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ON THE ATIYAH-DRINFELD-HITCHIN-MANIN CONSTRUCTION
FOR SELF-DUAL GAUGE FIELDS

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A B S T R A C T

The vector spaces A, B, C, in terms of which the general construction due to Atiyah, Drinfeld, Hitchin and Manin for self-dual gauge fields defined over some region of Euclidean space is phrased, are shown to be expressible in terms of the spaces spanned by the solutions of certain linear covariant differential equations depending on the gauge field. The corresponding linear maps between A and B, B and C are given with the properties required by ADHM and the results then necessary to verify the construction informally proved. The local problems associated with assuming the gauge field to obey the self-duality equations are separated from the global problems of assuring the required boundary conditions for a particular solution. With suitable global conditions C is shown to be the dual of A and a natural scalar product defined on B so as to reconstruct the gauge field in the standard form given by the construction. The procedure is explicitly applied to self-dual gauge fields given by the 't Hooft and axially symmetric Witten-Peng solutions and when specialized to the Prasad-Sommerfield monopole shown to lead to a derivation of the form of the construction obtained in that case by Nahm.

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1. INTRODUCTION

Given that non-Abelian gauge theories are an essential feature in our theoretical description of particle physics it seems desirable to explore their mathematical structure in some detail. In this context the analysis of possible solutions of the classical equations obeyed by the gauge field is of interest. An important set, although of course in no way is it the general case, are the solutions of the Euclidean self-duality equations. For a gauge field $A_\mu = A_\mu^a t_a$, $\{t_a\}$ being a set of matrix generators forming a basis for the Lie algebra of a gauge group $\mathcal{G}$, these read

$$F_{\mu\nu} = {}^* F_{\mu\nu} \equiv \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} F_{\alpha\beta},$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$
(1.1)

The classical field equations in this case are guaranteed to be satisfied by virtue of the Bianchi identity (throughout we only consider self-dual fields, for anti self-dual fields $F_{\mu\nu} = -{}^* F_{\mu\nu}$ trivial modifications are required).

Such self-dual gauge fields have been considered in the following contexts.

1) The original Belavin, Polyakov, Schwartz and Tyupkin (BPST) instanton¹⁾ and for $\mathcal{G} = \text{SU}(2)$ the multi-instanton generalizations due to 't Hooft²⁾ and with collinear $O(3)$ rotational symmetry to Witten and Peng³⁾. The latter case has been extended to a general gauge group $\mathcal{G}$ ⁴⁾. For multi-instanton solutions the gauge fields are defined on the one point compactification of Euclidean four-dimensional space, topologically identical to S^4 , so that the gauge field behaves as a pure gauge at infinity, i.e.,

$$|x| \rightarrow \infty, A_\mu \sim g \partial_\mu g^{-1}, g(\hat{x}) \in \mathcal{G}, \hat{x} = x/|x|.$$
(1.2)

These solutions are specified by an integer valued topological index

$$k = -\frac{1}{16\pi^2} \int d^4x \text{tr}(F_{\mu\nu} {}^* F_{\mu\nu})$$
(1.3)

($\text{tr}(t_a t_b) = -1/2 \delta_{ab}$) and correspond to minima of the Euclidean gauge theory action. For any given k and gauge group $\mathcal{G}$ the general solution depends on parameters ranging over some space, the dimensions of which is well understood⁵⁾, but whose other properties are unclear. The known explicit solutions²⁾⁻⁴⁾ correspond only to a subspace of the full parameter space for $k > 2$.

2) The so-called caloron solutions^{6),7)} which are multi-instanton gauge fields periodic in one Cartesian co-ordinate, thus being defined on $R^3 \times S^1$. They are appropriate in considering the quantized gauge field system at finite temperature.

3) The Prasad-Sommerfield monopole solution⁸⁾ for $\mathcal{G} = SU(2)$ of the Bogomolny equations⁹⁾, to which the self-duality equations (1.1) reduce when the gauge fields are independent of one Cartesian co-ordinate, e.g., x_4 , hence being defined as R^3 , and the associated component of the gauge field, $A_4 = \Phi$, behaves as a Higgs scalar with boundary condition

$$|\Phi| \rightarrow \infty, \quad |\Phi| \rightarrow C > 0, \quad |\Phi|^2 = -2\kappa(\Phi^2). \quad (1.4)$$

These solutions of the Bogomolny equations have been extended to describe a single monopole with spherical symmetry for general gauge groups $\mathcal{G}$ ¹⁰⁾ and recently multimonopole solutions have been found for $\mathcal{G} = SU(2)$ both when they are superposed¹¹⁾ and, albeit less explicitly, when they are separated¹²⁾ with the full set of parameters¹³⁾.

4) The so-called chromon solutions¹⁴⁾ which are self-dual excitations of a covariantly constant Abelian self-dual background. They are thus defined on R^4 with the Euclidean action density even asymptotically non-zero.

5) The minimum action field configurations on a hypertorus T^4 with twisted boundary conditions satisfy the self-duality equations and can be given explicitly for $\mathcal{G} = SU(n)$ ¹⁵⁾.

Besides the above the self-duality equations may be dimensionally reduced further¹⁶⁾ but interesting solutions are harder to find. In each case described above the global, and hence topological, aspects differ although techniques of solutions are similar. The caloron solutions^{6),7)} are expressed in terms of the 't Hooft representation of self-dual gauge fields²⁾, as can the PS monopole¹⁷⁾. Collinear instantons⁴⁾ and spherically symmetric monopoles¹⁰⁾ are closely related while recent multimonopole solutions^{11),12)} have been obtained in terms of a framework originally developed to describe instantons¹⁸⁾. However, despite these various results, the sets of solutions of the self-duality equations at present available are by no means complete.

As against this the approach of Atiyah, Drinfeld, Hitchin and Manin¹⁹⁾ (ADHM) offers a general construction for multi-instanton gauge fields on S^4 with many desirable features, in particular

- a) It is applicable for arbitrary topological index k and gauge group G , with minor modifications^{20),21)}, and all multi-instanton fields on S^4 are guaranteed in principle to be expressible in the form given by the construction with the full range of parameters. The gauge arbitrariness of the solutions occurs naturally and explicitly.
- b) For a self-dual gauge field A_μ given as a tensor product of differing self-dual gauge fields for differing gauge groups, or for a self-dual gauge field A_μ in other than the fundamental representation of the gauge group, an explicit connection of the formalism of the ADHM construction for A_μ can be made to that for the individual gauge fields in the tensor product, or to that for the corresponding gauge field in the fundamental representation²²⁾.
- c) The Greens function for the covariant Laplacian on flat space D^2 , $D_\mu = \partial_\mu + A_\mu$, is given very simply in terms of natural objects of the construction, and the result can be used as a basis for the Greens function for higher spin operators^{20),21)}.
- d) Solutions of the Dirac equation, and also higher spin operators, in the self-dual gauge field background are straightforwardly given and their normalization integrals on flat space can be computed^{20),23)}.
- e) A substantial body of results exists for the multi-instanton functional determinants that arise in semi-classical approximations to functional integrals about such self-dual background gauge fields²⁴⁾.

By virtue of (b) the results described in (c), (d) and (e) can be extended to tensor products and other representations of the gauge group. Despite these various advantages, and its inherent simplicity, the ADHM construction has not so far proved very successful in generating new explicit self-dual solutions with an unconstrained parametrization, save for a $k = 3$ $SU(2)$ multi-instanton solution²¹⁾. General solutions for arbitrary k would be desirable for computing the functional measure for gauge theories in the semi-classical approximation. In the hope of eventually finding such a wider class of self-dual gauge fields we have investigated the ADHM construction with a view to making it more constructive and also applying it to the monopole solutions described earlier.

The starting point for the construction is the observation of Ward²⁵⁾, see also Yang²⁶⁾, that self-duality, of $F_{\mu\nu}$, Eq. (1.1), implies that the connection A_μ is integrable on anti self-dual planes in complexified, compactified Euclidean four-dimensional space. Thus, for $n_{i\mu}$, $i = 1, 2$ two independent tangent vectors (for Euclidean space with metric $\delta_{\mu\nu}$ we make no distinction between tangent vectors and their duals) to such an anti self-dual plane S ,

$$n_i \cdot n_j = 0, \quad n_{1\mu} \wedge n_{2\nu} = -^*(n_{1\mu} \wedge n_{2\nu}) \quad (1.5)$$

we have then

$$[n_1 \cdot D, n_2 \cdot D] = 0 \quad (1.6)$$

which is the requisite integrability condition for

$$n_i \cdot D\rho|_{x \in S} = 0 \quad (1.7)$$

defining a covariantly constant scalar on S .

The anti self-dual planes $\{S\}$ in four dimensional space can be specified by points in CP^3 , given in homogeneous co-ordinates by complex four-vector ξ^α , arbitrary up to $\xi^\alpha \rightarrow \lambda \xi^\alpha$. Decomposing ξ into two-spinors

$$\xi^\alpha = (\eta_A, \chi^{A'}), \quad A, A' = 1, 2, \quad (1.8)$$

then the anti self-dual plane S_ξ corresponding to ξ is defined by the equation

$$\pi^{A'}(x) \equiv \chi^{A'} - x^{A'A} \eta_A = 0, \quad (1.9)$$

for $x^{A'A} = (x_\mu e_\mu)^{A'A}$, e_μ being an appropriate basis of 2×2 matrices or quaternions, described in more detail in Appendix A where the tangent vectors to S_ξ are shown to be expressible in terms of η so as to satisfy (1.5). In the analysis of self-dual gauge fields Ward²⁵⁾ was then led to consider the space E_ξ of all covariantly constant scalar fields on S_ξ which may be defined by

$$E_\xi = \{\rho\}/\sim, \quad n_i \cdot D\rho|_{x \in S_\xi} = 0, \quad (1.10)$$

$$\rho \sim \rho' \quad \text{if} \quad (\rho - \rho')|_{x \in S_\xi} = 0.$$

Assuming the gauge field A_μ is irreducible, i.e.,

$$D_\mu \rho = 0 \Rightarrow \rho = 0 \quad (1.11)$$

then E_ξ has the dimension d of the representation space of $\mathcal{G}$ on which A_μ acts. The vector bundle E thereby formed over CP^3 with fibres E_ξ then encodes the gauge connection A_μ and from E A_μ can be reconstructed up to the usual gauge arbitrariness²⁵⁾.

In the ADHM construction E_ξ is determined by linear algebra in terms of vector spaces A, B, C and linear maps, $f(\xi), g(\xi)$, homogeneous of degree one in ξ where

$$A \xrightarrow{f(\xi)} B \xrightarrow{g(\xi)} C, \quad g(\xi)f(\xi) = 0 \quad (1.12)$$

so that

$$E_\xi = \ker g(\xi) / \operatorname{im} f(\xi). \quad (1.13)$$

In the original proof of the construction¹⁹⁾ the spaces A, B, C are represented in term of sheaf cohomology groups, with which the theoretical physics community is perhaps not generally familiar (at least the author) although there are discussions designed specifically to introduce these ideas and then show the general existence of the construction²⁷⁾⁻²⁹⁾. With this powerful framework the global problem of defining the gauge field A_μ on S^4 with given topological index k , i.e., obtaining the limiting behaviour (1.2), is resolved simultaneously with the local problem of getting A_μ to be a solution of the self-duality equations.

