A 81-127

c.

ACADEMY OF SCIENCES OF THE USSR
P. N. LEBEDEV PHYSICAL INSTITUTE

0 NOV. 1981

Preprint No. 127

QUANTUM INTEGRALS OF MOTION
AND GRAVITY WAVE EXPERIMENT.
MEASUREMENTS IN PURE QUANTUM STATES

V.V.Dodonov, V.I.Man'ko, V.N.Budenko

High energy electrons laboratory
High energy and cosmic rays physics

Moscow 1981

CERN LIBRARIES, GENEVA

CM-P00067569

Abstract.

Quantum limitations which could arise in a measurement of a classical force acting on a quantum harmonic oscillator are studied in connection with the problem of increasing the sensitivity of a gravity wave experiment. After the authors of refs. [3-5] we draw an attention to the class of quantum nondemolition measurements free of quantum limitations for the accuracy of the measurement of an external force. It is shown that such measurements can be realized in observing the operators which are integrals of motion of the quantum system under study. The physical reasons responsible for the existence or nonexistence of a quantum limit of sensitivity for an arbitrary choice of an observable are exposed. They consist in a degree of the uncertainty of an initial state of the system. For integrals of motion this uncertainty can be reduced to zero as a result of the first exact measurement, and it remains zero for subsequent moments of time due to the properties of integrals of motion. Besides, it is shown that another choice of observables suitable for exact continuous measurements of an external force independently on the initial state of the system is also possible. An example of such an observable - the difference link operator - is given. The general rule for constructing such "optimal observables" can be derived from the quantum filtration theory developed in refs. [11-12]. It is shown using Ehrenfest's theorem that no quantum limitations exist in principle for the accuracy of the measurement of an external classical force acting on an

arbitrary quantum system.

Introduction.

The problem of the limiting sensitivity in a macroscopic experiment has attracted an appreciable attention of many physicists for recent years due to the significant progress of the experimental equipment and the need of carrying out some measurements of both fundamental and applied character in physics of gravitation, physics of elementary particles, geophysics, etc. The current state of this problem was analyzed in detail, e.g., in refs. [1-2] .

A new outburst of interest to this problem is connected with the gravitational waves experiment, namely, with the elucidation of the existence or nonexistence of quantum limits in discovering weak gravitational signals coming (presumably) to the Earth from the cosmos with the aid of macroscopic antennas [2-9] . Indeed, simple estimates by Thorne et al [3] show that the deformation of the gravitational detector (GD) of Weber's type caused by a splash of radiation from Virgo cluster should be less than the characteristic quantum dispersion of the coordinate of GD which is considered as quantum oscillator. A priori the possibility of performing measurements at such a low "quantum" level is not selfevident. It was shown in the recent papers by Braginsky et al [4] that the required high sensitivity could be achieved by means of a certain measurement procedure called "quantum nondemolition measurement" (QNDM).

Mathematical definitions of QNDM were given by Unruh [5] and Thorne et al [3]. Various schemes of devices admitting quantum non-demolition measurements were studied in refs. [3-8].

The aim of our paper is to discuss physical principles of QNDM. Usually, the analysis of the problem of discovering a weak force from the response of a quantum system is performed in the framework of the principle of the "inverse fluctuation influence" of a measuring device to the system under study, which was proposed by Braginsky in ref. [10]. Limitations of sensitivity arising due to this principle are significant when an experimenter is interested, e.g., in the continuous measurement of some canonical variable: coordinate, momentum, energy, etc. However, if the final aim of an experiment is to record some external force only, then these limitations can be avoided by using a new class of measurements, namely, QNDM. In essence, QNDM is reduced to the search of an observable which can be measured for a given time interval without changing the state of the quantum system under study. We give a systematic analysis of such observables and show, particularly, that they include the class of quantum time-dependent integrals of motion. Besides, we consider another class of measurements corresponding to the rules of the theory of optimum quantum filtration of a classical signal against a background of quantum fluctuations. The foundations of this theory were given in refs. [11-12].

In this paper we consider the case of pure quantum systems whose evolution (in the absence of an interaction with

a measuring apparatus) can be described completely in terms of vectors in Hilbert's space and the Schrödinger equation. Besides, we consider ideal devices and do not take into account a thermal or instrumental noise existing in real devices. In other words we confine ourselves to the elucidation of the possibility in principle of the exact measurement of a classical force acting on a quantum system, and do not go into details of a technical realization of the principle schemes considered. The concrete model discussed in this paper is, as in the majority of papers on the same subject, a quantum forced harmonic oscillator described by the Hamiltonian

$$\hat{H} = \frac{1}{2m} \hat{p}^2 + \frac{1}{2} m \omega^2 \hat{q}^2 - f(t) \hat{q}$$

Therefore the gravitational field is assumed to be a classical one. The justification of this model can be found, e.g., in refs. [1-4].

Preliminary estimation and examples.

