and in terms of Wilson loops are dual to each other.

The action $S(x)$ referred to above is the effective (renormalized) action of the theory. The central problem of determining this action is therefore connected to the problem of how to carry out in more detail the program outlined above. However, the non-renormalized action S_0 is well known, and it would be of interest to see how it can be interpreted within the present formalism. From the so-called Mandelstam formula it is known that the field strength $F^{\mu\nu}(x)$, assuming that it exists, can be expressed in terms of a "functional area derivative". (This is in fact one of the reasons for which we regard the distance $d(C_1, C_2)$ defined above as the natural one.) Thus if we consider a representation $\pi_x \in \chi$, with a corresponding gauge potential A_μ , we have

$$\begin{aligned} \frac{\delta \pi_x}{\delta \sigma_x^{\mu\nu}} &\equiv \lim_{d(1, \sigma_x^{\mu\nu}) \rightarrow 0} \frac{\pi_x(\sigma_x^{\mu\nu}) - \mathbb{1}}{d(1, \sigma_x^{\mu\nu})} \\ &= \pi_A(C_x) \frac{\delta \pi_A}{\delta \sigma_x^{\mu\nu}} \pi_A(C_x)^{-1} = i \pi_A(C_x) F_{\mu\nu}^\wedge(x) \pi_A(C_x)^{-1} \end{aligned} \quad (30)$$

where $\sigma_x^{\mu\nu} = C_x \sigma_x^{\mu\nu} C_x^{-1}$ is of the form previously discussed, with $\sigma_x^{\mu\nu}$ as a small loop in the μ, ν plane. $\pi_A(C_x)$ is the path ordered exponential of the integral of A_μ along C_x , and $(\delta \pi_A)/(\delta \sigma_x^{\mu\nu})$ is the same integral for the small loop $\sigma_x^{\mu\nu}$. It follows from this that the action can be written as

$$\begin{aligned} S_0(x) &= \frac{1}{4g^2} \int_{\mathbb{R}^4} \text{Tr} (F_{\mu\nu}^\wedge(x)^\dagger F^{\mu\nu}(x)) d^4x \\ &= \frac{1}{4g^2} \int_{\mathbb{R}^4} \text{Tr} \left\{ \pi_A(C_x) \left(\frac{\delta \pi_A}{\delta \sigma_x^{\mu\nu}} \right)^\dagger \pi_A(C_x)^{-1} \pi_A(C_x) \frac{\delta \pi_A}{\delta \sigma_x^{\mu\nu}} \pi_A(C_x)^{-1} \right\} d^4x \end{aligned} \quad (31)$$

$$\begin{aligned}
&= \frac{1}{4g^2} \int_{\mathbb{R}^4} \text{Tr} \left\{ \left(\frac{\delta \pi_x}{\delta \sigma_x^{\mu\nu}} \right)^+ \left(\frac{\delta \pi_x}{\delta \sigma_{x\mu\nu}} \right) \right\} d^4x \\
&\equiv \frac{1}{4g^2} \int_{\mathbb{R}^4} \text{Tr} \left\{ \left(\frac{\delta \chi}{\delta \sigma_x^{\mu\nu}} \right)^+ \left(\frac{\delta \chi}{\delta \sigma_{x\mu\nu}} \right) \right\} d^4x
\end{aligned} \tag{31}$$

Contd.

where g is the coupling parameter.

At the formal level the expression (31) seems to have a simple interpretation. Thus, if we assume the formal operation $\delta/\delta\sigma^{\mu\nu}(x)$ to correspond to a true differentiation then the integral in x and sum over μ, ν would naturally be interpreted as the sum over independent directions in tangent space. The action $S(\chi)$, which determines the weight of the representation χ in the functional integral, could then be interpreted as measuring the "size" of the representation, in analogy with the quantity

$$\sum_{\kappa} \text{Tr} J_{\kappa}^+ J_{\kappa} = \frac{1}{4} d(d^2 - 1) \tag{32}$$

for the generators J_{κ} of the rotation group, in an irreducible representation of dimension d .

