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CYLINDRICALLY SYMMETRIC SOLUTIONS OF THE NON-LINEAR
CHIRAL FIELD MODEL (σ MODEL)

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ABSTRACT

We show that the cylindrically symmetric solutions of the $2 + 1$ dimensional non-linear σ model can be analyzed by the inverse scattering method. We obtained exact "soliton-like" solutions and the cylindrically symmetric solutions of the three-dimensional model are also obtained. The reduction problem of the principal chiral model is discussed. The non-local conservation laws are calculated.

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1. INTRODUCTION

It is well known that the principal chiral field model and the $\vec{n}$ field model can be integrated by the inverse scattering method in $1 + 1$ dimensional space-time¹⁾. The multi-dimensional version of this model does not seem to be integrable, but we show that the cylindrically symmetric field configurations obey an exactly integrable equation. The aim of this paper is to calculate cylindrically symmetric solutions of the $2 + 1$ dimensional model. We give stationary solutions of the three dimensional Euclidean model and some solutions of the $3 + 1$ model.

The principal chiral field equation has the form:

$$\partial^\mu (g_\mu g^{-1}) = 0 \quad , \quad \mu = 0, 1, 2 \quad (1)$$

where g belongs to some Lie group G . Consider the cylinder co-ordinates (r, ψ) on the (x_1, x_2) plane. Assuming that g is independent of ψ we find

$$(g_t g^{-1})_t = (g_r g^{-1})_r + \frac{1}{r} g_r g^{-1} \quad (2)$$

This equation is the subject matter of our investigation. It is known that Eq. (1) is compatible with a certain class of restrictions on g which effectively "reduce" Eq. (1) to dynamics in various symmetric spaces. A complete list of such spaces can be found in Ref. 2). Here we restrict ourselves to an investigation of the Grassmann manifolds and as an example we consider the complex projective space $\mathbb{C}P^N$. The $\mathbb{C}P^N$ chiral model was formulated and its integrability proved in Ref. 1). Somewhat later, some interesting properties of this model, such as the existence of instanton solutions, were discovered³⁾.

The case of the $\mathbb{C}P^N$ model is the most interesting because it is equivalent to the Heisenberg model. The $\mathbb{C}P^1$ space can be parametrized by the $SU(2)$ group elements g satisfying the constraints

$$g^2 = -I$$

Such group elements can be represented in the form

$$g = i \vec{\sigma} \cdot \vec{n} \quad , \quad n^2 = n_1^2 + n_2^2 + n_3^2 = 1 \quad (3)$$

where σ^i are Pauli matrices.

Inserting (3) into (2) one finds

$$\vec{n}_{tt} - \frac{1}{r} \partial_r (r \vec{n}_r) + (n_t^2 - n_r^2) \vec{n} = 0 \quad (4)$$

Equation (4) is an extreme point of the following action functional

$$S = \int (n_t^2 - n_r^2) r dr dt \quad (5)$$

The change $t \rightarrow iz$ turns the space into a Euclidean one and Eq. (2) then describes stationary configuration in three Euclidean dimensions. We shall also give solutions of this latter equation. If one introduces the following co-ordinates in 3 + 1 dimension space-time

$$\begin{aligned} t &= s \cosh \theta, & x_1 &= s \sinh \theta \\ x_2 &= \rho \cos \psi, & x_3 &= \rho \sin \psi \end{aligned}$$

and if one requires that g be independent of ψ and θ one can obtain the equation

$$(g_s g^{-1})_s + \frac{1}{s} g_s g^{-1} = (g_\rho g^{-1})_\rho + \frac{1}{\rho} g_\rho g^{-1} \quad (6)$$

Note that the simple change of the co-ordinates $r = s\rho$, $t = 1/2(s^2 + \rho^2)$ reduces (6) to (2). Consequently, the solutions of Eq. (6) are simply related to those of (2).