First let us recall in what way the estimate for the minimum discoverable force is usually obtained. The average values of the operators of the coordinate and the momentum change in time according to classical equations

$$\frac{d}{dt} \langle \hat{q} \rangle = \frac{1}{m} \langle \hat{p} \rangle; \quad \frac{d}{dt} \langle \hat{p} \rangle = -m\omega^2 \langle \hat{q} \rangle + f(t) \quad (1)$$

m is the mass of an oscillator; ω - its frequency; $f(t)$ -

the external (classical) force. The solutions to these equations are the same both in quantum and classical cases:

$$\begin{aligned}\langle \hat{q}(t) \rangle &= \cos \omega t q^{(i)} + \frac{p^{(i)}}{m\omega} \sin \omega t + \delta_q(t); \\ \langle \hat{p}(t) \rangle &= -m\omega q^{(i)} \sin \omega t + p^{(i)} \cos \omega t + \delta_p(t)\end{aligned}\quad (2)$$

$$\begin{aligned}\delta_q(t) &= \frac{1}{m\omega} \int_0^t f(\tau) \sin \omega(t-\tau) d\tau; \\ \delta_p(t) &= \int_0^t f(\tau) \cos \omega(t-\tau) d\tau = m \frac{d}{dt} \delta_q(t)\end{aligned}\quad (3)$$

Symbols $q^{(i)}$ and $p^{(i)}$ mean the initial average values at $t=0$. In the resonance case, when $f(t) = F \sin(\omega t + \varphi)$, ω , $\varphi = \text{const}$, one has

$$\delta_q^{(res)}(t) = \frac{F}{2m\omega} \left\{ -t \cos(\omega t + \varphi) + \frac{1}{2\omega} [\sin(\omega t + \varphi) + \sin(\omega t - \varphi)] \right\} \quad (4)$$

Consequently, if $q^{(i)} = p^{(i)} = 0$, and the force acts on the oscillator for a rather long period of time $t \gg \omega^{-1}$, then the maximum displacement of the value $\langle \hat{q} \rangle$ is equal to $Ft/2m\omega$. However, one can say with certainty that this displacement is caused by the external force $f(t)$ only provided $|\delta_q(t)|$ is greater than $\sqrt{\sigma_q}$, where $\sigma_q = \langle (\hat{q} - \langle \hat{q} \rangle)^2 \rangle$ is the dispersion of the coordinate of the oscillator. One can easily check that the dispersions σ_q , $\sigma_p = \langle (\hat{p} - \langle \hat{p} \rangle)^2 \rangle$, and $\sigma_{pq} = \frac{1}{2} \langle \hat{p}\hat{q} + \hat{q}\hat{p} \rangle - \langle \hat{p} \rangle \langle \hat{q} \rangle$ do not depend on average values $\langle \hat{q} \rangle$ and $\langle \hat{p} \rangle$ and the force $f(t)$, and they evolve in

time according to the following equations:

$$\begin{aligned}\dot{\sigma}_p &= -2m\omega^2 \sigma_{pz}; \\ \dot{\sigma}_z &= \frac{2}{m} \sigma_{pz}; \\ \dot{\sigma}_{pz} &= \frac{1}{m} \sigma_p - m\omega^2 \sigma_z\end{aligned}\quad (5)$$

Equations (5) are the consequences of the Schrödinger equation.

The solutions to these equations are as follows (for the sake of simplicity we suppose $\sigma_{pz}(0) = 0$):

$$\begin{aligned}\sigma_z(t) &= \sigma_z(0) \cos^2 \omega t + \frac{\sigma_p(0)}{(m\omega)^2} \sin^2 \omega t; \\ \sigma_p(t) &= \sigma_p(0) \cos^2 \omega t + (m\omega)^2 \sigma_z(0) \sin^2 \omega t\end{aligned}\quad (6)$$

But the initial values $\sigma_z(0)$ and $\sigma_p(0)$ must satisfy the uncertainty relation $\sigma_p(0) \sigma_z(0) \geq \hbar^2/4$. Then $\sigma_p(0) \geq \hbar^2/4\sigma_z(0)$, and, consequently,

$$\sigma_z(t) \geq \sigma_z(0) \cos^2 \omega t + \frac{\hbar^2 \sin^2 \omega t}{4(m\omega)^2 \sigma_z(0)}$$

In turn, the right-hand side of this inequality is not less than its minimum (with respect to $\sigma_z(0)$) value. Thus one obtains the following inequality:

$$\sigma_z(t) \geq \frac{\hbar}{2m\omega} |\sin 2\omega t| \quad (7)$$

Therefore, if the moment t of the measurement of the displacement of the coordinate is quite arbitrary, then the ex-

ternal force can be registered with certainty provided $\delta_x^2 \geq \hbar/2m\omega$. Thus one arrives at the conclusion on the existence of the quantum limit for the minimum discoverable external force:

$$F_{min} \approx \frac{1}{t} (2m\hbar\omega)^{1/2} \quad (8)$$

Nevertheless, this restriction is not absolute. Indeed, eq.(7) shows that if the moments of the measurement of the coordinate are not arbitrary but satisfy the condition $t_K = K\pi/\omega$, $K = 1, 2, 3 \dots$, then the dispersion σ_x^2 can be made, in principle, equal to zero. Consequently, measuring the displacement of the coordinate of the oscillator at the moments $t_K = K\pi/\omega$ one can record an arbitrary weak external force. Thus we arrive at the concept of "stroboscopic measurements" by Braginsky et al [4].