However, when one tries to give the expression above meaning beyond the formal level, one runs into problems similar to those we have previously encountered, namely that Γ has locally too complex a structure. The identifications referred to above mean that the differentiations $\delta/\delta\sigma^{\mu\nu}(x)$ should be interpreted as generators of the group Γ . In some sense this is true, since the corresponding "small" loops $s_x^{\mu\nu}$ can be used to build up arbitrary loops $C \in \Gamma$. But the group Γ cannot be given the structure of an infinite Lie group since it does not contain any one-dimensional subgroups. This follows from the fact that it is not possible to exponentiate the "generators" referred to above.

Thus it seems impossible to introduce true group generators of the group Γ and therefore also to define tangent spaces and a differential structure on this group. The problem of giving meaning to the formal operations used on the above expressions should therefore also be seen in connection with a possible regularization of the theory. Of course, with a lattice cut-off this problem is completely circumvented, since $\delta/\delta\sigma^{\mu\nu}(x)$ is then replaced by its discrete version. However, in the more interesting case, where Γ is regularized in terms of continuous groups, these must presumably have a much simpler local structure than the full loop group Γ .

5. EXAMPLES

We will in this section illustrate the above ideas by some simple examples. In the first two examples we simplify the loop group Γ by introducing identifications between a large number of loops. The resulting groups are then finite-dimensional Lie groups, in fact, one-parameter groups, which can easily be handled. The third example includes the case of two simple loop groups on the lattice.

EXAMPLE 1: Loops in the plane $\mathbb{R}^2$

The loops considered here are confined to a two-dimensional plane, and we introduce an identification of loops covering the same area (counted with sign). The elements of Γ can then obviously be identified by the corresponding area a (Fig. 6). Thus the loop group is identical to the additive group of real numbers,

$$a \in \Gamma \simeq \mathbb{R} \quad (33)$$

The metric Γ is the standard one on $\mathbb{R}$

$$d(a_1, a_2) = |a_1 - a_2| \quad (34)$$

and the (invariant) measure is given by da .

The irreducible representations of Γ are given by the phase factors

$$a \rightarrow \exp\{i\chi a\}, \quad \chi \in \mathbb{R} \quad (35)$$

Thus, the corresponding gauge group is $G = U(1)$ and the representation space is $\Gamma = \mathbb{R}$. (The simplification which makes Γ an Abelian group excludes non-Abelian gauge groups G .)

An infinitesimal loop is represented by

$$da \rightarrow 1 + i\chi da \quad (36)$$

which means that χ corresponds to the field strength. The action is therefore simply given by

$$S_0(\chi) = \frac{P}{4} \chi^2 \quad (37)$$

and the (Euclidean) functional integrals then have the form

$$\langle f \rangle = \frac{1}{2} \sqrt{\frac{\beta}{\pi}} \int_{-\infty}^{+\infty} f(\chi) e^{-\frac{\beta}{4} \chi^2} d\chi \quad (38)$$

The Wilson loop average is simply given by the standard Fourier transform of $e^{-\beta/4 \chi^2}$

$$\begin{aligned} \langle W(a) \rangle &= \frac{1}{2} \sqrt{\frac{\beta}{\pi}} \int_{-\infty}^{+\infty} e^{i a \chi} e^{-\frac{\beta}{4} \chi^2} d\chi \\ &= \exp(-a^2/\beta) \end{aligned} \quad (39)$$

We note the strong exponential decay of $\langle W(a) \rangle$ with the area a of the loops.

EXAMPLE 2: Loops on a sphere S^2

The loop group Γ is here defined in a similar way as in the first example, by identifying loops covering the same area a . By a we then mean the minimal area on the sphere. The area a is therefore restricted to the interval

$$-2\pi \leq a \leq 2\pi \quad (40)$$

and the endpoints of the interval correspond to the same loops on the sphere. Thus the loop group can be identified as the group of phase factors $U(1) \simeq S^1$ (Fig. 7)

$$C = \exp\left\{\frac{i}{2} a\right\} \in \Gamma \simeq U(1) \quad (41)$$

Also, in this case the gauge group is restricted to $G \simeq U(1)$ and the irreducible representations are given by

$$a \rightarrow \exp\left\{\frac{i}{2} n a\right\}, \quad n \in \mathbb{Z} \quad (42)$$