2. CONSTRUCTION OF SOLITON SOLUTIONS

Equation (2) has the same form as the gravity field equation that has been integrated by Zakharov and Belinsky⁴⁾

$$\partial_\eta (\alpha g_\xi g^{-1}) + \partial_\xi (\alpha g_\eta g^{-1})$$

where $\alpha = r = \eta - \xi$, $t = \eta + \xi$. But here the field g takes its values from a different manifold. Equation (2) can be represented as a compatibility condition of the following pair of linear problems:

$$\psi_\xi = -\frac{r}{s-r} U \psi \quad (7)$$

$$\Psi_\eta = \frac{r}{r+r} V \Psi \quad (8)$$

Here and below

$$2\xi = t - r, \quad 2\eta = t + r, \quad U = g_\xi g^{-1}, \quad V = g_\eta g^{-1}$$

$$\gamma(\xi, \eta; \kappa) = \xi + \eta - \kappa + \sqrt{(\kappa - 2\xi)(\kappa - 2\eta)}$$

where κ is a spectrum parameter to be defined on the two-fold Riemann sheet Γ having branch points at $\kappa = 2\xi$ and $\kappa = 2\eta$. Our form of the compatibility condition seems to be more convenient for calculating partial solutions and differs from (4). In order to construct partial solutions by the method developed in Refs. 1) and 4) one must know some exact solution, g_0 , of Eq. (2) and the corresponding solution $\psi_0(\xi, \eta; \kappa)$ of the problems (7) and (8). Consider Eq. (2) in terms of the currents U, V :

$$\begin{aligned} (\alpha U)_\eta + (\alpha V)_\xi &= 0 \\ U_\eta - V_\xi + [UV] &= 0 \end{aligned} \quad (9)$$

Assuming that U, V are diagonal one can easily solve the system (9)

$$U_{ik} = \delta_{ik} \Omega_{i\xi}, \quad V_{ik} = \delta_{ik} \Omega_{i\eta}, \quad g_{0ik} = \delta_{ik} \exp \Omega_i$$

$$\begin{aligned} \Omega_\alpha &= \sum_\kappa \left(T_\alpha^1(\kappa) e^{i\kappa t} + T_\alpha^2(\kappa) e^{-i\kappa t} \right) \left(a_\alpha(\kappa) J_0(\kappa r) + \right. \\ &\quad \left. + b_\alpha(\kappa) N_0(\kappa r) \right) \end{aligned}$$

where J_0, N_0 are the Bessel and Neumann functions respectively. For two important partial cases the function ψ_0 has the form:

$$\Psi_0 = \text{const}, \quad \text{if} \quad U = V = 0 \quad (10)$$

$$\psi_{0jk} = \delta_{jk} \exp(i\omega_k(2t - \tau)), \text{ if } U_{jk} = V_{jk} = 2i\omega_k \delta_{jk} \quad (11)$$

$\omega_k = \text{const}$

The first solution (10) is connected with the function g_0 which is constant in space-time [in the case of the $\vec{n}$ field (3) it means that the vector $\vec{n}$ has the same direction in each point of space-time]. In the second case (11), the field rotates with constant velocity.

Following Refs. 1) and 4) we represent ψ in the form $\psi = \chi \psi_0$. A solution of the system (9) which corresponds to a function χ that is rational on the Riemann sheet Γ will be called a soliton solution. Such a function χ can be represented in the form:

$$\chi = I + \sum_i \frac{T_i(\xi, \eta)}{\gamma - \mu_i} \quad , \quad \mu_i = \gamma|_{\kappa_i} \quad (12)$$

where κ_i are some fixed points of Γ . The inverse matrix function χ^{-1} can be represented as:

$$\chi^{-1} = I + \sum_i \frac{S_i(\xi, \eta)}{\gamma - \nu_i} \quad , \quad \nu_i = \gamma|_{\kappa'_i} \quad (13)$$

It follows from $\chi\chi^{-1} = I$ that T_i and S_i satisfy the relations:

$$\begin{aligned} \left(I + \sum_{\kappa} \frac{T_{\kappa}}{\nu_i - \mu_{\kappa}} \right) S_i &= 0 \\ T_i \left(I + \sum_{\kappa} \frac{S_{\kappa}}{\mu_i - \nu_{\kappa}} \right) &= 0 \end{aligned} \quad (14)$$

The system (7), (8) implies the following equations for χ :

$$\begin{aligned} \chi_{\xi} \chi^{-1} - \frac{r}{\gamma - r} \chi U_0 \chi^{-1} &= -\frac{r}{\gamma - r} U \\ \chi_{\eta} \chi^{-1} + \frac{r}{\gamma - r} \chi V_0 \chi^{-1} &= \frac{r}{\gamma + r} V \end{aligned} \quad (15)$$