Nevertheless the ADHM construction is not only limited to this global requirement since Nahm has shown³⁰⁾ how it can be made to work for the simple PS monopole with all the usual desirable features. However, in this case the form to take for the spaces A, B, C and the associated linear maps f, g is not so clear cut and was not guaranteed to be possible by general results. Still, as stressed by Witten in an article³¹⁾ which stimulated this investigation, the ADHM construction may be considered purely locally for self-dual gauge fields defined over some open region of four-dimensional Euclidean space. The spaces A, B, C are then defined by the solutions of certain covariant linear differential equations with the gauge field A_μ as a background.

Thus in Section 2 of this paper we set out these equations, describe the associated linear maps f, g and prove the relevant results to ensure (1.13). It should be stressed that we make no pretensions to exact rigour. Notational details are collected in Appendix A. In Appendix B we give a brief sketch of the relationship between the various differential equations and the corresponding sheaf cohomology groups. Also in another Appendix C we briefly describe how the dimensionality of the space B may be determined. In Section 3 we show that with suitable extra global requirements, which ensure amongst other aspects,

$$\ker F(\xi) = 0 \quad (1.14)$$

a positive definite scalar product $\langle, \rangle$ can be defined on B , and also C is the dual of A , so that the standard form of the construction is realized,

$$A_\mu(x) = \langle v(x), \partial_\mu v(x) \rangle, \quad \langle v(x), v(x) \rangle = 1_g, \quad (1.15)$$

with $v(x) \in B$ satisfying certain linear equations given later.

In Sections 4 and 5 we apply these results to the 't Hooft²⁾ and Witten, Peng³⁾ self-dual solutions respectively. It can be shown³²⁾ that these two forms of the $SU(2)$ multi-instanton solution are gauge equivalent but there are interesting differences in formulation which make them possibly worth considering separately. In each case the PS monopole is considered and the results of Nahm³⁰⁾ obtained in a less inspired fashion. A few brief final remarks are offered by way of conclusion.

2. SPACES AND MAPS FOR THE ADHM CONSTRUCTION

With the notational details set out in Appendix A the vector spaces A , B , C are defined, following partially Witten's outline³¹⁾, in reverse order.

Space C

This is spanned by solutions of the Dirac equation

$$D_{AA'} \psi^{A'} = 0, \quad (2.1)$$

or, equivalently, using the spinor derivatives introduced in (A.10)

$$d_u^{A'} \psi^{B'} - d_u^{B'} \psi^{A'} = 0, \quad (2.2)$$

for $\psi^{A'}(x)$ independent of u_B . A linearly independent complete set of solutions of (2.1), or (2.2), is assumed to be assembled as a row vector $\psi^{A'}$ so, with this basis for C , an arbitrary element is expressible as

$$\psi^{A'} w', \quad (2.3)$$

for some column vector w' , taken to be in a space $\tilde{C}$.

Space B

Just as discussed for C, the elements of B are given in terms of a suitable complete basis, with $v \in \tilde{B}$, by

$$\phi v, \Omega_B^{A'} v \quad (2.4)$$

with the pair $(\phi, \Omega_B^{A'})$ forming row vectors of linearly independent solutions of

$$d_u^{A'} \phi = \Omega_B^{A'} z^B, \quad z = (u, xu), \quad (2.5a)$$

$$D_{AA'} \Omega_B^{A'} = 0, \quad (2.5b)$$

for $(\phi, \Omega_B^{A'})$ independent of u . Consistency of Eqs. (2.5) requires that the field strength $F_{\mu\nu}$ corresponding to the gauge field A_μ , $D_\mu = \partial_\mu + A_\mu$, is self-dual. Since C is complete we can write

$$\Omega_B^{A'} = \psi^{A'} A_B', \quad (2.6)$$

where $A_B' : \tilde{B} \rightarrow \tilde{C}$. Further defining, for z^B as in (2.5a),

$$A'(z) = A_B' z^B = \Delta'^B(x) u_B \quad (2.7)$$

then Δ'^B is linear in x , explicitly, with $A_B' = (a'^B, b'_B)$,

$$\Delta'^B(x) = a'^B + b'_B x^{B'B}. \quad (2.8)$$

With Δ' in the form (2.8), Eqs. (2.5) can be alternatively expressed as

$$D^{AA'} \phi = \psi^{A'} \Delta'^A. \quad (2.9)$$

Space A

Similarly the elements of A may be represented with $w \in \tilde{A}$, by

$$\lambda_{B'} w, \quad (2.10)$$

with $\lambda_{B'}$ a row vector of a complete set of linearly independent solutions of

$$D_{AA'} D^2 \tilde{\lambda}^{A'} = 0. \quad (2.11)$$

In order to discuss the map $f : A \rightarrow B$ an alternative characterization of A , involving some redundancy, is necessary, Define

$$\lambda_B = (\lambda^B, \lambda_{B'}), \quad \lambda^B(x) = -\lambda_{B'}(x)x^{B'B}, \quad (2.12)$$

and then

$$d_u^A \lambda_B = P_{B\gamma}^A z^\gamma, \quad (2.13)$$

with u arbitrary, $P_{B\gamma}^A$ independent of u and

$$P_{B'C'}^A = \frac{1}{2} D^2 \tilde{\lambda}^A \varepsilon_{B'C'}. \quad (2.14)$$

From (2.12)

$$\lambda_B x^{BD} = 0, \quad z^B = x^{BD} u_D \quad (2.15)$$

and applying d_u^A to this using (2.13) gives

$$-2 \tilde{\lambda}^A \varepsilon^{DE} = P_{B\gamma}^A x^{BD} x^{\gamma E}, \quad (2.16)$$

so that we require, consistent with (2.14),

$$P_{B\gamma}^A = -P_{\gamma B}^A. \quad (2.17)$$

Further from (2.12), (2.13), (2.14) and (2.17) we can show

$$D^2 \lambda_B = 2 P_{BA'}^A$$

and hence, applying $D_{AA'}$ to (2.13)

$$(D_{AA'} P_{B\gamma}^A) z^\gamma = 0. \quad (2.18)$$

But (2.11) implies $D_{AA'} P_{B'C'}^A = 0$ and, in conjunction with (2.18), it is now easy to see that

$$D_{AA'} P_{B\gamma}^A = 0. \quad (2.19)$$

The alternative description of A is then in terms of the pair $(\lambda_\beta, P_{\beta\gamma}^{A'})$ obeying (2.12), (2.13), (2.17) and (2.19), which are equivalent to the simple Eq. (2.11).

However, this set of equations is clearly similar in form to Eqs (2.5) and since B is furthermore complete then, as in (2.6),

$$\lambda_\beta = \phi A_\beta, \quad P_{\beta\gamma}^{A'} = \Omega_\gamma^{A'} A_\beta = \psi^{A'} A'_\gamma A_\beta, \quad (2.20)$$

where $A_\beta : \tilde{A} \rightarrow \tilde{B}$ and from (2.17)

$$A'_\gamma A_\beta = -A'_\beta A_\gamma. \quad (2.21)$$

If Δ^B is introduced in terms of A_β just as in (2.7), and so is also expressible in the form (2.8) for $A_\beta = (a^B, b_B)$, then (2.21) becomes equivalently

$$\Delta'^A \Delta^B = \varepsilon^{AB} c, \quad (2.22)$$

with $c(x) : \tilde{A} \rightarrow \tilde{C}$ quadratic in x . From (2.12) or (2.15)

$$\phi \Delta^B = 0. \quad (2.23)$$

Alternatively, requiring the maximal linearly independent set of solutions of (2.23) for all x may be taken as the definition of $\Delta^B(x) = a^B + b_B x^{B'}$ since, as in the connection between (2.16) and (2.15), applying $D^{A'A}$ to (2.23) ensures the condition (2.22) and then gives the relation

$$\psi^{A'} c = 2\tilde{\lambda}^{A'} = 2\phi \tilde{b}^{A'}. \quad (2.24)$$

Equation (2.24) does not entail any further constraints; applying $D_{AA'}$ to both sides gives an identity, using (2.1), (2.5) or (2.13), and $\partial_{AA'} c = 2\Delta_{A'}^B b_A$ from its definition (2.22).

The linear maps f, g in (1.12) required for the construction are now straightforwardly defined, for $A \rightarrow B$ by

$$\lambda_\beta w, \quad P_{\beta\gamma}^{A'} w \xrightarrow{f(\xi)} \lambda_\beta \xi^\beta w, \quad P_{\beta\gamma}^{A'} \xi^\beta w = \phi v, \quad \Omega_\gamma^{A'} v, \quad (2.25)$$

$$v = A(\xi) w,$$

and for $B \rightarrow C$ by

$$\begin{aligned} \phi v, \Omega_{\beta}^{A'} v &\xrightarrow{g(\xi)} \Omega_{\beta}^{A'} \xi^{\beta} v = \psi^{A'} w', \\ w' &= A'(\xi) v. \end{aligned} \quad (2.26)$$

Obviously $g(\xi)f(\xi) = 0$ as a result of (2.17) or (2.21), $A'(\xi)A(\xi) = 0$.

The analysis of $\ker g(\xi)$ and $\operatorname{im} f(\xi)$, which appear in (1.13) is simple: $\ker g(\xi)$. Suppose, for some v ,

$$\Omega_{\beta}^{A'} \xi^{\beta} v = 0, \text{ or } A'(\xi) v = 0, \quad (2.27)$$

then, if $\Omega_{\beta}^{A'} = (\Omega_{\beta}^{A'B}, \Omega_{\beta}^{A'})$, from (2.5), and the definitions (1.8), (1.9),

$$d_{\eta}^{A'} \phi v = -\Omega_{\beta}^{A'} v \pi^{\beta}. \quad (2.28)$$

As discussed in Appendix A, $d_{\eta}^{A'} A' = 1, 2$ form a basis for tangential derivatives to the anti self-dual plane S_{ξ} , on which $\pi^{B'}(x) = 0$. Hence (2.28) shows that ϕv is covariantly constant on S_{ξ} .

$\operatorname{im} f(\xi)$. Suppose $(\phi v, \Omega_{\alpha} v) \in \operatorname{im} f(\xi)$, or $v = A(\xi)w$ for some w , then from (2.12) and (1.8), (1.9)

$$\phi v = \lambda_{B'} w \pi^{B'}, \quad (2.29)$$

so ϕv vanishes on S_{ξ} .