However, the stroboscopic measurement has many evident disadvantages. Firstly, it can record only the existence of an external force, but it cannot be used for studying the time dependence of this force. Secondly, the stroboscopic procedure can be applied only in the one-dimensional case, when the motion (without an external force) is strictly periodic. Already in the case of a two-dimensional oscillator with non-commensurable frequencies this procedure falls. Therefore the stroboscopic measurement should be considered as a "mental experiment" only, and it can hardly be used in practice.

Thorne et al [3] proposed to observe not the coordinate of the oscillator but some linear combinations of the con-

dinate and momenta with time-dependent coefficients.

Namely, they proposed to measure the following observables:

$$Q_0(t) = q \cos \omega t - \frac{p}{m\omega} \sin \omega t \quad (9)$$

$$P_0(t) = p \cos \omega t + m\omega q \sin \omega t \quad (10)$$

(in their paper these quantities were designated as X_1 and X_2).

Let us show that measuring $Q_0(t)$ one can indeed in principle detect an arbitrary weak external force. Using eqs. (2), (3) and (9) one can easily obtain the relation

$$\langle \hat{Q}_0(t) \rangle = q^{(i)} - \frac{1}{m\omega} \int_0^t f(\tau) \sin \omega \tau d\tau \equiv q^{(i)} - \delta_Q(t) \quad (11)$$

As to the dispersion $\sigma_Q(t) = \langle \hat{Q}_0^2(t) \rangle - \langle \hat{Q}_0(t) \rangle^2$, it can be shown, using eqs. (5) and (9), that it does not depend on time: $\frac{d}{dt} \sigma_Q \equiv 0$. Consequently, the dispersion of the quantity $Q_0(t)$ coincides with the dispersion of the coordinate at the initial moment: $\sigma_Q(t) = \sigma_q(0)$. The dispersion $\sigma_q(0)$ is not affected to any quantum limitations, so that it can be made equal to zero in principle. This means that the variable $Q_0(t)$ can be measured, in principle, with an arbitrary accuracy at any moment of time. Therefore, measuring the difference $Q_0(t) - q^{(i)}$ one can exactly determine the value $\delta_Q(t)$ and, consequently, the external force

$f(t)$. It is significant that this can be done in an arbitrary moment of time, i.e. in a continuous manner.

The last statement requires in fact a thorough inspection in view of its compatibility with the basic postulates of quantum mechanics referring to the process of measurement. This aspect is discussed in the next section.

Fundamental assumptions.

We proceed from the orthodox treatment of quantum mechanics given by von Neumann [13] (see also, for example, refs. [14-15] and especially book [16] in which different points of view on the problem of measurements in quantum mechanics are represented).

The first important assumption is that every hermitian operator corresponds to some physical observable. Usually there are no objections to this assumption, although it is by no means trivial. Indeed, it implies that one can invent, in principle, a device measuring the observable corresponding to any given hermitian operator, for example, $\hat{q}^n \hat{p}^m + \hat{p}^m \hat{q}^n$. Intuitively this is not obvious (see also ref. [15]). It is essential for our purposes that the first assumption enables to construct a device measuring the variable $Q_0(t)$. We shall not discuss the concrete scheme of such a device given in ref. [3] , but confine ourselves by the assumption that it can be made.

The second assumption concerns the process of a measure-

ment. Let the quantum system be in an eigenstate $|\varphi_\lambda\rangle$ of a hermitian operator $\hat{A}$: $\hat{A}|\varphi_\lambda\rangle = \lambda|\varphi_\lambda\rangle$. Then any device measuring the observable A must yield in its output the unique and quite definite value: λ . We suppose the existence of ideal devices which can measure the observables a) instantly (practically for an arbitrary small period of time); b) without disturbing the initial eigenstate of the quantum system, i.e. in such a manner that the state of the system after the measurement would be the same state $|\varphi_\lambda\rangle$ that before it.

Generally speaking, the state of the system in the process of measurement in most cases changes (as it takes place, for example, in the usually considered schemes of the measurement of the momentum with the aid of the diffraction grille or Compton's scattering [14]). Moreover, some authors (see, e.g., [17]) believe that nondisturbing measurements cannot be made in real experiments with the exception of measurements of coordinates. Evidently, the measurement can be nondisturbing only if the interaction Hamiltonian describing the interaction between the quantum system under study and the measuring device commutes with the operator $\hat{A}$; the simplest example of such a Hamiltonian is $\hat{H}_{int} = \hat{A} \hat{Y}$, where the operator $\hat{Y}$ acts only on the state of the device. However, it is not clear, whether the interactions corresponding to the given mathematical Hamiltonians can be arranged in nature in all cases. Therefore, although the concept of

nondisturbing measurements ("first kind observations" by von Neumann, or "ideal measurements" by Messiah) does not contradict the principles of quantum mechanics, one should remember that this concept may turn after all only a nice mathematical toy having nothing in common with the reality.