The right-hand side of each equation has only one pole on Γ (at $\gamma = \pm\alpha$). The left-hand side of the equation has no pole at κ_i, κ'_i iff

$$T_i D_{1,2}(\kappa_i) \left(I + \sum_{\kappa} \frac{S_{\kappa}}{\mu_i - \nu_{\kappa}} \right) = 0, \quad \left(I + \sum_{\kappa} \frac{T_{\kappa}}{\nu_i - \mu_{\kappa}} \right) D_{1,2}(\kappa'_i) S'_i = 0 \quad (16)$$

where

$$\mathcal{D}_1(x) = \partial_\xi + \frac{r}{\gamma - r} V_0$$

$$\mathcal{D}_2(\kappa) = \partial_\eta - \frac{r}{\gamma + r} V_0$$

One can obtain non-trivial solutions of (14) by the standard method of Ref. 1).
Assuming

$$T_i = |x_i\rangle \langle t_i|, \quad S_i = |s_i\rangle \langle y_i| \quad (17)$$

where the symbol $| \cdot \rangle$ represents a column vector, $\langle \cdot |$ is a row vector. Inserting (17) into (14) we find linear algebraic equations for $|x_i\rangle$ and $\langle y_i|$:

$$|s_i\rangle + \sum_k \frac{|x_k\rangle \langle t_k| s_i\rangle}{\nu_i - \mu_k} = 0$$

$$\langle t_i| + \sum_k \frac{\langle t_i| s_k\rangle \langle y_k|}{\mu_i - \nu_k} = 0$$

Inverting the matrix

$$\| M_{ik} \| = \frac{\langle t_i| s_k\rangle}{\mu_i - \nu_k}$$

we find

$$|x_i\rangle = \sum_k |s_k\rangle (M^{-1})_{ki} \quad (18)$$

$$\langle y_i| = \sum_k (M^{-1})_{ik} \langle t_k| \quad (19)$$

Thus we have the vectors $|x_i\rangle, \langle y_i|$ as functions of $|s_\alpha\rangle, \langle t_\alpha|$. Equations (16) enable us to define the dependence of $|s\rangle, \langle t|$ on the co-ordinate [see Ref. 1)]:

$$\begin{aligned} |s_i\rangle &= \psi_0(\xi, \eta; \kappa_i') |s_{i0}\rangle \\ \langle t_i| &= \langle t_{i0} | \psi_0^{-1}(\xi, \eta, \kappa_i) \end{aligned} \quad (20)$$

where $|s_{i0}\rangle$ and $\langle t_{i0}|$ are any constant vectors, and $\psi_0(\xi, \eta; \kappa_i)$ is the solution of Eqs. (7), (8) for known U_0, V_0 .

It is now easy to calculate soliton-like solutions

- 1) Using the known solutions U_0, V_0 of Eqs. (9) we solve Eqs. (7), (8) for the functions $\psi_0(\xi, \eta, \kappa)$.
- 2) Choosing two sets $\{\kappa_i\}$ and $\{\kappa_i'\}$ of Γ and arbitrary constant complex vectors $|s_{i0}\rangle$ and $\langle t_{i0}|$ we define $\langle t_i|$ and $|s_i\rangle$ using (20).
- 3) Forming the matrix M_{ij} (18) and inverting it we obtain $|x_i\rangle$ and $\langle y_i|$.
- 4) Inserting the calculated vectors in to (17) we define the matrices T_i, S_i and then we use (12), (13) to calculate X and X^{-1} .

It follows from (7), (8) that the function

$$g = X \psi_0 \Big|_{\kappa=\infty}, \quad (\chi|_{\kappa=\infty} = 0) \quad (21)$$

satisfies Eq. (2). The solution (21) has the sets $\{\kappa_i\}, \{\kappa_i'\}$ and $|s_{i0}\rangle, \langle t_{i0}|$ as free parameters.