Thus by (1.10) $\ker g(\xi)$ and $\operatorname{im} f(\xi)$ have the properties required for (1.13). To prove (1.13) it is necessary further to show that B is large enough to include all covariantly constant scalars on S_{ξ} , and A large enough relative to B to ensure that E_{ξ} , as defined in (1.13), depends only on ξ . These requirements are the consequence respectively of the corollary to lemma 1 and lemma 2, which are now demonstrated,

Lemma 1: $g(\xi)$ is onto

Proof: for any $\psi^{A'} \in C$ it can be expressed in the form

$$\psi^{A'} - d_{\eta}^{A'} \rho = \wedge^{A'}_{B'} \pi^{B'}. \quad (2.30)$$

The content of (2.30) is that, since the Dirac equation in the form (2.2) in terms of $d_{\eta}^{A'}$ is the requisite integrability condition, on the plane S_{ξ} $\psi^{A'} = d_{\eta}^{A'} \rho$, by virtue of (A.11) using a version of the Poincaré lemma. Requiring the Dirac equation (2.1) to hold in general gives

$$-(D_{AA'} \Lambda^{A'}_{B'}) \pi^{B'} = (2 \Lambda^{A'}_{A'} + D^2 \rho) \eta_A ; \quad (2.31)$$

so that

$$2 \Lambda^{A'}_{A'} + D^2 \rho = \mu_{B'} \pi^{B'}. \quad (2.32)$$

Now let

$$\hat{\rho} = \rho - \nu_C \pi^C, \quad \hat{\Lambda}^{A'}_{B'} = \Lambda^{A'}_{B'} - d^{A'}_{\eta} \nu_{B'}, \quad (2.33)$$

where

$$D^2 \nu_{C'} = \mu_{C'}, \quad (2.34)$$

supposing only that D^2 can be inverted, although not necessarily uniquely.
With the redefinition (2.33), (2.30) becomes

$$\psi^{A'} = d^{A'}_{\eta} \hat{\rho} + \hat{\Lambda}^{A'}_{B'} \pi^{B'}, \quad (2.35)$$

but in this case

$$D_{AA'} \hat{\Lambda}^{A'}_{B'} = 0 \quad (2.36)$$

since, by the definition of $\nu_{C'}$ in (2.34), $2 \hat{\Lambda}^{A'}_{A'} + D^2 \hat{\rho} = 0$. Assuming

$$-d^{A'}_{\eta} \hat{\rho} = \hat{\Lambda}^{A'}_{\beta} \xi^{\beta} \quad (2.37)$$

for arbitrary u then defines $\hat{\Lambda}^{A'B}$, with the decomposition $\hat{\Lambda}^{A'}_{\beta} = (\hat{\Lambda}^{A'B}, \hat{\Lambda}^{A'}_{B'})$.
Applying $D_{AA'}$ to (2.37) finally shows, with (2.36), that $D_{AA'} \hat{\Lambda}^{A'}_{\beta} = 0$. Hence $(-\hat{\rho}, \hat{\Lambda}^{A'}_{\beta}) \in B$ and from (2.35) and (2.37)

$$\hat{\Lambda}^{A'}_{\beta} \xi^{\beta} = \psi^{A'}. \quad (2.38)$$

Thus for any w' there is a v , $w' = A'(\xi)v$.

Corollary: for ρ covariantly constant on S_{ξ} there is a v so that $(\rho - \phi v)|_{x \in S_{\xi}} = 0$, $A'(\xi)v = 0$.

If $\psi^{A'} = 0$ in (2.30) then it becomes the equation defining a general covariantly constant scalar on S_ξ . The preceding argument then shows that there is a $(-\hat{\rho}, \hat{\Lambda}_B^{A'}) \in B$ with $(\rho - \hat{\rho})|_{x \in S_\xi} = 0$, by (2.33), and further, by (2.38)

$$\hat{\Lambda}_B^{A'} \xi^B = 0. \quad (2.39)$$

Lemma 2: if $(\phi v, \Omega_B^{A'} v) \in \ker g(\xi)$ and $\phi v|_{x \in S_\xi} = 0$ then $(\phi v, \Omega_B^{A'} v) \in \text{im } f(\xi)$.

Proof: If ϕv vanishes on S_ξ then it can be assumed to be of the form

$$\phi v = \nu_{A'} \pi^{A'} \quad (2.40)$$

for some $\nu_{A'}$ and (2.28) becomes

$$(d_{\eta}^{A'} \nu_{B'} + \Omega_{B'}^{A'} v) \pi^{B'} = 0. \quad (2.41)$$

Hence

$$d_{\eta}^{A'} \nu_{B'} = -\Omega_{B'}^{A'} v + X^{A'} \tilde{\pi}_{B'} \quad (2.42)$$

defines $X^{A'}$ with the property $\tilde{d}_{\eta A'} X^{A'} = 0$ so we can solve

$$D^2 \nu_{\eta A} = D_{AA'} X^{A'}. \quad (2.43)$$

Now $\nu_{B'}$ may be redefined

$$\nu_{B'} = \lambda_{B'} + \nu \tilde{\pi}_{B'} \quad (2.44)$$

so that (2.42) becomes

$$d_{\eta}^{A'} \lambda_{B'} = -\Omega_{B'}^{A'} v - T^{A'} \tilde{\pi}_{B'} \quad (2.45)$$

where

$$T^{A'} = -X^{A'} + d_{\eta}^{A'} \nu, \quad D_{AA'} T^{A'} = 0. \quad (2.46)$$

Thus, since from (2.45),

$$D^2 \tilde{\lambda}^{B'} = 2T^{B'} \quad (2.47)$$

we see that λ_B obeys (2.11) and may be used to construct the corresponding $\lambda_B, P_{B\gamma}^{A'}$. Equation (2.45) then implies

$$\Omega_{B'}^{A'} v = - P_{B'\gamma}^{A'} \xi^\gamma$$

and, in conjunction with (2.5a), we finally get

$$\Omega_{B'}^{A'} v = P_{\gamma B'}^{A'} \xi^\gamma, \quad \phi v = \lambda_{B'} \pi^{B'} = \lambda_B \xi^B \quad (2.48)$$

with $(\lambda_B, P_{B\gamma}^{A'}) \in A$. ■

Hence the spaces A, B, C as defined at the start of this section, with the consequential maps f, g, are sufficient to give (1.13). For the subsequent discussion two further results are important.

Lemma 3: $\ker f(\xi) = \text{space of solutions of } D^2 r = 0$.

Proof: Suppose $A(\xi)w = 0$ or equivalently

$$\lambda_B \xi^B w = 0, \quad P_{B\gamma}^{A'} \xi^\gamma w = 0 \quad (2.49)$$

for some w. The condition on $\lambda_B w$ in (2.49), in conjunction with (2.12), requires

$$\lambda_{B'} w = r \tilde{\pi}_{B'} \quad (2.50)$$

for a scalar r. Applying $d_\eta^{A'}$ to (2.50) then gives, from (2.13) and (2.49)

$$P_{B'C'}^{A'} w = -d_\eta^{A'} r \xi_{B'C'} \quad (2.51)$$

and since this must satisfy the Dirac equation as in (2.19)

$$D^2 r = 0. \quad (2.52)$$

Conversely solutions of (2.52) may be used, via (2.50) and (2.51), to generate elements of $\ker f(\xi)$. ■

Lemma 4: $g(\xi)f(\bar{\xi})$ is onto
 $\bar{\xi}^\alpha$ is defined in (A.8), and, if ξ^α is decomposed as in (1.8), we may then write

$$\xi = (\eta, q, \eta), \quad \bar{\xi} = (\bar{\eta}, q, \bar{\eta}) \quad (2.53)$$

with $q = q_\mu e_\mu$, q_μ a real four-vector.

Proof: Suppose $\eta \neq 0$ (for $\eta = 0$ a similar proof can easily be given), then for $\psi^{A'} \in C$ define λ_B , through

$$\begin{aligned} & -\bar{\eta}\eta |x-q|^2 (x-q)^{A'A} D^2 \frac{(\overline{x-q})_{AB'}}{|x-q|^2} \tilde{\chi}^{B'}(x) \\ &= \bar{\eta}\eta (-|x-q|^2 D^2 \delta_B^{A'} + 2(\overline{x-q})_{AB'} D^{A'A} + 4\delta_B^{A'}) \tilde{\chi}^{B'}(x) \\ & \quad = 2\psi^{A'}(x) \end{aligned} \quad (2.54)$$

where the solution is required to be regular at $x = q$. Since $\psi^{A'}$ obeys the Dirac equation, λ_B satisfies (2.11) so that it is possible to construct $(\lambda_B, P_{\beta\gamma}^{A'}) \in A$. Inserting (2.13), (2.14) and (2.16) into (2.54) then gives the desired result

$$P_{\beta\gamma}^{A'} \bar{\xi}^\beta \xi^\gamma = \psi^{A'}. \quad (2.55)$$

Corollary: $\ker g(\xi)f(\bar{\xi}) = \text{space of solutions of } D^2 s_B = 0$.

If $P_{\beta\gamma}^{A'} \bar{\xi}^\beta \xi^\gamma w = 0$ then (2.54) holds for $\lambda_{B,w}$ with $\psi^{A'} = 0$. The solution of the resulting homogeneous equation can obviously be written

$$\tilde{\chi}^{B'}(x) w = (x-q)^{B'B} s_B(x), \quad D^2 s_B = 0. \quad (2.56)$$

By (2.7), (2.22) and (2.53) we have

$$A'(\xi)A(\bar{\xi}) = \bar{\eta}\eta c(q) \quad (2.57)$$

so lemma 4 ensures that $w' = c(q)w$ for any w' , and if $c(q)w = 0$ $\lambda_B w$ is of the form entailed by (2.56).

The spaces A, B, C have so far been defined in this section in terms of general solutions of four-dimensional partial differential equations which will involve arbitrary functions of three variables. With suitable global conditions on the gauge field A_μ they may be considerably restricted while still ensuring the results that are necessary for the validity of the construction in the form (1.13). For self-dual fields on S^4 , represented by $A_\mu(x)$ obeying the boundary condition (1.2), then solutions of the equations defining A, B, C are strongly constrained by requiring them to be globally defined, as sections, on this compact manifold. The conditions, corresponding to (1.2), are equivalent to imposing, on flat space, the asymptotic behaviour

$$\begin{aligned}\psi^{A'}(x) &\sim \frac{x^{A'A} \gamma_A}{|x|^4} g(\hat{x}), \quad \phi(x) \sim \beta g(\hat{x}), \\ \tilde{\chi}^{A'}(x) &\sim \frac{x^{A'A} \alpha_A}{|x|^2} g(\hat{x}), \quad r(x) \sim \frac{\delta}{|x|^2} g(\hat{x}), \\ s_B(x) &\sim \frac{\epsilon_B}{|x|^2} g(\hat{x}).\end{aligned}\tag{2.58}$$

Simple positivity arguments, with Eqs. (2.52) or (2.56), then show that r or s_B must vanish, $\ker f(\xi) = \ker g(\xi)f(\bar{\xi}) = 0$, so $c(q)$ is invertible. Further the spaces A, B, C are finite dimensional and by application of index theorems, or in Appendix C for B , then for $\mathcal{G} = \text{SU}(n)$,

$$\dim A = \dim C = k, \quad \dim B = 2k + n.\tag{2.59}$$

Thus in this case A'_α, A_β are just finite matrices.