Strictly speaking, the statement that the result of the measurement of the observable Λ in the eigenstate $|\varphi\rangle$ is always equal to λ , is in fact also only an idealization, since the results obtained in the papers of ref. [18] show that the exact measurement of an observable whose operator does not commute with the operators of the conserved additive quantities (momentum, energy, angular momentum) is impossible. Nonetheless approximate measurements can be performed with an arbitrary given accuracy.

The situation with the possibility of instant nondisturbing measurements is even more complicated because the problem of existence or nonexistence of such measurements is intimately related to the problem of the correct interpretation of the time-energy uncertainty relation. Some distinguished physicists deny categorically the possibility of instant nondisturbing measurements saying that such measurements are incompatible with quantum mechanics (see, e.g., refs. [17, 19]), while others believe that such measurements are quite admissible [20]. We do not intend to go into details of this complicated problem here, but simply assume that instant nondisturbing measurements do not contradict the principles of quantum mechanics.

Obviously, if such measurements are impossible, then the operators like (9) cannot be used for continuous detecting an external force. Moreover, in such a case the idea of continuous measurements itself becomes meaningless.

Quantum integrals of motion and QNEM.

Now we want to clarify the physical significance of the operators $\hat{Q}_0(t)$ (9) and $\hat{P}_0(t)$ (10) and to show that they are nothing but quantum integrals of motion of a harmonic oscillator. Besides, we intend to discuss in detail the general relations between integrals of motion and QNEM.

The operator $\hat{I}(t)$ is called the integral of motion provided its average values in every quantum state $|\psi(t)\rangle$ satisfying the Schrödinger equation do not depend on time (see, e.g., refs. [14,17]):

$$\frac{d}{dt} \langle \psi(t) | \hat{I}(t) | \psi(t) \rangle = 0 \quad (12)$$

Using the Schrödinger equation $i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi$ and taking into account the hermiticity of the Hamiltonian operator one can obtain the usual equation for the integral of motion

$$(i\hbar \frac{\partial}{\partial t} \hat{I}(t) - [\hat{H}, \hat{I}]) \psi(t) = 0 \quad (13)$$

Sometimes only those integrals of motion which do not depend on time explicitly (energy, momentum, angular momentum) are considered. However such operators do not exhaust

all the variety of integrals of motion, and in many cases explicitly time-dependent integrals of motion prove to be very useful [21]. Just such a situation takes place in the case under study.

Indeed, one can easily check that both operators (9) and (10) satisfy eq.(13). Consequently they are integrals of motion. They can be called the operators of the initial coordinate and momentum, since at $t=0$ they coincide with the operators $\hat{q}$ and $\hat{p}$, respectively.

It should be emphasized (to prevent a possible misunderstanding) that we use the Schrödinger picture. The reasons for the choice of this representation are as follows. Firstly, we follow the traditional interpretation of quantum mechanics in most standard textbooks (see, e.g. [14,17]). Secondly, the process of a measurement is usually described just in terms of the reduction of the wave function of a system. Sometimes Heisenberg's picture is more preferable than Schrödinger's one (see, e.g. the discussion on the relations between both representations in ref. [22]), but the analysis of the problem of measurements in nonrelativistic quantum mechanics can be performed in the most convenient and natural way, in our opinion, just in the framework of Schrödinger's description, in which the time evolution is attributed to wave functions while the operators of coordinates and momenta do not depend on time, and, consequently, the devices measuring the corresponding observables are explicitly the same at all moments of time.

The postulates discussed in the previous section related to individual measurements. Suppose some observable described by a hermitian operator $\hat{A}$ admits a continuum measurement. This implies that after an instant measurement at a moment t the state of a system under investigation is not changed, i.e. that the measurement does not interfere with the free evolution of the system governed by the Schrödinger equation. According to basic postulates, this can be only provided the state $|\psi(t)\rangle$ is an eigenstate of the operator $\hat{A}$ at all moments of time. Since the state $|\psi(t)\rangle$, generally speaking, changes in time, the operator $\hat{A}$ and its eigenvalue λ corresponding to the state $|\psi(t)\rangle$ must also depend on time in a general case. Consequently, the following relation must hold:

$$\hat{A}(t)|\psi(t)\rangle = \lambda(t)|\psi(t)\rangle \quad (14)$$

Introducing the evolution operator $\hat{U}(t)$ satisfying the Schrödinger equation $i\hbar \partial \hat{U} / \partial t = \hat{H} \hat{U}$ and the initial condition $\hat{U}(0) =$ unity operator, one can express the state $|\psi(t)\rangle$ in terms of the initial state $|\psi(0)\rangle$ as follows, $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$. Then equation (14) can be written as follows,

$$\hat{A}(t)\hat{U}(t)|\psi(0)\rangle = \lambda(t)\hat{U}(t)|\psi(0)\rangle$$

Multiplying both parts of this equation by $\hat{U}^{-1}(t)$ one obtains:

$$\hat{U}^{-1}(t)\hat{A}(t)\hat{U}(t)|\psi(0)\rangle = \lambda(t)|\psi(0)\rangle$$

Thus we see that $|\psi(0)\rangle$ is the eigenstate of two operators:

$\hat{A}(0)$ and $\hat{U}^{-1}(t) \hat{A}(t) \hat{U}(t)$. From textbooks [14, 17] we know that the necessary and sufficient condition of the possibility of a simultaneous exact measurement of two different observables is the commutativity of the operators corresponding to these observables. Consequently, the condition of the possibility of the continuous measurement of the operator $\hat{A}(t)$ can be written as follows,

$$[\hat{A}(0), \hat{U}^{-1}(t) \hat{A}(t) \hat{U}(t)] = [\hat{U}(t) \hat{A}(0) \hat{U}^{-1}(t), \hat{A}(t)] = 0 \quad (15)$$

The simplest possibility to satisfy this relation is to impose the following condition on the operator $\hat{A}(t)$:

$$\hat{A}(t) = \hat{U}(t) \hat{A}(0) \hat{U}^{-1}(t) \quad (16)$$

But this condition means that $\hat{A}(t)$ is an integral of motion, since

$$\begin{aligned} \langle \Psi(t) | \hat{A}(t) | \Psi(t) \rangle &= \langle \Psi(0) | \hat{U}^{\dagger}(t) \hat{U}(t) \hat{A}(0) \hat{U}^{-1}(t) \hat{U}(t) | \Psi(0) \rangle \\ &= \langle \Psi(0) | \hat{A}(0) | \Psi(0) \rangle = \text{const} \end{aligned}$$

(we have used the unitarity of the evolution operator: $\hat{U}^{\dagger}(t) = \hat{U}^{-1}(t)$). Thus we have shown that integrals of motion admit continuous measurements. The physical reason of this is that any integer degree of an integral of motion: $\hat{A}^n(t)$ - is an integral of motion too (since $[\hat{U} \hat{A}(0) \hat{U}^{-1}]^n = \hat{U} [\hat{A}(0)]^n \hat{U}^{-1}$), so that not only average values of $\hat{A}(t)$ are conserved in time, but all higher moments, including the dispersion $\sigma_A^2 = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2$, are also conserved. After the first measu-

rement of $\hat{A}(0)$ the system turns into an eigenstate of this operator in which the dispersion $\sigma_A(0)$ equals zero. Since it is conserved, it will be equal to zero for all subsequent moments of time, so that the results of subsequent measurements of $\hat{A}(t)$ can be predicted with certainty.

Note that the fact, that the uncertainties in measurements of the operators $\hat{A}(0)$ and $\hat{A}(t)$ related with eq.(16) coincide, was mentioned many years ago by Wigner [15]. Nonetheless the clear understanding of the fact that integrals of motion are ideal and natural tool for continuous measurements was not reached in the previous papers devoted to this problem.

Now let us return to the problem of discovering a weak external force. Evidently, since the external force $f(t)$ is unknown, one can use for solving this problem not integrals of motion of the complete Hamiltonian

$$\hat{H} = \frac{1}{2m} \hat{p}^2 + \frac{1}{2} m \omega^2 \hat{q}^2 - f(t) \hat{q} \equiv \hat{H}_0 - f(t) \hat{q} \quad (17)$$

(since the explicit forms of these operators depend on the unknown function $f(t)$), but only integrals of motion of the "free" Hamiltonian $\hat{H}_0$. It is not evident beforehand that such operators can help us (since they are related to another Hamiltonian). Fortunately, for a harmonic oscillator the integrals of motion - operators of initial coordinates and momenta (i.e. operators $\hat{u} \hat{q} \hat{u}^{-1}$ and $\hat{u} \hat{p} \hat{u}^{-1}$) in the presence of

an external force differ from the same operators in the absence of an external force only in an additive C -number function: one can easily check that the generalization of the integral of motion $\hat{Q}_0(t)$ (9) to the case $f(t) \neq 0$ is [21]

$$\hat{Q}'_0(t) = \hat{Q}_0(t) + \delta_Q(t) \quad (17)$$

where $\delta_Q(t)$ is given by eq.(11). Therefore the operators $\hat{Q}_0(t)$ and $\hat{Q}'_0(t)$ have the same eigenfunctions, so that the measurement of the operator $\hat{Q}_0(t)$ (9) can be performed in a continuous manner even in the presence of an external force.

Note that the accuracy of the measurement does not depend on the initial eigenstate or the initial average value $\bar{Q}^{(i)}$ of the operator $\hat{Q}_0(t)$, since the dispersion of the operator $\hat{Q}_0(t)$ does not depend on its average value. Moreover, this statement is valid for an arbitrary integral of motion of the type $\chi(\hat{Q}_0)$ or $\varphi(\hat{P}_0)$, where χ and φ - arbitrary real functions. Indeed, the eigenstates of the operators $\chi(\hat{Q}_0)$ and $\hat{Q}'_0(t)$ (17) obviously coincide. Therefore, if at the initial moment the system was in an arbitrary eigenstate of the operator $\chi(\hat{Q}_0(0))$, it will be in the eigenstate of the operator $\chi(\hat{Q}_0(t))$ for all subsequent moments of time. Consequently, although the average values $\langle \chi(\hat{Q}_0(t)) \rangle$ will change under the influence of the external force, but the dispersions of the operators $\chi(\hat{Q}_0(t))$ will identically equal to zero, so that the force can be measured exactly independently on the choice of the initial eigenstate. However, such a situation does not take place for an arbitrary integral of

motion.