The simplest one-soliton solution calculated by the preceding scheme has the form

$$g = \left(I - \frac{M_1 - V_1}{M_1} P_1 \right) g_0, \quad g_0 = \psi_0 \Big|_{\chi=0} \quad (22)$$

where

$$\begin{aligned} P_1^{\alpha\beta} &= \frac{S_1^\alpha \bar{T}_1^\beta}{\sum_\delta S_1^\delta \bar{T}_1^\delta}, & S_1^\alpha &= \sum_\beta \psi_0^{\alpha\beta}(\xi, \eta; \kappa_i') S_{i0}^\beta \\ \bar{T}_1^\alpha &= \sum_\beta \psi_0^{-1\beta\alpha}(\xi, \eta; \kappa_i) \bar{T}_{i0}^\beta \end{aligned}$$

As a function ψ_0 one can take the function (10) or (11) or any other exact solutions of Eqs. (7) and (8).

The two soliton-like solutions are represented by

$$g = (I + A_1 + A_2 + \frac{1}{\Delta} [\tau (\mu_1 - \nu_1)(\mu_2 - \nu_2)(A_1 + A_2) +$$
(23)

$$+ \mu_1(\mu_2 - \nu_1)A_1A_2 + \mu_2(\mu_1 - \nu_2)A_2A_1]) g_0$$

where

$$A_i = \frac{\nu_i - \mu_i}{\mu_i} P_i, \quad P_i^{\alpha\beta} = \frac{S_i^\alpha \bar{t}_i^\beta}{\sum_\delta \bar{t}_i^\delta S_i^\delta},$$

$$\Delta = (\mu_1 - \nu_2)(\mu_2 - \nu_1) - \tau (\mu_1 - \nu_1)(\mu_2 - \nu_2)$$

$$\tau = t_2(P_1 P_2), \quad S_i^\alpha = \sum_\beta \psi_0^{\alpha\beta}(k_i) S_{i0}^\beta, \quad \bar{t}_i^\alpha = \sum_\beta \bar{t}_{i0}^\beta (\psi_0^{-1}(k_i))^\beta$$

Of course, one can find multi-soliton solutions using our scheme, but it is not done here. The formulae (22), (23) are sufficient to understand the behaviour of the N soliton solutions. The analysis of solutions on symmetric spaces will be done in Section 4.

3. REDUCTIONS OF THE PRINCIPAL CHIRAL FIELD MODEL

The solutions of Eq. (2) were found in the previous section to be elements of the general linear group $GL(N, \mathbb{C})$. It is well known that the chiral field equation may be restricted to an arbitrary Lie group or symmetric space. The aim of this section is to construct solutions belonging to a given group G (or symmetric space $\tilde{G}$). In other words, we have to find constraints on the free parameters of the solution (22) such that $g \in G$ ($g \in \tilde{G}$).

We start with a discussion of the reduction to a Lie group. It is more convenient to do this in terms of Lie algebra. The currents U, V are the elements of the respective Lie algebras and obey Eq. (9). Due to the Levy-Maltsev theorem [see for example Ref. 5)] each finite dimensional Lie algebra can be decomposed in the sense of a linear space, into a direct sum of a soluble radical R and a semi-simple subalgebra S :

$$\mathcal{G} = R \oplus S \tag{24}$$

The semi-simple part S is a direct sum of the simple subalgebras

$$S = \sum_k S'_k \quad (25)$$

We therefore represent the currents U, V by

$$U = U_R + \sum_k U_k; \quad U_R, V_R \in R \quad (26)$$

$$V = V_R + \sum_k V_k; \quad U_k, V_k \in S'_k$$

It follows from the commutation relations

$$[R, R] \subset R; [R, S_k] \subset R; [S_k, S_l] = \delta_{kl} S_k \quad (27)$$

that Eq. (9) can be decomposed in the following way:

$$[r U_k]_\eta + [r V_k]_\xi = 0 \quad (28)$$

$$U_k \eta - V_k \xi + [U_k, V_k] = 0 \quad (29)$$

$$(r U_R)_\eta + (r V_R)_\xi = 0 \quad (30)$$

$$U_R \eta - V_R \xi + [U_R, V_R] + \sum_k \{ [U_k, V_R] + [U_R, V_k] \} = 0 \quad (31)$$

Equations (28), (29) for elements belonging to the simple subalgebras separate from the rest. One can prove [see Ref. 6)] that the system restricted to the radical is linear. Thus the non-trivial examples of non-linear equations are connected with the simple Lie algebras. It is natural to restrict ourselves to the investigation of the chiral field model connected with the simple Lie algebras (groups).