For $\mathcal{G} = \text{Sp}(n)$, or $\text{O}(n)$, there are extra reality conditions that can be imposed. For $\text{Sp}(n)$ [for $\text{O}(n)$ the modifications are obvious] the group elements obey

$$g^\dagger g = 1_g, \quad g^T \sigma g = \sigma, \quad \sigma = \sigma^* = -\sigma^T, \quad \sigma^2 = -1\tag{2.60}$$

so $g^* \sigma = \sigma g$. The reality conditions are then

$$\begin{aligned}\sigma \tilde{\psi}^{A'} &= \psi^{A'} \sigma_C, \quad \sigma_C^2 = 1, \\ \sigma \phi^* &= \phi \sigma_B, \quad \sigma \tilde{\Omega}^{A'} \bar{\xi}^* = \Omega^{A'} \bar{\xi} \sigma_B, \quad \sigma_B^2 = -1, \\ \sigma \lambda^* \bar{\xi}^* &= \lambda \bar{\xi} \sigma_A, \quad \sigma \tilde{P}^{A'} \bar{\xi}^* \wedge z^* = P^{A'} \bar{\xi} \wedge \bar{z} \sigma_A, \quad \sigma_A^2 = 1,\end{aligned}\tag{2.61}$$

for some $\sigma_{A,B,C} = \sigma_{A,B,C}^*$ and with the notations $\Omega \cdot \xi = \Omega_\alpha \xi^\alpha$, $\lambda \cdot \xi = \lambda_\alpha \xi^\alpha$, $P^{A'} \cdot \xi \wedge z = P_{\alpha\beta}^{A'} \xi^\alpha z^\beta$. From (2.61) we get

$$\begin{aligned}\sigma_C A'(\xi)^* &= A'(\bar{\xi}) \sigma_B, \quad \sigma_B A(\xi)^* = A(\bar{\xi}) \sigma_A, \\ \sigma_C \bar{\Delta}' &= -\tilde{\Delta}' \sigma_B, \quad \sigma_B \bar{\Delta} = -\tilde{\Delta} \sigma_A.\end{aligned}\tag{2.62}$$

3. RECONSTRUCTION OF THE GAUGE FIELDS

As mentioned in the introduction the ADHM construction provides, besides the vector bundle E , a succinct prescription for the gauge field, (1.15), in terms of the vector spaces and their maps, once suitable scalar products have been defined. To do this it is necessary to require that the spaces A, B, C are so restricted that, as for S^4 , $\ker f(\xi) = \ker g(\xi)f(\bar{\xi}) = 0$. By virtue of lemma 4, $c(x)$ is then invertible for all x in the region over which the self-dual gauge field is defined, as is now assumed henceforth.

To derive a formula for A_μ introduce $F(x)$: $\tilde{C} \rightarrow \tilde{C}^*$ with the properties

$$F^\dagger = F, \quad F \text{ positive definite.} \quad (3.1)$$

We may then construct

$$I = \phi^\dagger \phi + \tilde{\Delta}'^\dagger F \tilde{\Delta}', \quad L = Fc, \quad (3.2)$$

where $I = I^\dagger$ ($\tilde{\Delta}'^\dagger = (\tilde{\Delta}')^T$). From now on we also prefer to suppress spinorial indices, displaying $e_\mu = \partial_\mu x$ and $\bar{e}_\mu$ explicitly). Using (2.22) in the form

$$\tilde{\Delta}' \Delta = 1_2 c \quad (3.3)$$

we obtain from the definition (3.2)

$$I \Delta = \tilde{\Delta}'^\dagger L. \quad (3.4)$$

Now if we compute $\partial_\mu I$, in terms of its definition (3.2), with the aid of (2.9) and (2.24) expressed now as

$$D_\mu \phi = -\frac{1}{2} \tilde{\varphi} e_\mu \tilde{\Delta}', \quad \tilde{\varphi} = 2\phi b c^{-1}, \quad (3.5)$$

and also their Hermitian conjugates since $A_\mu = -A_\mu^\dagger$, together with

$$\begin{aligned} \partial_\mu F &= -F \partial_\mu c c^{-1} - \partial_\mu L c^{-1}, \\ 1_2 \partial_\mu c &= \bar{e}_\mu \tilde{b}' \Delta + \tilde{\Delta}' b e_\mu, \end{aligned} \quad (3.6)$$

and also the derivative of (3.4)

$$I b e_\mu + \partial_\mu I \Delta = \tilde{b}'^\dagger L e_\mu + \tilde{\Delta}'^\dagger \partial_\mu L \quad (3.7)$$

then we find

$$P^\dagger \partial_\mu I P = 0 \quad (3.8)$$

where P is the projection operator

$$P = 1 - \Delta c^{-1} \tilde{\Delta}' \quad (3.9)$$

Suppose further that I is invertible, this requires that $Iu = 0$ has no non-trivial solutions. From (3.2) $Iu = 0$ implies $\phi u = 0$, $F\tilde{\Delta}'u = 0$ ($\tilde{\Delta}'^t w' = 0 \Rightarrow w' = 0$), or $u = \Delta w$, $Fcw = Lw = 0$, but if F is suitably chosen this requires $cw = 0$ or $w = 0$, by hypothesis. Then

$$PI^{-1}\phi^\dagger = I^{-1}\phi^\dagger \quad (3.10)$$

since $\tilde{\Delta}'I^{-1}\phi^\dagger = 0$ and (3.8) can be re-expressed as

$$\phi \partial_\mu I^{-1}\phi^\dagger = 0. \quad (3.11)$$

Also

$$\phi = \phi I^{-1}I = \phi I^{-1}\phi^\dagger \phi \quad (3.12)$$

so if ϕ is irreducible this gives

$$\phi I^{-1}\phi^\dagger = 1_g, \quad (3.13)$$

which is confirmed by $D_\mu(\phi I^{-1}\phi^\dagger) = 0$ ($D_\mu X = \partial_\mu X + [A_\mu, X]$) as a result of (3.11) and, with (3.5),

$$(D_\mu \phi) I^{-1}\phi^\dagger = 0. \quad (3.14)$$

Rewriting (3.14) then realizes a formula for A_μ ,

$$A_\mu = -\partial_\mu \phi I^{-1}\phi^\dagger = \phi I^{-1}\partial_\mu \phi^\dagger \quad (3.15)$$

using (3.11) again.

The result (3.15) expresses the gauge field in terms of ϕ , the solution of linear equations and the map Δ' . The standard ADHM construction is obtained if F is so chosen that I, L in (3.2) are independent of x . The appropriate choice is given by

$$\psi^\dagger \psi = -\partial^2 F. \quad (3.16)$$

For ψ defined on S^4 with the asymptotic behaviour of (2.58), or with other restrictions so that $-\partial^2$ is invertible, then F is uniquely defined with the properties (3.1). Using (3.16) and the definition (3.2) we may compute $\partial^2 I$ along the same lines as earlier with (3.5), (3.6), (3.7) and also

$$D^2 \phi = -2\hat{\psi}\hat{b}' \quad (3.17)$$

to obtain

$$\partial^2 I = 2\tilde{\Delta}'^+ c^{+-1} \bar{e} \cdot \partial L^+ \tilde{b}' + P^+ \partial_\mu I \Delta \bar{e}_\mu c^{-1} \tilde{b}' + h.c. \quad (3.18)$$

The result (3.18) also entails the analogous equation which is obtained by applying ∂^2 to $F = Lc^{-1}$ and then using (3.16), with ψ given by (3.15), and also on the right-hand side (3.6) together with its derivative. Clearly a consistent solution of (3.18) is for I, L to be constant. We have been unable to analyze (3.18), or to re-express it in such a form, that, with the appropriate boundary conditions such as those following from (2.58) and (3.2), (3.16)

$$I = O(1), L = O(1), |x| \rightarrow \infty \quad (3.19)$$

this result is necessary, as should be the case. A few further remarks are made in Appendix B but from now on we assume that

$$I: \tilde{B} \rightarrow \tilde{B}^*, L: \tilde{A} \rightarrow \tilde{C}^*, \quad (3.20)$$

are constant, as may be checked in particular applications.

In this situation it is possible to define a natural, positive definite, scalar product on $\tilde{B}$ and make $\tilde{C}$ the dual of $\tilde{A}$

$$\begin{aligned} \langle v, u \rangle &= v^+ I u, \\ \langle w', w \rangle &= w'^+ L w. \end{aligned} \quad (3.21)$$

By virtue of (3.4)

$$\begin{aligned} \langle v, \Delta w \rangle &= \langle \tilde{\Delta}' v, w \rangle, \\ \langle v, A(\xi) w \rangle &= - \langle A'(\bar{\xi}) v, w \rangle. \end{aligned} \quad (3.22)$$

With this structure the ADHM construction requires the finding of $v(x)$ satisfying

$$\begin{aligned} \langle v(x), \Delta(x) w \rangle &= 0 \quad \forall w \in \tilde{A}, \\ \langle v(x), v(x) \rangle &= 1_g \end{aligned} \quad (3.23)$$

and then, as in (1.15),

$$A'_\mu = \langle v, \partial_\mu v \rangle \quad (3.24)$$

defines, as easily verified^{20),21)}, a self-dual gauge field. From (3.4) and (3.13) the general solution of (3.23) is given by

$$v(x)^\dagger = g(x) \phi(x) I^{-1}, \quad g(x) g(x)^\dagger = 1_g, \quad (3.25)$$

and thus from (3.15)

$$A'_\mu = A_\mu^g = g A_\mu g^{-1} + g \partial_\mu g^{-1}, \quad (3.26)$$

thereby demonstrating the reconstruction of the original gauge field up to the gauge transformation $g(x) \in \mathcal{G}$.

The bases chosen for $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ are of course arbitrary and the results are invariant under

$$\begin{aligned} w' &\rightarrow P w', \quad v \rightarrow Q v, \quad w \rightarrow R w, \\ \Delta &\rightarrow Q \Delta R^{-1}, \quad \Delta' \rightarrow P \Delta Q^{-1}, \end{aligned} \quad (3.27)$$

for P, Q, R members of the general set of linear transformations on $\tilde{C}, \tilde{B}, \tilde{A}$. In consequence of (3.27)

$$I \rightarrow Q^{-1\dagger} I Q^{-1}, \quad L \rightarrow P^{-1\dagger} L R^{-1}. \quad (3.28)$$

The freedom in (3.28) can be used to set

$$I = 1, \quad L = 1 \quad \Rightarrow \quad \tilde{\Delta}' = \Delta^\dagger \quad (3.29)$$

and then, in (3.27), the remaining arbitrariness is restricted to $Q^\dagger Q = 1$, $P^\dagger R = 1$.

For $\mathcal{G} = \text{Sp}(n)$ [or $O(n)$] then with (2.61) and (2.62) further conditions can be imposed. Define.

$$w'_\sigma = \sigma_C w'^*, \quad v_\sigma = \sigma_B v^*, \quad w_\sigma = \sigma_A w^* \quad (3.30)$$

and then the bilinear forms

$$\begin{aligned} [v, u] &= -\langle v_\sigma, u \rangle = v^T J u, \\ [w', w] &= \langle w'_\sigma, w \rangle = w'^T K w, \end{aligned} \quad (3.31)$$

where from (2.61) and (2.62) with (3.2) and (3.16)

$$\begin{aligned} J &= -\sigma_B^T I = \phi^T \sigma \phi + \Delta'^T F \tilde{\Delta}' = -J^T, \\ K &= \sigma_C^T L = F_C, \quad \tilde{\psi}^T \sigma \psi = -\partial^2 F, \quad F^T = F \end{aligned} \quad (3.32)$$

Thus (3.31) gives a skew symmetry form on $\tilde{B}$, $[v, u] = -[u, v]$. Also from (3.22)

$$[v, A(\xi)w] = [A'(\xi)v, w] \quad (3.33)$$

or

$$J \Delta(x) = \Delta'(x)^T K. \quad (3.34)$$

With a suitable basis as with (3.29) we can choose $\sigma_C = \sigma_B = K = 1$ and $J = -\sigma_B^T = \sigma_B$. The spaces $\tilde{A}$, $\tilde{C}$ are then real while $\tilde{B}$ can be taken to be quaternionic.