The space of representations is therefore discrete, $\hat{\Gamma} = \mathbb{Z}$. As the quantity corresponding to the field strength we have $\frac{1}{2}n$, and the action is therefore given by

$$S_0(n) = \frac{\beta}{16} n^2 \quad (43)$$

The functional integral reduces to a sum

$$\langle f \rangle = \sum_{n=-\infty}^{+\infty} f_n e^{-\frac{\beta}{16} n^2} / \sum_n e^{-\frac{\beta}{16} n^2} \quad (44)$$

with a similar expression for the Fourier transformation defining the Wilson loop average

$$\langle W(a) \rangle = \sum_n e^{\frac{i}{2} n a} e^{-\frac{\beta}{16} n^2} / \sum_n e^{-\frac{\beta}{16} n^2} \quad (45)$$

We note that the integer n can be interpreted as an instanton number and due to the simplification of Γ introduced above, only one configuration (representation) exists in each instanton sector. The " θ vacuum" would in this example correspond to the following modification of the action

$$S_0(n) \rightarrow S'_0(n) = \frac{\beta}{16} (n^2 + \theta n) \quad (46)$$

EXAMPLE 3: Lattice groups

The full loop group on a two-dimensional lattice is the free group generated by a set of loops associated with each of the plaquettes of the lattice. The two simplest subgroups, which we will examine below, are the groups of loops running around one and two plaquettes, respectively.

The loops of the group associated with a single plaquette are characterized by the winding number n of the loops around the plaquette (Fig. 8). Thus the loop group is identical to the additive group of integers

$$n \in \Gamma \approx \mathbb{Z} \quad (47)$$

The irreducible unitary representations of this group are defined by the phase factors

$$n \rightarrow \exp \{in\chi\} , \chi \in [0, 2\pi] \quad (48)$$

and the space of representations can consequently be identified as the circle $\hat{\Gamma} = S$. (Note that there is a duality relation - in the sense of Fourier transformations - between the spaces Γ and $\hat{\Gamma}$ in this example and the corresponding spaces in Example 2.)

The action we take to be the Wilson action

$$S_0(\chi) = \frac{\beta}{2} (1 - \cos \chi) \quad (49)$$

We then get the expression for the functional integrals

$$\langle f \rangle = Z^{-1} \int_0^{2\pi} f(\chi) \exp \left\{ -\frac{\beta}{2} (1 - \cos \chi) \right\} d\chi \quad (50)$$

with Z as the normalization constant

$$Z = \int_0^{2\pi} \exp \left\{ -\frac{\beta}{2} (1 - \cos \chi) \right\} d\chi \quad (51)$$

The Wilson loop average in particular gets the form

$$\langle W(n) \rangle = Z^{-1} \int_0^{2\pi} \exp \{ in\chi - 2\beta(1 - \cos \chi) \} d\chi \quad (52)$$

For the loop group defined on two plaquettes things get much more complicated. The loop group is then the free group on two generators (Fig. 9) and this group possesses irreducible representations of arbitrary high dimensions. It is well known that this group is not of type I and hence the present theory of Fourier transformations cannot be applied. This illustrates the fact, which we have previously commented on in Sect. 3, that the lattice regularization of Γ does not seem to be appropriate within the present framework.

APPENDIX

Let G be an Abelian locally compact group. All unitary irreducible representations π are one-dimensional and are characterized completely by $\chi(g) = \text{Tr} \pi(g)$ (normalized so that $\chi(1) = 1$). The set of irreducible characters is a group with respect to multiplication and can in a natural way be given a topology such that it becomes a locally compact group $\hat{G}$.

The Fourier transformation F maps functions on G into functions on $\hat{G}$ and is defined by

$$F(\varphi)(\chi) = \int_G \varphi(g) \chi(g) dg$$

where dg is the Haar measure on G .

The Fourier transformation is actually an isometric operator from $L^2(G)$ onto $L^2(\hat{G})$.

The inverse is given in terms of the Fourier transformation on $\hat{G}$ by

$$\begin{aligned} \bar{F}(\psi)(g) &= \int_{\hat{G}} \psi(\chi) \chi(g)^* d\chi \\ &= \int_{\hat{G}} \psi(\chi) \chi(g^{-1}) d\chi \end{aligned}$$

where $d\chi$ is the Haar measure on $\hat{G}$.

For G non-Abelian and locally compact this can be generalized in the following way. Up to unitary equivalence a representation π is characterized by

$$\chi(g) = \text{Tr} \pi(g)$$

the trace being taken in any orthonormal basis in the representation space.