To construct a solution for the given algebra (group) one must know what conditions separate this algebra (group) from the complex matrix Lie algebra $[GL(N, \mathbb{C}) \text{ group}]$. The classical Lie groups (which correspond to the infinite series of the Cartan classification) are separated from the $GL(N, \mathbb{C})$ as a subgroup preserving some bilinear (symmetric or antisymmetric) form B

$$g^T B g = B \quad (32)$$

or the complex bilinear form

$$g^+ H g = H$$

There is a classification of such forms and groups corresponding to them [see Refs. 2) and 5)].

The reduction of the chiral field model for such a group was considered in Ref. 7). It follows from (7), (8) that the function $\psi(\xi, \eta; \kappa)$ possesses the following symmetry properties

$$B^{-1} \psi^T B = \psi^{-1} \quad (33)$$

The function χ has the same properties. As a result of the symmetry the poles of χ and χ^{-1} coincide, i.e., the sets $\{\kappa_i\}$ and $\{\kappa_i'\}$ are the same; the method for calculating the solution should be modified as in Ref. 7).

In the case of (33) we have

$$H^{-1} \chi^+(\bar{\kappa}) H = \chi^{-1}(\kappa) \quad (34)$$

i.e., the poles of χ and χ^{-1} are complex conjugated pairs $[\bar{\kappa}_i = \kappa_i', \gamma(\kappa_i) = \gamma(\kappa_i')]$. Moreover, it follows from (34) that the vectors $|s_{i0}\rangle$ and $\langle t_{i0}|$ are connected by the relation

$$H |s_{i0}\rangle = |t_{i0}\rangle$$

It is easy to see that this relation will be true for $|s_i\rangle$ and $\langle t_i|$ at each of the co-ordinate points (ξ, η) .

If H is a unit matrix then (34) means that g is unitary. Using the unitary solution of (2) one can construct the special unitary solution $\tilde{g} \in SU(N)$:

$$\tilde{g} = [\det(g)]^{-1/N} g$$

of Eq. (2). One can find the restriction for $\kappa_i, \kappa_i', |s_{i0}\rangle |t_{i0}\rangle$ such that g belongs to a given classical group. The symmetric spaces can be characterized as sets of g satisfying the following relation

$$g g^* = I \quad (35)$$

where g^* means some involution (maybe the identity) in a Lie group. For instance, one can represent (32) as (35), assuming that $g^* = B^{-1} g^t B$. This involution changes the order of multiplication. For this kind of involution one can prove

$$g U^* + U g = 0 \quad , \quad g V^* + V g = 0 \quad (36)$$

where U^* , V^* are the corresponding involutions of the Lie algebra. It follows from (36), (7) and (8) that

$$g \Psi^*(\sigma(\kappa)) = \Psi(\kappa) J \quad (37)$$

where J is a constant matrix subject to the condition (35). The σ indicates the change of Riemann sheet when the involution $*$ does not contain the complex conjugation. In the case when the involution $*$ contains the complex conjugation, σ indicates the change of the sheet along with an operation of a complex conjugation. Obviously, the poles of the function ψ at $\{\kappa_i\}$ imply the poles of the function ψ^* at $\{\sigma(\kappa_i)\}$.

The complex Grassman manifold can be parametrized by the general linear group elements satisfying the following relations

$$g g^+ = I \quad , \quad g^2 = I \quad , \quad g \in GL(N, \mathbb{C}) \quad (38)$$

Indeed,

$$g = I - 2P \quad , \quad P^2 = P \quad , \quad P^+ = P \quad (39)$$

Thus each g can be parametrized by the Hermitean projector P . Each projection operator P is characterized by the dimension k of its image (the space onto which it projects). It is obvious that throughout the evolution the number k cannot change, therefore one may assume in (39)

$$g = I - 2P_k \quad (k = \dim(\operatorname{Im} P_k))$$

In the special case, $K = 1$, the Grassmann manifold is just the projection space $\mathbb{CP}^{N-1}$ of dimension $N - 1$. In this case the projector P_1 can be represented by

$$P_1 = |n\rangle\langle n| \quad (40)$$

where $|n\rangle$ is a unit vector

$$\langle n | n \rangle = 1$$

Inserting (39), (40) into (1) one can obtain the $\mathbb{CP}^N$ non-linear σ model.