4. 't HOOFT SOLUTION

The local solution of the self-duality equation (1.1) discovered by 't Hooft⁽²⁾, for $\mathfrak{G} = \text{SU}(2)$, is expressible in our notation by

$$A_\mu = \frac{1}{2} \eta_{\mu\sigma} \partial_\sigma \ln \varphi, \quad \partial^2 \varphi = 0, \quad (4.1)$$

where it should be recognized that $\eta_{\mu\sigma}$, and often $e_\mu, \bar{e}_\mu$ subsequently, although having exactly the properties listed in (A.1) are acting to or from two-dimensional spaces other than those for primed and unprimed spinors exemplified in (A.2).

With the solution (4.1) it is straightforward to locally solve the equations defining the spaces A, B, C. It is of some assistance for this purpose to note, from (4.1),

$$\begin{aligned} \varphi^{\pm} D_\mu \varphi^{-\pm} &= \partial_\mu - \frac{1}{2} \bar{e}_\mu \cdot \partial \ln \varphi e_\mu, \quad \varphi^{-\pm} D_\mu \varphi^{\pm} = \partial_\mu + \frac{1}{2} \bar{e}_\mu e_\mu \cdot \partial \ln \varphi, \\ \varphi^{\pm} D^2 \varphi^{-\pm} &= \partial^2 - \bar{e}_\mu \cdot \partial \ln \varphi e_\mu \cdot \partial. \end{aligned} \quad (4.2)$$

The Dirac equation, written as

$$D_\mu \tilde{\psi} e_\mu = 0 \quad (4.3)$$

which defines the space C , was solved by Grossman³³⁾ obtaining

$$\tilde{\psi} = \varphi^\pm \bar{e} \cdot \partial \left(\frac{f}{\varphi} \right), \quad \partial^2 \rho = 0. \quad (4.4)$$

Regarding ρ now as representing a row vector of solutions, as is also then $\tilde{\psi}$, in (4.4) it is easy to obtain, corresponding to (3.16),

$$\text{tr}_2(\tilde{\psi}^\dagger \tilde{\psi}) = -\partial^2 F, \quad F = d - \frac{1}{\varphi} \rho^T \rho. \quad (4.5)$$

Here tr_2 designates a trace on spinorial indices and d is a solution of $\partial^2 d = 0$, necessary in order to later ensure that (4.5) holds globally with the required boundary conditions on F .

For the space B we first solve (3.17). Using (4.2), a consistent ansatz is

$$\phi = \varphi^{-\pm} \bar{e} \cdot \partial \sigma \quad (4.6)$$

so that (3.17) becomes

$$D^2 \phi = \varphi^\pm \bar{e} \cdot \partial \left(\frac{\partial^2 \sigma}{\varphi} \right) = -2 \varphi^\pm \bar{e} \cdot \partial \left(\frac{f}{\varphi} \right) \tilde{b}' \quad (4.7)$$

or

$$\partial^2 \sigma + 2 \varphi l = -2 \rho \beta', \quad \tilde{b}' = \beta' 1_2, \quad (4.8)$$

for l a constant. The space B is naturally here represented as the product of a two-dimensional space and that spanned by the linearly independent basis formed by the row vector σ of solutions of (4.8). If (4.6) is inserted into Eq. (3.5) for $D_\mu \phi$ we get

$$(\partial_\mu - \frac{1}{2} \bar{e} \cdot \partial \ln \varphi e_\mu) \bar{e} \cdot \partial \sigma = -\frac{1}{2} \varphi \bar{e} \cdot \partial e_\mu \left(\frac{f}{\varphi} \tilde{\Delta}' \right) - \bar{e}_\mu \rho \tilde{b}' \quad (4.9)$$

or, consistent with (4.8),

$$\rho \tilde{\Delta}' = -\bar{e} \cdot \partial \sigma - \phi \epsilon, \quad \epsilon(x) = j + l \bar{x} \quad (4.10)$$

where j , like l , is a constant vector. Given σ, ρ (4.10) defines $\tilde{\Delta}'(x) = \tilde{a}' + x \tilde{b}'$.

For the space A solving (2.11), or

$$D_\mu D^2 \lambda e_\mu = 0, \quad (4.11)$$

is simple since from (4.4), (4.6) and (4.7) we need only assume

$$\lambda = \phi b = \phi^{-\frac{1}{2}} \bar{e} \cdot \partial \tau, \quad \partial^2 \partial^2 \tau = 0. \quad (4.12)$$

Regarding τ as constituting a row vector of solutions of (4.12) then

$$\sigma \beta = \tau + c, \quad b = \beta 1_2 \quad (4.13)$$

with c another constant vector.

When global multi-instanton fields on S^4 are considered, the general solution for ϕ in (4.1), with topological charge k , is²⁾

$$\phi = \sum_0^k \frac{\lambda_i^2}{|x_i|^2}, \quad x_i = y_i - x. \quad (4.14)$$

The apparent singularities in A_μ for $x_i \rightarrow 0$ are gauge artefacts. With ϕ given in (4.14) and imposing the asymptotic behaviour in (2.58) for ψ then in (4.4) we can only take³³⁾

$$\rho_i = \frac{1}{|x_i|^2}, \quad i = 0, 1, \dots, k. \quad (4.15)$$

The singularity at $x_i = 0$ is cancelled in (4.4) by the factor $1/\phi$ so the solutions are globally valid. However, the solutions from (4.4) and (4.15) are linearly dependent since $\sum_0^k \rho_i \lambda_i^2 = \phi$; k independent solutions may be obtained by considering only ρ_j , $j = 1, \dots, k$. In (4.4) d is then uniquely determined by the necessity of cancelling $|x_j|^{-2}$ singularities in F with $F(x) = O(|x|^{-2})$ as $|x| \rightarrow \infty$,

$$d_{jj'} = \delta_{jj'} \frac{p_j}{\lambda_j^2} . \quad (4.16)$$

The consequent $F(x)$ is regular for all x .

A natural basis for possible solutions of (4.8), consistent with the boundary requirements (2.58), is correspondingly provided by

$$\sigma_i = \ln|x_i| , \quad i = 0, 1, \dots, k \quad (4.17)$$

defining the vector ϕ in (4.6). Then from (4.10)

$$\tilde{\Delta}'_{ji} = \delta_{ji} \bar{x}_i - \delta_{0i} \frac{\lambda_j^2}{\lambda_0^2} \bar{x}_0, \quad j = 1, \dots, k, \quad \epsilon_i = -\delta_{0i} \frac{\bar{x}_0}{\lambda_0^2} . \quad (4.18)$$

In (4.12) a suitable basis of solutions, consistent with the required behaviour at the singularities $x_i \rightarrow 0$ and also $|x| \rightarrow \infty$ in (2.58), is

$$\tau_j = \ln|x_0| - \ln|x_j| . \quad (4.19)$$

From (4.13) this fixes b and then $\phi\Delta = 0$ gives

$$\Delta_{ij} = \delta_{ij} x_i - \delta_{i0} x_0 . \quad (4.20)$$

Together, $\tilde{\Delta}'$ in (4.18) and Δ in (4.20), satisfy the condition (3.3) which defines c and then (3.2) determines

$$L_{jj'} = \delta_{jj'} \frac{1}{\lambda_j^2} , \quad I_{ii'} = \delta_{ii'} \frac{1}{\lambda_i^2} 1_i \quad (4.21)$$

independent of x as expected. These results are essentially the standard expressions for the 't Hooft solution in the ADHM framework^{(20), (23)}.

The PS monopole⁽⁸⁾ was expressed in the form (4.1) by Manton⁽¹⁹⁾ using the complex solution

$$\phi(x) = \frac{\sinh Cr}{Cr} e^{iCt}, \quad r = |x|, \quad t = x_4 \quad (4.22)$$

which leads to $A_\mu(x)$, independent of t , centred at the origin and obeying (1.4), albeit in a non-Hermitian gauge. Consistent with the form of ϕ in

(4.21) and the symmetries of the PS monopole we restrict the scalar functions that appear in the solutions of the equations defining the function spaces A, B, C for the 't Hooft solution to expressions of the form

$$f(r) e^{i\delta t} \quad (4.23)$$

Thus in (4.4) we take

$$\rho(x, \delta) = -\frac{i}{r} \sinh \delta r e^{i\delta t} \quad (4.24)$$

and in (4.6), by solving $\partial^2 \sigma = 2\rho$ from (4.8) with $\ell = 0$,

$$\sigma(x, \delta) = -\frac{i}{\delta} \cosh \delta r e^{i\delta t} \quad (4.25)$$

so that (4.6) becomes

$$\phi(x, \delta) = \phi(x)^{-t} e^{i\delta x}, \quad x = t1_2 + i\tilde{x} \cdot \underline{g} \quad (4.26)$$

From (4.10), with $\varepsilon = 0$, we then get

$$\tilde{\Delta}'(x) = -i \frac{\overleftarrow{\delta}}{\delta x} - \bar{x} \quad (4.27)$$

and requiring (2.23)

$$\Delta(x) = -i \frac{\overleftarrow{\delta}}{\delta x} \quad x \quad (4.28)$$

which ensure (3.3) is satisfied and give

$$c(x) = -\tilde{D}_x^2 + r^2, \quad \tilde{D}_x = \frac{\overleftarrow{\delta}}{\delta x} - it \quad (4.29)$$

To complete the definition of the spaces A, B, C it is necessary to define the scalar products between the bases parametrized by γ and vectors in their dual spaces $\tilde{A}, \tilde{B}, \tilde{C}$ so as to form general elements as in (2.3), (2.4) or (2.10). With respect to these scalar products we may then define the adjoints of $\tilde{\Delta}', \Delta$ acting on $\tilde{B}, \tilde{C}$. The natural restriction on γ are given by noting that with (4.24) inserted into (4.4) to define $\psi(x, \gamma)$

$$\psi(x, 0) = \psi(x, C) = 0 \quad (4.30)$$

and since C is required to be the dual of A we also impose

$$w(0) = w(C) = 0. \quad (4.31)$$

Then for (2.3) we take

$$\int_0^C \psi(x, \gamma) w'(\gamma) d\gamma, \quad (4.32)$$

and analogously for (2.4) and (2.10), so that the conditions (4.30), (4.31) allow the differential operators $\tilde{\Delta}', \Delta$ to be integrated by parts without any boundary terms to give their adjoints $\tilde{\Delta}'^a, \Delta^a$. Further, (4.31) ensures

$$\Delta^a(q)w = 0 \Rightarrow w = 0, \quad \Delta^a(q) = i \frac{\partial}{\partial \gamma} - q \quad (4.33)$$

as necessary for the analysis of Section 3 to be valid.