In this case, the set of irreducible unitary characters $\hat{G}$ cannot be given the structure of a group in any natural manner, but if the group G is separable and of type I (this means that the algebra generated by the left regular

representation of G is of type I in the sense of von Neumann algebras²⁰⁾ one can introduce a Borel structure and a measure $d\chi$ on $\hat{G}$ and define a Fourier transformation $\mathcal{F}$ which we will now describe (for details see Refs 18) and 19)).

For any $\chi \in \hat{G}$ one chooses $\pi_\chi \in \chi$ (where we identify χ with the equivalence class of representations with character χ) in a measurable way. Then for a complex valued function ϕ on G we set

$$\mathcal{F}(\phi)(\chi) = \int \phi(g) \pi_\chi(g) dg$$

Here dg is a left invariant measure on G . Thus $\mathcal{F}(\phi)$ is a matrix-valued function on $\hat{G}$ such that $\mathcal{F}(\phi)(\chi)$ is a matrix in the representation space of π_χ .

It can be shown that there exists a measure $d\chi$ on $\hat{G}$ and a matrix-valued function $K(\chi)$ with the property that

$$\pi_\chi(g)^{-1} K(\chi) \pi_\chi(g) = \Delta(g)^{-1} K(\chi)$$

where $\Delta(g)$ is the modular function on G , such that if one defines the inverse Fourier transform $\bar{\mathcal{F}}(T)$ of a matrix-valued function $T(\chi)$ by

$$\bar{\mathcal{F}}(T)(g) = \int_{\hat{G}} \text{Tr}(T(\chi) \pi_\chi(g^{-1})) d\chi$$

then we have the Fourier inversion formula

$$\begin{aligned} \phi(g) &= \bar{\mathcal{F}}(\mathcal{F}(\phi)(\chi) K(\chi))(g) \\ &= \int_{\hat{G}} \text{Tr}(\mathcal{F}(\phi)(\chi) K(\chi) \pi_\chi(g^{-1})) d\chi \end{aligned}$$

The function $K(\chi)$ reflects the non-unimodularity of the group, that is the left- and right invariant Haar measures are not equal but related by Δ . Especially if G is compact and hence unimodular then $K(\chi) = \mathbb{1}_\chi$, and $\hat{G}$ becomes a discrete space. In this case the above results are equivalent to the theorem of Peter and Weyl (see Ref. 21)).

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FIGURE CAPTIONS

- Fig. 1 : Definition of the group of loops Γ : a) loops which differ only by pieces which are run back and forth are identified; b) group multiplication is defined by the standard composition of loops.
- Fig. 2 : A small loop of the form $s = C_x \sigma_x C_x^{-1}$ which is used in the definition of the metric on Γ . It is associated with a well-defined minimal area located in the neighbourhood of the point x .
- Fig. 3 : Definition of the distance $d(l, C)$. The loop C is deformed into the trivial loop l by means of small loops with a well-defined minimal area. The distance is measured as the sum of the areas of the small loops.
- Fig. 4 : Examples of paths of shortest length in Γ . The loop C in these cases consists of two circles of radius r , traversed in opposite directions, and with a separation h between the circles.
a) deformation of C corresponding to minimal area for $h \ll r$;
b) a deformation of C corresponding to minimal area for $h \gg r$.
- Fig. 5 : Two possible deformations of a loop C into the trivial loop l , where both deformations correspond to paths of shortest length $d(l, C)$ in Γ . The distance $d(C', C'')$ between the intermediate loops is equal to $d(l, C)$. This ambiguity in the definition of shortest paths (geodesics) in Γ is present for arbitrarily small loops C .
- Fig. 6 : Identification of loops in the two-dimensional plane. The loops are identified by the area which they cover.
- Fig. 7 : Identification of loops on the sphere. The loops are identified by their area, modulo 4π (i.e., the total area of the sphere). The corresponding loop group is equivalent to the group of phase factors $U(1)$.
- Fig. 8 : The one-plaquette loop group. The loops are identified by the winding number n of the loops around the plaquette.
- Fig. 9 : The two-plaquette group. The loops A and B are the two generators of the group.

Fig. 1

Fig. 2

Fig. 3

Fig.4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9