If $N = 1$ then g may be represented by

$$g = \vec{\sigma} \cdot \vec{n}, \quad n_1^2 + n_2^2 + n_3^2 = 1 \quad (41)$$

where $\vec{n}$ is a real three-dimensional vector and σ^α are Pauli matrices. Inserting (41) into (2) one arrives at the Heisenberg model. Returning to the involutive condition (38) one may choose the function $\psi(\kappa)$ to satisfy the following constraints:

$$\psi^+(\kappa) = \psi^{-1}(\kappa) \quad (42)$$

$$g\psi(\sigma(\kappa)) = \psi(\kappa) \quad (43)$$

Here σ means the change of the list of Γ . In terms of $\gamma: \gamma \{\sigma(\kappa)\} = r^2/\gamma(\kappa)$. Thus, if the function ψ has a pole at κ , then it has a pole at $\kappa_2 = \sigma\kappa_1$ as well. It follows from (42) that ψ^{-1} has poles at $\kappa_1^1 = \bar{\kappa}_1$ and $\kappa_2^1 = \sigma\bar{\kappa}_1$. The relations (42), (43) lead to restrictions on the residues at the poles, i.e., on the vectors $|s_{i0}\rangle$ and $|t_{i0}\rangle$

$$\begin{aligned} |s_{10}\rangle &= |t_{10}\rangle \\ |s_{20}\rangle &= |t_{20}\rangle = J |t_{10}\rangle \end{aligned} \quad (44)$$

In the Grassmann case only one vector (for instance $|t_{10}\rangle$) is independent of the solution (23). We have assumed, in deriving (44), that the partial solution $\psi_0(\xi, \eta; \kappa_0)$ satisfies (42), (43).

In a similar fashion one can calculate the multi-soliton-like solution which belongs to other symmetric spaces, e.g., the equation which was solved by Belinsky and Zakharov⁴⁾ may be interpreted as a chiral field model defined on the symmetric space $SL(2, \mathbb{R})/SO(2)$. The reduction problem for exceptional groups, has not yet been solved. Shubov drew our attention to the fact that the model (1) admits reduction for the affine symmetric spaces, which have also been classified. Here we will not touch on this problem.

4. THE ANALYSIS OF SOLITON-LIKE SOLUTIONS OF THE $U(N)$ CHIRAL MODEL AND THE $\vec{n}$ FIELD MODEL

The one soliton-like solution (22) is unitary provided $s_{10}^\alpha = t_{10}^\alpha$ and κ_1 is complex conjugate of κ' . The two soliton solution (23) is unitary provided $|s_1\rangle = |h_1\rangle$ and $\kappa_1 = \bar{\kappa}'_1$. Using the method described above one can calculate the simplest solution of the $\vec{n}$ field model for the "rotating" vacuum

$$g_0 = \begin{pmatrix} 0 & e^{2i\omega t} \\ e^{-2i\omega t} & 0 \end{pmatrix}, \quad \psi_0 = \begin{pmatrix} 0 & e^{i\omega(2t-r)} \\ e^{-i\omega(2t-r)} & 0 \end{pmatrix} \quad (45)$$

The solution may be written in the following form

$$n_3 = \frac{(\bar{\mu}-\mu)(\mu\bar{\mu}-r^2)}{2\mu\bar{\mu}} \cdot \frac{(\mu-\bar{\mu})(\mu\bar{\mu}+r^2)\text{sh}A\cos B + i(\mu\bar{\mu}-r^2)(\mu+\bar{\mu})\text{ch}A\sin B}{(\mu\bar{\mu}-r^2)^2\text{ch}^2A - r^2(\mu-\bar{\mu})^2\cos^2B} \quad (46)$$

$$n_+ = \frac{e^{-2i\omega t}}{4\mu\bar{\mu}} \cdot \frac{[(\mu+\bar{\mu})\text{ch}A + (\mu-\bar{\mu})\text{sh}A]^2(\mu\bar{\mu}-r^2)^2 - (\mu-\bar{\mu})^2[(\mu\bar{\mu}+r^2)\cos B - i(\mu\bar{\mu}-r^2)\sin B]^2}{(\mu\bar{\mu}-r^2)^2\text{ch}^2A - r^2(\mu-\bar{\mu})^2\cos^2B} \quad (47)$$

where

$$A = i\omega(\mu-\bar{\mu}) + \ln|q|, \quad B = \omega(2t - \mu - \bar{\mu}) + \arg(q), \quad q \in \mathbb{C} \\ \mu = t - \kappa + \sqrt{(t-\kappa)^2 - r^2}, \quad n_+ = n_1 + i n_2.$$