Thus the ADHM construction for the PS monopole can be obtained once I, L are determined, but a small modification of (3.2), (3.16) is necessary here to cope with the non-Hermitian form of A_μ . Replacing $\phi^\dagger, \psi^\dagger$ we need the corresponding solutions ϕ^c, ψ^c of the equations involving $\tilde{D}_\mu = \tilde{D}_\mu - A_\mu$. Thus from (4.4) and (4.6) we take

$$\tilde{\psi}^c = \varphi^{\frac{1}{2}} e^{\frac{1}{2} \partial \left(\frac{p}{\varphi} \right)}, \quad \phi^c = \varphi^{-\frac{1}{2}} e^{\frac{1}{2} \partial \sigma}, \quad (4.34)$$

with ρ, σ still given by (4.23) and (4.24). In (4.5) d is now determined by the boundary conditions on $F(x; \gamma', \gamma)$ which follow from (4.30)

$$F(x; C, \gamma) = F(x; 0, \gamma) = F(x; \gamma', C) = F(x; \gamma', 0) = 0 \quad (4.35)$$

so that, assuming $0 \leq \gamma, \gamma' \leq C$

$$F(x; \gamma', \gamma) = e^{i(\gamma' + \gamma - C)t} \frac{C}{r} \left\{ \frac{\sinh \gamma' r \sinh \gamma r}{\sinh Cr} - \theta(\gamma' + \gamma - C) \sinh(\gamma' + \gamma - C)r \right\}. \quad (4.36)$$

By using (4.29) and (4.30) in (3.2), with $\tilde{\Delta}'^\dagger \rightarrow \tilde{\Delta}'^c, \tilde{\Delta}'^c, \tilde{\Delta}'^c(x) = -i(\partial/\partial \gamma) - x$,

$$L(\gamma', \gamma) = I(\gamma', \gamma) = C \delta(\gamma' + \gamma - C). \quad (4.37)$$

Finally (3.15) with (4.26) and, from (4.34),

$$\phi^c(x, \delta) = \phi(x)^{-\frac{1}{2}} e^{i\delta \bar{x}} \quad (4.38)$$

gives

$$A_\mu(x) = \frac{1}{C} \int_0^C d\delta \phi(x, C-\delta) \partial_\mu \phi^c(x, \delta) \quad (4.39)$$

reproducing the initial 't Hooft expression for the PS monopole given by (4.1) and (4.22). The result (4.39) is just the appropriate modification of Nahm's³⁰⁾ formula for this non-Hermitian gauge. From (4.26) and (4.38)

$$\begin{aligned} \phi_\sigma^c &= \phi^\dagger U, \quad \phi_\sigma = U^{-1} \phi^c, \\ \phi_\sigma(x, \delta) &= \phi(x, C-\delta), \quad U(x) = e^{C\sigma \cdot \hat{x}} \end{aligned} \quad (4.40)$$

so that

$$-A_\mu^\dagger = U A_\mu U^{-1} + U \partial_\mu U^{-1} \quad (4.41)$$

demonstrating that this A_μ is gauge equivalent to a Hermitian gauge field.

5. WITTEN-PENG SOLUTION

The general ansatz describing an $SU(2)$ gauge field for which there is an $O(3)$ rotational symmetry about the four axis given by Witten³⁾ can be expressed as^{*)}

$$\begin{aligned} A_\mu dx_\mu &= -\frac{1}{4} (1-s) \sigma_- \omega_+ - \frac{1}{4} A \sigma_r dz \\ &+ \frac{1}{4} (1-s^*) \sigma_+ \omega_- + \frac{1}{4} A^* \sigma_r d\bar{z} \end{aligned} \quad (5.1)$$

where $s(r, t)$, $A(r, t)$, and with standard spherical polar co-ordinates r, θ, φ

$$\begin{aligned} z &= r + it, \quad \bar{z} = r - it, \quad \sigma_\pm = \sigma \cdot (\hat{\theta} \pm i \hat{\varphi}), \quad \sigma_r = \sigma \cdot \hat{x}, \\ \omega_\pm &= d\theta \pm i \sin\theta d\varphi. \end{aligned} \quad (5.2)$$

^{*)} This is identical with Witten's expression with the substitutions $s = -\phi_2 - i\phi_1$, $A = A_0 + iA_1$.

Simple calculation from (5.1) shows that

$$\begin{aligned} \frac{1}{2} F_{\mu\nu} dx_\mu \wedge dx_\nu &= \frac{1}{4} (s^* s - 1) \sigma_r w_+ \wedge w_- + \frac{1}{4} E \sigma_r dz \wedge d\bar{z} \\ &+ \frac{1}{4} d_2 s \sigma_- dz \wedge w_+ + \frac{1}{4} d_{\bar{2}} s \sigma_- d\bar{z} \wedge w_+ \\ &- \frac{1}{4} d_{\bar{2}} s^* \sigma_+ d\bar{z} \wedge w_- - \frac{1}{4} d_2 s^* \sigma_+ dz \wedge w_- \end{aligned} \quad (5.3)$$

with

$$E = \partial_{\bar{2}} A + \partial_2 A^*, \quad d_2 = \partial_2 + \frac{1}{2} A q, \quad d_{\bar{2}} = \partial_{\bar{2}} - \frac{1}{2} A^* q \quad (5.4)$$

where acting on s , $q = 1$, on s^* , $q = -1$. Since

$$\begin{aligned} *(dz \wedge d\bar{z}) &= r^2 w_+ \wedge w_-, \\ *(dz \wedge w_\pm) &= \pm dz \wedge w_\pm, \quad *(d\bar{z} \wedge w_\pm) = \mp d\bar{z} \wedge w_\pm \end{aligned} \quad (5.5)$$

the self-duality equations (1.1) become

$$\begin{aligned} d_{\bar{2}} s &= 0, \quad d_2 s^* = 0, \\ E &= r^{-2} (s^* s - 1). \end{aligned} \quad (5.6)$$

Witten solved Eqs (5.6) by writing

$$\begin{aligned} A &= 2 \partial_2 \psi, \quad A^* = 2 \partial_{\bar{2}} \psi, \quad s = e^\psi f, \\ \psi &= -\ln \frac{1 - g^* g}{2r} - \frac{1}{2} \ln \frac{f^* f}{g'^* g'}, \end{aligned} \quad (5.7)$$

where $f(z)$, $g(z)$ are analytic, for $\text{Re } z \geq 0$, in the region over which gauge fields are defined with $|g| < 1$. Validity of the solution requires for $r = 0$, $|g| = 1$ and also the zeros of f and g' coincide so

$$f = g' h \quad (5.8)$$

for $h(z)$ analytic with no zeros in $\text{Re } z \geq 0$. The freedom in h corresponds to the residual gauge arbitrariness present in the ansatz (5.1), as expressed by the gauge transformation $A_\mu \rightarrow A_\mu^g$, in (3.26), where

$$g(x) = \exp \left[-\frac{1}{2} \ln \frac{h(z)}{|h(z)|} \sigma_r \right]. \quad (5.9)$$

For definiteness we mostly consider the gauge so that

$$s(0, t) = 1 \quad (5.10)$$

where the gauge field A_μ is regular at $r = 0$. This may be accomplished in (5.7) and (5.8) by choosing

$$g = \frac{n}{d}, \quad n(z) = d(-\bar{z})^*, \quad h = -d^2, \quad (5.11)$$

for $d(z)$ analytic, with no zeros in $\text{Re } z \geq 0$. The resulting solutions give multi-instanton gauge fields with topological index k when for $|x| \rightarrow \infty$

$$d(z) \sim z^{k+1}, \quad e^{-\psi} = \frac{|d|^2 - |n|^2}{2r} = O(|z|^k). \quad (5.12)$$

Solutions of the Dirac equation, and the other equations relevant for the ADHM construction, when A_μ is given by (5.1) and (5.7) can be obtained by introducing appropriate ansätze consistent with the rotational symmetry about the four-axis. Thus we write

$$\begin{aligned} \tilde{\psi} &= \frac{1}{r} (W P_- + \bar{W} P_+), \\ \phi &= Z P_- + \bar{Z} P_+, \end{aligned} \quad (5.13)$$

for $W(r, t)$, $Z(r, t)$, similarly $\bar{W}$, $\bar{Z}$, and $P_\pm = \frac{1}{2}(1 \pm \sigma_r)$. The resulting action of the covariant derivative on ϕ is given by

$$\begin{aligned} dx_\mu D_\mu \phi &= d_2 (d_2 Z P_- + d_2 \bar{Z} P_+) + d_{\bar{2}} (d_{\bar{2}} Z P_- + d_{\bar{2}} \bar{Z} P_+) \\ &\quad - \omega_+ \frac{1}{4} (Z - s \bar{Z}) \sigma_- + \omega_- \frac{1}{4} (\bar{Z} - s^* Z) \sigma_+ \end{aligned} \quad (5.14)$$

with $d_z, d_{\bar{z}}$ just as in (5.4) but acting on Z $q = 1/2$, on $\bar{Z}$ $q = -1/2$, and also analogously for $\tilde{\psi}$. The Dirac equation (4.3) then becomes

$$2 d_{\bar{z}} W = \frac{1}{r} s \bar{W}, \quad (5.15)$$

together with its conjugate, $z \leftrightarrow \bar{z}$, $W \leftrightarrow \bar{W}$, $s \leftrightarrow s^*$. Further if

$$\begin{aligned}\tilde{\Delta}' &= -i\delta' P_+ + i\bar{\delta}' P_-, \\ \delta'(z) &= \alpha' + \beta' z, \quad \bar{\delta}'(\bar{z}) = -\alpha' + \beta' \bar{z}\end{aligned}\tag{5.16}$$

equation (3.5) breaks up into

$$Z - s \bar{Z} = W \delta',\tag{5.17a}$$

$$d_{\bar{z}} Z = 0,\tag{5.17b}$$

$$d_z Z = -\frac{1}{2r} W \bar{\delta}',\tag{5.17c}$$

and their obvious conjugates (henceforth the conjugate equations will generally be implicitly assumed). These equations are not independent since (5.17c) follows from (5.17a,b) using (5.6) and (5.15). This dependence may be used to determine the solutions of the Dirac equation since, from (5.17),

$$W \beta' = -d_z Z + \frac{1}{2r} (Z - s \bar{Z}).\tag{5.18}$$

The form of the right-hand side of (5.18) guarantees that $W \beta'$ satisfies (5.15) using only (5.17b) and (5.6). A general solution of (5.15) is then given by

$$W = -d_z U + \frac{1}{2r} (U - s \bar{U}), \quad d_{\bar{z}} U = 0.\tag{5.19}$$

From (5.17b) and (5.19) we can write

$$U = e^{\frac{1}{2}\psi} \mu, \quad Z = e^{\frac{1}{2}\psi} \nu, \quad \mu(z), \nu(z)\tag{5.20}$$

for μ, ν analytic functions. If $\sigma(r, t)$ is defined through

$$\frac{1}{r^2} s = -2 d_z \left(\frac{\sigma}{r} \right)\tag{5.21}$$

the result (5.19) can be simplified as

$$\frac{1}{r} W = -d_2 \left\{ \frac{1}{r} (u - \sigma \bar{u}) \right\}. \quad (5.22)$$

In the non-singular gauge given by (5.10) and (5.11) to ensure regularity at $r = 0$ we require

$$\sigma(0, t) = 1, \quad \mu(y) = \bar{\mu}(-y), \quad \nu(y) = \bar{\nu}(-y). \quad (5.23)$$

For solutions on S^4 with the asymptotic behaviour (5.12) the requirements (2.58) are satisfied by

$$\mu(z) = O(z^{k-1}), \quad \nu(z) = O(z^k) \quad (5.24)$$

so that μ, ν can be taken as $k, k+1$ row vectors consisting of a complete set of linearly independent polynomials in z of degree $k-1, k$ [e.g., $\nu_i(z) = z^i, i = 0, \dots, k$]. If

$$\mu = d, \quad \bar{\mu} = n^* \quad \text{or} \quad \mu = n, \quad \bar{\mu} = d^*$$

then from (5.20) and (5.22) $W = \bar{W} = 0$, recognizing that $\sigma = n/d^*$ is a solution for (5.21) in the non-singular gauge. Using this then, comparing (5.18) and (5.19) and assuming (5.24), we deduce

$$\mu \beta' = \nu - \omega \ell, \quad (5.25)$$

for $\omega = d - (-1)^k n, \omega(z) \sim cz^k, \ell$ a constant vector. The remaining condition (5.17a) then serves to define the $k \times (k+1)$ matrix δ' since it entails, using (5.25),

$$\mu \delta' = -\epsilon, \quad \epsilon = j\omega + \ell\lambda \quad (5.26)$$

for j another constant vector and $\lambda(z) = \omega(z)z - cd(z) = O(z^k)$. Together (5.25) and (5.26) determine α', β' uniquely.