For example, let us consider another integral of motion of a free oscillator: its energy $\hat{E} \equiv \hat{H}_0$. In the case of a forced oscillator the energy $\hat{E}$ is not conserved; the conserved operator coinciding with $\hat{E}$ at the initial moment $t=0$ is

$$\begin{aligned} \hat{K}(t) &= \frac{1}{2m} [\hat{p}'_0]^2 + \frac{1}{2} m \omega^2 [\hat{q}'_0]^2 \\ &= \frac{1}{2m} (\hat{p} - \delta_p(t))^2 + \frac{1}{2} m \omega^2 (\hat{q} - \delta_q(t))^2 \\ &= \hat{E} - \frac{1}{m} \delta_p(t) \hat{p} - m \omega^2 \delta_q(t) \hat{q} + \Phi(t); \end{aligned} \quad (18)$$

$$\begin{aligned} \Phi(t) &= \frac{1}{2m} \delta_p^2 + \frac{1}{2} m \omega^2 \delta_q^2 \\ &= \frac{1}{2m} \int_0^t \int_0^t f(\tau) f(\tau') \cos \omega(\tau - \tau') d\tau d\tau' \\ &= \frac{1}{2m} \left| \int_0^t f(\tau) e^{i\omega\tau} d\tau \right|^2 \end{aligned} \quad (18a)$$

where $\delta_q(t)$ and $\delta_p(t)$ are given by eqs.(3).

Let the initial state be the n -th eigenstate of a free oscillator: $\langle E^{(i)} \rangle = \hbar \omega (n + \frac{1}{2})$, $\sigma_{EE}(0) = \langle \hat{E}^2(0) \rangle - \langle \hat{E}(0) \rangle^2 = 0$. Then for all moments of time $\langle \hat{K}(t) \rangle = \hbar \omega (n + \frac{1}{2})$, since $\hat{K}(t)$ is the integral of motion. Calculating average values of both sides of eq.(18) and taking into account eqs.(2)-(3) and initial conditions $\langle \hat{p}(0) \rangle = \langle \hat{q}(0) \rangle = 0$ one can obtain the following relation:

$$\Delta E = \langle \hat{E}(t) \rangle - \langle \hat{E}(0) \rangle = \Phi(t) \quad (19)$$

The dispersion of the energy $\sigma_{EE}(t)$ can be obtained from the

following system of equations (it can be derived immediately from the Schrödinger equation):

$$\frac{d}{dt} \sigma_{EE} = \frac{2}{m} f(t) \sigma_{pE} \quad (20)$$

$$\frac{d}{dt} \sigma_{pE} = -m\omega^2 \sigma_{qE} + \frac{f(t)}{m} \sigma_p \quad (21)$$

$$\frac{d}{dt} \sigma_{qE} = \frac{1}{m} \sigma_{pE} + \frac{f(t)}{m} \sigma_{pq} \quad (22)$$

The designations are as follows,

$$\sigma_{pE} = \frac{1}{2} \langle \hat{p} \hat{E} + \hat{E} \hat{p} \rangle - \langle \hat{p} \rangle \langle \hat{E} \rangle;$$

$$\sigma_{qE} = \frac{1}{2} \langle \hat{q} \hat{E} + \hat{E} \hat{q} \rangle - \langle \hat{q} \rangle \langle \hat{E} \rangle$$

The initial values of σ_p , σ_q , and σ_{pq} satisfy the relations $\sigma_p = (m\omega)^2 \sigma_q = m\hbar\omega(n + \frac{1}{2})$, $\sigma_{pq} = 0$. According to eqs.(5), these relations will hold for all moments of time. The initial conditions for σ_{pE} and σ_{qE} are as follows, $\sigma_{pE}(0) = \sigma_{qE}(0) = 0$ (this is a simple consequence of the fact that the eigenfunctions of a harmonic oscillator in the coordinate representation are either even or odd). Therefore the solutions of eqs.(21)-(22) are given by formulas (2)-(3) in which one should make the substitutions $\langle \hat{q} \rangle \rightarrow \sigma_{qE}$; $\langle \hat{p} \rangle \rightarrow \sigma_{pE}$; $f(t) \rightarrow f(t) E^{(i)}$. As the final result one obtains that the dispersion of energy at the moment t equals

$$\sigma_{EE}(t) = 2 E^{(i)} \Phi(t) \quad (23)$$

If the spectrum of the energy was continuous, then an arbitrary weak external force could be detected in principle, since both the average value ΔE and the dispersion σ_{EE} could change only under the influence of the force. However, since the real energy spectrum of the quantum oscillator is discrete, one can register the external force with the appreciable probability only provided either $\Delta E \sim \hbar\omega$ or $\sigma_{EE} \sim (\hbar\omega)^2$ [2].