1) The case of the trivial vacuum

In the case of the trivial vacuum the one soliton solution for the $U(N)$ chiral model has the following form

$$g = (1 - \frac{\bar{\mu} - \mu}{\mu} P_0) g_0$$

where P_0 is a constant Hermitean projector. The solutions of the $\vec{n}$ field model may be obtained from (46), (47) if $\omega = 0$. This solution represents a solitary wave that moves towards the origin and is subsequently reflected back. This wave looks like a ring of finite thickness. Far away from the ring the field vanishes like a power in r . Outside the ring the field approaches the vacuum direction and inside it approaches a different direction. The change of direction of the $\vec{n}$ field is localized in the ring. Far from the centre the wave moves with the velocity of light. The energy of the simplest solution is finite in the $\vec{n}$ field case and logarithmically divergent for the chiral field.

The two soliton solution of the $U(N)$ model is given by Eq. (23) provided one takes into account the reduction constraints. This solution represents the non-linear superposition of two concentric solutions. When the solitons are far from the origin of the co-ordinates the distance between them is equal to $\text{Re}(\kappa_1 - \kappa_2)$.

ii) The case of the rotating vacuum

The $U(2)$ chiral model has a soliton solution in the form (45) with P not constant

$$P = \begin{bmatrix} \frac{e^Y}{2\cosh Y} & \frac{e^{iX}}{2\cosh Y} \\ \frac{e^{-iX}}{2\cosh Y} & \frac{e^{-Y}}{2\cosh Y} \end{bmatrix} \quad \begin{aligned} X &= \arg\left(\frac{S_{10}'}{S_{10}^2}\right) + (\omega_1 - \omega_2)\left(2t - \frac{\mu + \bar{\mu}}{2}\right) \\ Y &= (\omega_1 - \omega_2)\frac{\mu - \bar{\mu}}{2i} + \ln\left|\frac{S_{10}^2}{S_{10}'}\right| \end{aligned} \quad (48)$$

This soliton solution looks like a pair of waves. The first one is completely analogous to the wave in the trivial vacuum case which has a velocity of light, whereas the second wave has a velocity less than that of light

$$v = \sqrt{1 - \frac{\sigma^2}{a^2}}, \quad \sigma = \text{Im} \kappa_1, \quad a^2 = \left(\sigma - \frac{1}{\omega_1 - \omega_2} \ln\left|\frac{S_{10}^2}{S_{10}'}\right|\right)^2$$

and has a finite thickness

$$\Delta R \approx t \frac{\sqrt{1-v^2}}{\sigma}$$

which increases in time. Far from the ring the field energy vanishes exponentially.

$$r^2 - t^2 \left(1 - \frac{\sigma^2}{a^2}\right) = a^2 - \sigma^2 \quad (49)$$

The amplitude of the wave decreases in time as $\sigma/\sqrt{1-v^2}$. The projector P (48) outside of the ring becomes $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ whereas inside the ring it becomes $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Far away from the origin this wave looks like the plane wave in the $1+1$ model¹⁾ whose amplitude and thickness are slowly changing parameters. Note that the rotating vacuum is degenerate in the sense of Ref. 1). Therefore there are neither decays nor fusions of solitons in the $SU(N)$ model.

iii) The solitons of the three-dimensional model

To obtain a soliton solution of the three-dimensional model it is sufficient to change $t \rightarrow iz$ (in the case of the constant vacuum). The main feature of these solutions is connected with the branch points of the square root. Therefore, to obtain the single-valued solution one should make a cut in the three-dimensional space. The branching points of the root form the ring $r = |\text{Re } \kappa|$, $z = \text{Im } \kappa$. We choose this cut as a disc whose boundary coincides with the ring. The field $\vec{n}$ will have some jump on the disc and we interpret this fact as a consequence of the magnetic moment of the disc. Our solution (47) ($\omega \equiv 0$) minimizes the following Hamiltonian