Analogous to (5.16) Δ may be represented as

$$\begin{aligned} \Delta &= -i \delta P_- + i \bar{\delta} P_+, \\ \delta(z) &= \alpha + \beta z, \quad \bar{\delta}(\bar{z}) = -\alpha + \beta \bar{z}, \end{aligned} \quad (5.27)$$

so that the condition $\phi\Delta = 0$ becomes from (5.13) and (5.20)

$$\gamma \delta = 0. \quad (5.28)$$

With the conditions (5.24) this determines δ as a $(k+1) \times k$ matrix up to $\delta \rightarrow \delta R^{-1}$, $R \in GL(k, \mathbb{C})$. From (5.25) and (5.28) with (5.24) we get

$$\mu \beta' \delta = -\omega l \alpha, \quad l \beta = 0, \quad \mu \delta' \beta = -\epsilon \beta = -\omega j \beta \quad (5.29)$$

and considering the large z behaviour we must have $l \alpha = j \beta$, hence

$$\mu \beta' \delta = \mu \delta' \beta. \quad (5.30)$$

Since (5.30) is true for all z then necessarily

$$\beta' \alpha = \alpha' \beta. \quad (5.31)$$

This suffices to ensure the basic constraints (2.22) and (3.3) with

$$c = \bar{\delta}' \delta = \delta' \bar{\delta}. \quad (5.32)$$

The definition of scalar products in order to reconstruct the gauge field may be short circuited by observing that in the non-singular gauge $\psi(z, \bar{z}) = \psi(-\bar{z}, -z)$ and $e^{-\psi}$ in (5.12) is a polynomial of degree k in both $z, \bar{z}$ so that it can be represented

$$e^{-\psi} = \gamma I^{-1} \gamma^{\dagger} = \bar{\gamma} I^{-1} \bar{\gamma}^{\dagger} \quad (5.33)$$

for a unique, constant, positive definite, Hermitian matrix I^{-1} . Thus, from (5.20),

$$Z I^{-1} Z^{\dagger} = \bar{Z} I^{-1} \bar{Z}^{\dagger} = 1. \quad (5.34)$$

and (5.17b) then gives

$$\frac{1}{4} A^* = \partial_{\bar{z}} Z I^{-1} Z^{\dagger} = -Z I^{-1} \partial_{\bar{z}} Z^{\dagger}. \quad (5.35)$$

Assuming

$$\delta' I^{-1} \bar{Z}^{\dagger} = \bar{\delta}' I^{-1} Z^{\dagger} = 0 \quad (5.36)$$

we may obtain from (5.17b) and (5.17c)

$$S = Z I^{-1} \bar{Z}^\dagger, \frac{1}{4} A = -\partial_z Z I^{-1} Z^\dagger = Z I^{-1} \partial_z Z^\dagger. \quad (5.37)$$

The conditions (5.36), which should follow from the general construction of I described in Section 3, are here implicitly implied by the consistency of the results (5.35) and (5.37) for A and A^* , together with the conjugate equations, using the invariance of ψ for $z \leftrightarrow -\bar{z}$. With A_μ, ϕ given by (5.1), (5.13), then (5.35) and (5.37), with their conjugates, are equivalent to the standard form (3.15) of the ADHM construction. It is perhaps of interest to note that the entire dependence on the parameters of the gauge field is contained in the scalar ψ and the "metric" I .

The PS monopole⁸⁾, in its standard gauge, is expressible in the non-singular gauge here by

$$F = 1, e^{-\psi} = \frac{\sinh Cr}{Cr}, g(z) = e^{-Cz}, h(z) = \frac{e^{Cz}}{C}. \quad (5.38)$$

Assuming only dependence on t of the form $e^{i\gamma t}$ we take necessarily

$$-\mu(z) = \nu(z) = e^{\gamma z}, \quad (5.39)$$

so that from (5.25), (5.26) with $\epsilon = 1$ and (5.28)

$$\delta'(z) = \delta(z) = \frac{\overleftarrow{\partial}}{\partial z} - z. \quad (5.40)$$

The resulting $\tilde{\Delta}', \Delta$ from (5.16), (5.27) are exactly the same as (4.27) and (4.28). In this case

$$W|_{\gamma = \pm \frac{1}{2}C} = 0, \quad \tilde{\psi}|_{\gamma = \pm \frac{1}{2}C} = 0, \quad (5.41)$$

so in a similar fashion to Section 4 we restrict $-\frac{1}{2}C \leq \gamma \leq \frac{1}{2}C$ and impose $w(\pm \frac{1}{2}C) = 0$. To construct F we use

$$\begin{aligned} \epsilon_2(\tilde{\psi}^\dagger \tilde{\psi}) &= \frac{1}{r^2} (W^\dagger W + \bar{W}^\dagger \bar{W}) \\ &= \frac{1}{4r} \nabla^2 \left\{ \frac{1}{r} (u^\dagger u + \bar{u}^\dagger \bar{u} - \sigma^* \bar{u}^\dagger u - \sigma u^\dagger \bar{u}) \right\} \end{aligned} \quad (5.42)$$

for $\nabla^2 = 4\partial_z\partial_{\bar{z}}$, $\partial^2 \rightarrow r^{-1}\nabla^2 r$. Since we may take $\sigma = \cosh Cr$ then F defined through (4.5) and required to be regular at $r = 0$ and vanish at $\gamma', \gamma = \pm \frac{1}{2}C$ is

$$F(x; \gamma, \gamma) = -e^{i(\gamma - \gamma')t} \frac{C}{2r} \left\{ \sinh|\gamma' - \gamma|r + \frac{1}{\sinh Cr} [\cosh(\gamma' + \gamma)r - \cosh Cr \cosh(\gamma' - \gamma)r] \right\}. \quad (5.43)$$

Using (3.2) or equivalently

$$I = \bar{I} = \bar{Z}^\dagger \bar{Z} + \delta'^\dagger F \delta' \quad (5.44)$$

we get

$$L(\gamma, \gamma) = I(\gamma, \gamma) = C \delta(\gamma' - \gamma) \quad (5.45)$$

and it is trivial to verify that (5.33) holds. In this case we have recovered exactly Nahm's³⁰⁾ version of the ADHM construction for the PS monopole.

6. CONCLUSION

Ultimately our analysis of the ADHM construction is of any utility only if it is of some assistance in extending the range of solutions, described in the introduction, which can be cast into the ADHM framework. Specifically it would be desirable to consider solutions described by the higher ansatz of Atiyah and Ward¹⁸⁾, of which the 't Hooft solution is the first, which have been used to obtain multimonopole solutions of the Bogomolny equations, and also monopole solutions for larger gauge groups¹⁰⁾ which are obtained in an extension of the SU(2) Witten solution of Section 5. It is possible that self-dual gauge fields are described by a completely integrable system³⁴⁾, which involves a linearization of the full non-linear equations. The inverse problem of constructing the ADHM framework which has been discussed here is also basically a linear programme. On the other hand the ADHM framework may possibly be extendable, in some part, to describe the complete equations of motion of gauge fields, even when coupled to sources³⁵⁾.

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APPENDIX A

The basis for 2×2 matrices, or quaternions, provided by e_μ , or its conjugate $\bar{e}_\mu$, used here has the properties

$$e_\mu \bar{e}_\nu = \delta_{\mu\nu} 1_2 + \eta_{\mu\nu}^+ , \quad \bar{e}_\mu e_\nu = \delta_{\mu\nu} 1_2 + \eta_{\mu\nu}^- ,$$

$$\eta_{\mu\nu}^\pm = \pm^* \eta_{\nu\mu}^\pm . \quad (A.1)$$

They may be represented explicitly by $e_\mu = (ig, l_2)$, $\bar{e}_\mu = (-ig, l_2)$, and in terms of primed and unprimed spinorial indices are written

$$(e_\mu)^{A'A} , (\bar{e}_\mu)_{AA'} , \quad (e_\mu)^{A'A} (\bar{e}_\mu)_{BB'} = 2 \delta_B^A \delta_{B'}^{A'} . \quad (A.2)$$

Thus any four-vector q_μ may be encoded as a quaternion, $q = e_\mu q_\mu$ or $\bar{q} = \bar{e}_\mu q_\mu$. The spinorial indices may be raised and lowered with the two-dimensional anti-symmetric symbol ϵ , $\epsilon_{12} = \epsilon^{12} = 1$, so

$$\tilde{u}^A = \epsilon^{AB} u_B , \quad \tilde{S}_A = S^B \epsilon_{BA} ,$$

$$\tilde{v}_{A'} = \epsilon_{A'B'} v^{B'} , \quad \tilde{t}^{A'} = t_{B'} \epsilon^{B'A'} . \quad (A.3)$$

For the spinors u, v we may also define their conjugates

$$\bar{u}^A = u_A^* , \quad \bar{v}_{A'} = v^{A'}{}^* \quad (A.4)$$

where from (A.3) and (A.4)

$$\tilde{\bar{u}} = u , \quad \bar{\tilde{u}} = u , \quad \tilde{\bar{u}} = -\bar{\tilde{u}} , \quad \tilde{u} u' = -\tilde{u}' u , \quad (A.5)$$

and similarly for v . Also useful is the relation

$$\bar{q}_{AA'} = -\epsilon_{A'B'} q^{B'B} \epsilon_{BA} \quad (A.6)$$

so, for instance, $\tilde{q}u = \tilde{u}\bar{q}$.

The tangent vectors to the anti self-dual plane S_ξ may, from (1.8) and (1.9), be represented by

$$n_i^{A'A} = \lambda_i^{A'} \tilde{\eta}^A \quad (A.7)$$

using $\tilde{\eta}\eta = 0$ and where λ_i , $i = 1, 2$, are two independent primed spinors.