In the resonance case $\Phi(t) \approx F^2 t^2 / 8m$ ($t \gg \omega^{-1}$), so that the second of these relations leads to the estimate [2]

$$F_{\min} \sim \frac{2}{t} \left(\frac{m \hbar \omega}{n} \right)^{1/2}; \quad n \gg 1$$

Consequently, the accuracy of the measurement depends on the initial energy eigenvalue $E^{(i)} = \hbar\omega(n + \frac{1}{2})$. The unrestricted accuracy can be achieved only in the limit case $n \rightarrow \infty$.

This example shows distinctly the advantages of integrals of motion with the continuous spectrum (like $\hat{Q}_0(t)$ or $\chi(\hat{Q}_0)$) in comparison with the integrals of motion with the discrete spectrum (like $\hat{E}$): in the first case to achieve the high accuracy of the measurement one can take an arbitrary eigenstate of the integral of motion as the initial state of the oscillator (this can be achieved in principle by means of the first measurement of $\hat{Q}_0$; the result of the observation - the eigenvalue $q^{(i)}$ - is unimportant in this case) while in the second case one has to know how to construct the eigenstates of the operator $\hat{E}$ with the given eigenvalue $E^{(i)}$.

We emphasize once more that we use the Schrödinger representation. In the Heisenberg representation condition (15) takes the form [3,5]

$$[\hat{A}_H(t), \hat{A}_H(t')] = 0 \quad (15a)$$

because in this representation the state of a system is described by the same vector at all moments of time. In particular, the integrals of motion in the Heisenberg picture do not depend explicitly on time at all: $\hat{I}_H(t) \equiv \hat{I}_H(0)$. So that condition (15a) for them is fulfilled in a trivial manner.

Note that the stroboscopic procedure [4] is the special case of a more general method of measuring integrals of motion $\hat{Q}_0$ and $\hat{P}_0$, since at the moments $t_K = \pi K/\omega$ the operator $\hat{Q}_0(t_K)$ coincides with the operator $(-1)^K \hat{Q}$.

Extension of the class of admissible measurements:
a difference link.

The main result of the previous sections is that the problem of continuous nondemolition measurements is reduced from the viewpoint of principle to finding those observables whose dispersions are determined completely by their own values at the initial moment and do not depend on the dispersions of other observables. Then, if the initial state is prepared in such a manner that the dispersion of the observable found equals zero, it will be zero at all subsequent moments of time (provided an external force is absent). The examples of such

observables are integrals of motion $\hat{Q}_0$ and $\hat{P}_0$ - operators of initial coordinates and momenta. It should be noted, that although no restrictions on the accuracy of measurements of these observables exist, they cannot be measured exactly in the strict sense of this word (in accordance with the results of ref. [18]), because it is impossible to construct the exact eigenstates of the operators $\hat{Q} = \hat{Q}_0(0)$ and $\hat{P} = \hat{P}_0(0)$, due to an infinite energy of these states.

Dealing with the operator $\hat{Q}_0$ one has to solve two problems. The first one is to construct a real device measuring this observable (some concrete schemes were discussed in ref. [3]). The second problem consists in preparing the initial state with the minimal possible dispersion of the coordinate. Both problems are in fact very nontrivial. The aim of the present section is to show that besides integrals of motion there exist other observables suitable for the continuous measurement of an external force which are more convenient for the practical use.

In this section we use Heisenberg's picture.

Let us consider the following variable corresponding in the classical case to the difference link with a time delay:

$$\hat{y}(t; \tau) = \hat{q}(t + \tau) - \hat{q}(t) \quad (24)$$

Here τ is a constant parameter. Solving Heisenberg's equations of motion (their solutions are given in fact by eqs.(2)) one can obtain the following explicit formula for the operator

$$\hat{y}(t; \tau) :$$

$$\begin{aligned} \hat{y}(t; \tau) = & \hat{q}(t) (\cos \omega \tau - 1) + \frac{\hat{p}(t)}{m\omega} \sin \omega \tau \\ & + \frac{1}{m\omega} \left[\int_0^{t+\tau} \sin \omega(t+\tau-z) f(z) dz \right. \\ & \left. - \int_0^t \sin \omega(t-z) f(z) dz \right] \end{aligned} \quad (25)$$

Consequently, if the delay τ satisfies the relation

$$\omega \tau = 2\pi \ell; \quad \ell = 1, 2, 3, \dots \quad (26)$$

then the operator $\hat{y}(t; \tau)$ becomes a C -number:

$$\hat{y}\left(t; \frac{2\pi \ell}{\omega}\right) = \frac{1}{m\omega} \int_t^{t + \frac{2\pi \ell}{\omega}} f(z) \sin \omega(t-z) dz \quad (27)$$

Therefore, it can be measured exactly at all moments of time, since its dispersion is obviously equal to zero. Note that the operator $\hat{y}(t; 2\pi \ell / \omega)$ satisfies condition (15a), although it is not an integral of motion. The most attractive feature of the observable $\hat{y}(t; 2\pi \ell / \omega)$ is that it does not require any preliminary preparation of the initial state of the oscillator. This invariance with respect to the initial state is explained by the fact that the quantum fluctuations of the observables $\hat{q}(t+\tau)$ and $\hat{q}(t)$ are not independent, but correlated, so that their mutual correlation $\langle \hat{q}(t+\tau) \hat{q}(t) + \hat{q}(t) \hat{q}(t+\tau) \rangle \neq 0$ diminishes the dispersion of $\hat{y}(t; \tau)$ to zero in the case $\tau = 2\pi \ell / \omega$.