$$H = \int \frac{1}{2} (\nabla h)^2 d^3x + \int dS (\vec{M} \vec{n}) \quad (50)$$

$$\vec{M} = - \frac{(|q|^2 + 1) 2 R^2}{(R^2(|q|^2 + 1) - r^2(q + \bar{q})^2) \sqrt{R^2 - r^2}} \begin{bmatrix} 0 \\ |q|^2 - 1 \\ i(\bar{q} - q) \end{bmatrix} \Theta(R - r)$$

where $\vec{M}$ is the magnetic vector of the disc, $R = |\kappa|$ is its radius and $\Theta(X) = 1$ if $X > 0$ and 0 otherwise. The N soliton solution is a field formed by N discs distributed along the z axis.

Equation (2) also describes the nematic liquid crystal. So we can interpret our solution as a crystal field, but then the vector M is naturally considered to lie on the surface of the disc and the $(\vec{M}, \vec{n})$ term describes the interaction energy between the crystal and the surface of the disc.

iv) Changing the variables

$$t \rightarrow \frac{1}{2} (t^2 - x^2 + y^2 + z^2)$$

$$r^2 \rightarrow (t^2 - x^2)(y^2 + z^2)$$

we find the solution of the $3 + 1$ dimensional model. In the $3 + 1$ dimensional Minkowsky space the branch points of the solution fill out the two-dimensional surface

$$2r^2 = |K| + |Re K|$$

$$2x^2 = 2t^2 + |K| - |Re K| \quad (Im K \neq 0)$$

At each fixed moment of time the branch manifold consists of two circumferences. Also, we should cut the space to avoid a branch in the solution. Finally, the solution represents two magnetized hyperbolic plates which fly away from one another with the velocity of light.

5. CONSERVATION LAWS

To obtain the conservation laws of the model we start from another representation of Eq. (2). We represent it as a compatibility condition of the following linear problems:

$$[(\mu - \mu') \partial_r - (\mu + \mu') \frac{\mu}{r} \partial_\mu - g_t g^{-1} + \mu^{-1} g_r g^{-1}] \psi = 0 \quad (51)$$

$$[\mu (\partial_r - \frac{\mu}{r} \partial_\mu) + \partial_t - g_t g^{-1}] \psi = 0 \quad (52)$$

We are looking for the function $\psi(\mu; r, t)$ in the following form:

$$\Psi(\mu; r, t) = \sum_{k=0}^{\infty} \psi_k(r, t) \mu^{-k} \quad (53)$$

Inserting (53) into (51) we find

$$\psi_{k+1,r} - \psi_{k-1,r} + \frac{k+1}{r} \psi_{k+1} + \frac{k-1}{r} \psi_{k-1} - g_t g^{-1} \psi_k + g_r g^{-1} \psi_{k-1} = 0 \quad (54)$$

We normalize the function ψ at $r = 0$

$$\psi(\mu, r, t) = I \Big|_{\rho=0} \quad (55)$$

Thus, $\psi_0 = I$

$$\psi_{1,r} + \frac{\psi_1}{r} - g_t g^{-1} = 0$$

and

$$\psi_1 = \frac{1}{r} \int_0^r r' g_t g^{-1} dr' + \frac{C_1}{r} \quad (56)$$

Taking into account (55) we find that $C_1 = 0$. Finally we have a recursion relation

$$\psi_{k+1} = \frac{1}{r^{k+1}} \int_0^r dr' r'^{k+1} (g_t g^{-1} \psi_k - g_r g^{-1} \psi_{k-1} - \frac{k-1}{r'} \psi_{k-1} + \psi_{k-1,r'}) \quad (57)$$

It follows from (52) that

$$\psi_{k,t} + \psi_{k+1,r} + \frac{k+1}{r} \psi_{k+1} - g_t g^{-1} \psi_k = 0$$

Putting $r \rightarrow \infty$ and assuming that all integrals are convergent, one finds that

$$Q_k = \lim_{r \rightarrow \infty} r^k \psi_k(r, t)$$

are independent of t and represent the infinite set of non-local conservation laws:

$$\begin{aligned} Q_1 &= \int_0^\infty g_t g^{-1} r dr, \\ Q_2 &= \int_0^\infty g_t g^{-1} \int_0^r g_t g^{-1} r' dr' r dr \\ &\dots \end{aligned}$$

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