Since $n_i \bar{n}_j = 0$ then, from (A.1), this is easily seen to imply (1.5)

For each point in CP^3 represented by a four vector ξ^α , decomposed into unprimed and primed spinors according to (1.8), its conjugate may be defined by

$$\bar{\xi}^\alpha = (\tilde{\eta}_A, \tilde{\chi}^{A'}), \quad \bar{\bar{\xi}}^\alpha = -\xi^\alpha. \quad (A.8)$$

By using e_μ , or $\bar{e}_\mu$, the covariant derivative $D_\mu = \partial_\mu + A_\mu$ may be re-expressed so as to act from unprimed to primed spinors $D^{A'A}_\mu$, or vice versa $D_{AA'}$ (dropping the bar on D for convenience in this case). For self-dual gauge fields satisfying (1.1) then from (A.1)

$$D_{AA'} D^{A'B} = D^2 \delta_A^B \quad (A.9)$$

as often used in the text. With an unprimed spinor u we may define a spinorial derivative

$$d_u^{A'} = D^{A'A} u_A \quad (A.10)$$

which for $A' = 1, 2$ provides two independent tangential derivatives for the anti self-dual planes whose tangent vectors are given in terms of u as in (A.7). The self-duality condition then takes the form

$$d_u^{A'} d_u^{B'} - d_u^{B'} d_u^{A'} = 0 \quad (A.11)$$

which expresses the integrability on the anti self-dual planes just as (1.6). It is useful to note

$$\partial_u^{A'} v^{B'} = 0, \quad v^{B'} = x^{B'B} u_B \quad (A.12).$$

APPENDIX B

In the analysis of the ADHM construction in terms of sheaf cohomology^(2),28),29) for self-dual gauge fields on S^4 it is possible to identify the spaces A, B, C with certain H^1 sheaf cohomology groups for sheaves over CP^3 . Thus

$$C = H^1(E(-1)), \quad B = H^1(E \times \Omega^1), \quad A = H^1(E \times \Omega^2), \quad (B.1)$$

with notation for which a partial explanation is contained below.

Homogeneous co-ordinates for CP^3 are provided by a complex four-vector

$$z = (u, v), \quad z \sim \lambda z \quad (B.2)$$

and the projection $CP^3 \rightarrow S^4$ is given by

$$u \neq 0, \quad z = (u, xu), \quad (B.3a)$$

$$v \neq 0, \quad z = (\xi v, v) \quad (B.3b)$$

where $x = x_\mu e_\mu$, $\xi = x_\mu \bar{e}_\mu / |x|^2$ are co-ordinates for the regions V_0 ($\xi \neq 0$) V_∞ ($x \neq 0$) covering S^4 . Sections of vector bundles with connection A_μ over S^4 are joined by an appropriate group transformation on the intersection $V_0 \cap V_\infty$, for instance for the scalar ϕ this is the gauge transformation $g_{0\infty}$ so that

$$\phi(x) = g_{0\infty} \phi_\infty(\xi) \quad (B.4)$$

For a covering of CP^3 provided by $\{U_i\}$ ($i = 1, \dots, 4$ is sufficient) the Čech H^1 cohomology groups are defined as the additive group of functions (technically sheaves) within certain classes defined on $U_i \cap U_j$ which are cocycles

$$c_{ij} = -c_{ji}, \quad (\delta c)_{ijk} = c_{ij} + c_{jk} + c_{ki} = 0 \quad (B.5)$$

modulo coboundaries

$$(\delta c)_{ij} = c_i - c_j$$

where c_i is defined on U_i .

The cocycles for the cohomology groups listed in (B.1) are sections of the vector bundle E , described in the introduction, which have form respectively

$$h_{ij}, \quad \omega_{ij} = \Omega_{ij,\alpha} dz^\alpha, \quad \rho_{ij} = \frac{1}{2} P_{ij,\alpha\beta} dz^\alpha \wedge dz^\beta, \quad (B.6)$$

where the following properties hold

$$h_{ij}(z), \Omega_{ij,\alpha}(z), P_{ij,\alpha\beta}(z) \text{ homogeneous of degree } -1 \text{ in } z, \quad (B.7a)$$

$$\Omega_{ij,\alpha}(z) z^\alpha = 0, \quad P_{ij,\alpha\beta}(z) z^\beta = 0, \quad (B.7b)$$

$$P_{ij,\alpha\beta} = -P_{ij,\beta\alpha}, \quad (B.7c)$$

[(B.7b) is necessary to ensure that the one and two forms in (B.6) are over CP^3] and also a holomorphy requirement. For the subset of CP^3 described in (B.3a), where in each U_i we may take $z \rightarrow (u, x)$, with $x(u, v)$ regular and u providing co-ordinates for CP^1 , this may be expressed as

$$d_u^A h_{ij} = 0, \quad d_u^A \Omega_{ij,\alpha} = 0, \quad d_u^A P_{ij,\alpha\beta} = 0, \quad (B.8)$$

$$h_{ij}, \Omega_{ij,\alpha}, P_{ij,\alpha\beta} \text{ analytic in } u, \text{ homogeneous of degree } -1.$$

For the region described in (B.3b) it is necessary to use the transition function $g_{0\infty}$, as in (B.4), e.g., $h_{\infty ij} = g_{0\infty}^{-1} h_{ij}$, to define cocycles regular at $\xi = 0$. The equations corresponding to (B.8) are then in terms of $d_{vA} = D_{\infty AA'}^{\xi} v^{A'} = \xi_{AA'} / |\xi|^4 g_{0\infty}^{-1} d_u^{A'} g_{0\infty}$.

The twistor programme³⁶⁾ provides a machinery for passing from H^1 cohomology groups on CP^3 to solutions of field equations on S^4 . In the present case, as argued by Madore et al.²⁹⁾, then, if U_i, U_j cover the whole of $CP^1 = \{[u]\}$, due to the triviality of the corresponding cohomology on CP^1 it is possible to split the cocycles

$$h_{ij} = h_i - h_j, \quad \Omega_{ij,\alpha} = \Omega_{i,\alpha} - \Omega_{j,\alpha}, \quad (B.9)$$

$$P_{ij,\alpha\beta} = P_{i,\alpha\beta} - P_{j,\alpha\beta},$$

so that $h_i, \Omega_{i,\alpha}, P_{i,\alpha\beta}$ are regular in u throughout the intersection of U_i with CP^1 . The cocycle condition (B.5) then forces (B.9) to be valid for all i, j . Following Ref. 27) then from (B.8)

$$d_u^{A'} h_i = \psi^{A'}, \quad d_u^{A'} \Omega_{i,\alpha} = \Omega_{\alpha}^{A'}, \quad d_u^{A'} P_{i,\alpha\beta} = P_{\alpha\beta}^{A'}, \quad (B.10)$$

where $\psi^{A'}(x), \Omega_{\alpha}^{A'}(x), P_{\alpha\beta}^{A'}(x) = -P_{\beta\alpha}^{A'}(x)$, of degree 0 in u , are in fact independent of u since they are holomorphic in u over the whole of CP^1 . Furthermore, since from (A.11) $\tilde{d}_{uA}, d_u^{A'} = 0$, they obey the Dirac equation as in (2.1), (2.5b) and (2.19). Similarly, from (B.7b),

$$\Omega_{i,\alpha} z^{\alpha} = \phi, \quad P_{i,\alpha\beta} z^{\beta} = \lambda_{\alpha}, \quad (B.11)$$

giving $\phi(x), \lambda_{\alpha}(x)$, also independent of u , such that $\lambda_{\alpha} z^{\alpha} = 0$. Applying $d_u^{A'}$ to (B.11), with (B.10), just gives the equations (2.5b) and (2.13). Thus the equations whose solutions describe the spaces A, B, C in Section 2 are just those corresponding to the cohomology groups in (B.1). It is straightforward to obtain the transition functions linking the solutions defined over the two regions V_0, V_{∞} covering S^4 , thus for ϕ (B.4) holds. The asymptotic conditions (2.58) are obtained by shrinking V_{∞} to just the point $\xi = 0$ with $g_{0\infty} \rightarrow g(\hat{x})$.

By analogous arguments to the above the absence of solutions of $D^2 r = 0$ corresponds to $H^1(E(-2)) = 0$ which plays a crucial role in the cohomological analysis.

Representative cocycles for the cohomological groups in (B.1) can be easily found with the results of Section 2. Define, with $B_i: \tilde{A} \rightarrow \tilde{B}$,

$$Y_i(z) = B_i(A'(z)B_i)^{-1}, \quad P_i(z) = \frac{1}{z} (1 - Y_i(z)A'(z)), \quad (B.12)$$

which are of degree -1 in z , then if U_i is defined by $z^i \neq 0, \det(A'(z)B_i) \neq 0$, we can take

$$h_i = \phi Y_i, \quad \Omega_{i,\alpha} = \delta_{i\alpha} \phi P_i + \phi Y_i A'_{\alpha}, \\ P_{i,\alpha\beta} = \delta_{i\alpha} \phi P_i A_{\beta} - \delta_{i\beta} \phi P_i A_{\alpha} + \phi Y_i A'_{\alpha} A_{\beta} \quad (B.13)$$

which satisfy (B.10) and (B.11). Choosing $B_i \rightarrow A_{\gamma}$, since $A'(z)A(\bar{z})$ is non-singular, it can be seen that these U_i provide a covering of CP^3 .

The scalar products of Section 3 are given cohomologically by

$$\begin{aligned} H^1(E(-1)) \times H^1(E \times \Omega_1^2) &\simeq H^2(\Omega^2) \simeq \mathbb{C}, \\ H^1(E \times \Omega') \times H^1(E \times \Omega') &\simeq H^2(\Omega^2) \simeq \mathbb{C}. \end{aligned} \tag{B.14}$$

Lacking the techniques for passing from $H^2(\Omega^2)$ to field equations on S^4 we have been unable to verify that for $\mathfrak{g} = \text{Sp}(n) \times K, J$ defined in (3.32) provide for this case the unique constant coefficients for a representative cocycle of $H^2(\Omega^2)$ realizing (B.14).

APPENDIX C

The Eqs (2.5) defining the space B are not all independent. If $\Omega_B^{A'} = (\Omega^{A'B}, \Omega_B^{A'})$ it is sufficient to solve

$$D^2 \phi = 2\Omega^{A'}_{A'}, \quad D_{AA'} \Omega^{A'}_{B'} = 0, \quad (C.1)$$

since, given ϕ , Eq. (2.5a) determines $\Omega^{A'B}$ so that (2.5b) holds, assuming (A.9) or that the gauge field is self-dual. Counting the solutions of (C.1) can be achieved by recognizing that for every solution $\psi^{A'}$ of the Dirac equation (2.1) there are two possible $\Omega^{A'}_{B'}$. The corresponding ϕ may be found by inverting D^2 . In addition it is necessary to include all solutions of $D^2 \phi = 0$ with, for fields defined on S^4 , ϕ having the asymptotic behaviour of (2.58). If d is the dimension of the representations there are d such solutions. This may be seen by considering the Greens function for D^2 which satisfies

$$-D^2 G(x, y) = \delta^4(x - y). \quad (C.2)$$

With (1.2) this can be written in the form

$$G(x, y) = \frac{h(x, y)}{4\pi^2 |x - y|^2}, \quad h(x, y) \underset{|y| \rightarrow \infty}{\sim} u(x) g(\hat{y})^{-1}. \quad (C.3)$$

From (C.2) and (C.3)

$$D^2 u = 0, \quad u(x) \underset{|x| \rightarrow \infty}{\sim} g(\hat{x}), \quad (C.4)$$

giving d solutions. Thus for topological charge k and $\mathcal{G} = \text{SU}(n)$ we get the counting in (2.59).

More systematically Madore et al.²⁷⁾ have shown that Eqs. (C.1) can be expressed in a homogeneous covariant form on S^4 , with the standard metric, so that normal index theorems can be applied.

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