Obviously, if one measured separately $\hat{q}(t)$ and $\hat{q}(t+\tau)$ and after this constructed their difference, then the inter-

ference between $\hat{q}(t)$ and $\hat{q}(t+\tau)$ would be completely destroyed so that the resulting dispersion would not be equal to zero; sooner it would be $2\sigma_q(t)$. It is necessary to have a device measuring immediately the variable $\hat{q}(t; \tau)$. Whether such a device exists?

It seems to us that the answer is affirmative. Indeed, the operator $\hat{q}(t; \tau)$ is hermitian; therefore, if we seriously believe in postulates of quantum mechanics formulated earlier, then we have to recognize that this operator corresponds to an observable which can be measured in an experiment.

For example, the measurement of the operator $\hat{Q}_0(t)$ by no means implies that one measures separately the coordinate and momentum and after that constructs the necessary combination of these observables; no, the measuring device responds to the whole combination $(\hat{q} \cos \omega t - \hat{p} \sin \omega t / m\omega)$ (see the scheme of such a device in ref. [3]). But if such a measurement is possible for the operator $\hat{Q}_0(t)$, why it could not be possible for the operator $\hat{q}(t; \tau)$? (Note that τ is a parameter, so that $\hat{q}$ depends only on the single time t , and in this sense it does not differ from any other operator in Heisenberg's picture). Besides this formal reasoning, we can adduce some arguments of the physical nature.

Firstly, we would like to note that, from the viewpoint of physics, an experiment on the observation of $\hat{q}(t+\tau) - \hat{q}(t)$ is nothing but a typical interference experiment, similar, e.g., to the experiment on the observation of the interference due to the diffraction on two slits in Rayleigh's interfe-

rometer. The difference is that in the interferometer one uses the space delay (the interference lines arise because of different optical path lengths from the slits to the points of observation on the screen), while in the case of the difference link we use the time delay, but from the point of view of principle both experiments are of the same nature. For example, one can draw a parallel with the interferometer in the following manner. If the medium between the slits and the screen is uniform (this situation corresponds to the case when an external force is absent), then the resulting amplitude equals zero at the points for which the path lengths difference equals an integer number of wave lengths (when the time delay satisfies eq.(26)). But if the refractive index becomes a function of coordinates (an external time-dependent force is switched on) then the resulting amplitude at the same points becomes different from zero; measuring this amplitude one can obtain the information on the variation of the refractive index in different points of space (respectively, measuring $\hat{y}(t; 2\pi\rho/\omega)$ one obtains the value of the integral in the right-hand side of eq.(27) which allows to estimate the amplitude of an external force).

Another argument is even more important, especially from the practical point of view. It is known from radiophysics that the difference link with the time delay τ , which a random process $g(t)$ is put through, is equivalent to the filtration with the spectral transformation coefficient

$$K(\omega) = e^{i\omega\tau} - 1;$$

$$|K(\omega)| = \left| \sin \frac{\omega\tau}{2} \right| = \left| \sin \frac{(\omega - \omega_0)\tau}{2} \right| \quad (28)$$

(ω_0 is the oscillator eigenfrequency corresponding to the centre of the filter). Thus one can suppose that the measurement of $\hat{q}(t)$ can be performed by means of the continuous observation of the coordinate $\hat{q}(t)$, but not directly, but through a "light-filter" with the characteristic (28). This is the measurement of spectral Fourier-components lying out of the filter attenuation band with the width of the order of $\Delta\omega \sim \tau^{-1}$. Consequently (and this is very important), such a measurement is not the measurement of the coordinate itself, so that it does not disturb the state of the oscillator. If an external force is absent, an observer cannot obtain any information concerning the trajectory $\hat{q}(t)$ using such a filter, since in such a case the spectral components of $\hat{q}(t)$ lie inside the filter attenuation band. On the contrary, if an external force is present, some of spectral components appear out of the attenuation band, and they can be detected. Note that the measurement of some separate spectral components of the operator $\hat{q}(t)$ does not disturb other spectral components.

We would like to draw an attention of the reader to the following important consequences.

- 1) The example considered above shows the perspectiveness of the spectral approach to the problem of detecting weak forces (see also ref. [9]). The spectral methods are much more

sensitive with respect to the crude integral criterion of the "inverse fluctuation influence" of refs. [3,5,10].

2) It should be emphasized that the difference link operator $\hat{y}(t; \tau)$ is only one of the simplest examples of variables admitting exact measurements of external forces irrespective of the state of the quantum system. The general rule for finding such operators can be obtained from the quantum theory of optimal filtration of weak signals developed in papers [11-12]. In particular, some results of the application of this theory to the gravity wave antenna of the given structure (coupled oscillators, passive and active variants) can be found in ref. [6] (see also ref. [8]). According to [11-12], to select a determinate function $S'(t)$ against the background of the quantum Gaussian variable $\hat{f}(t)$ satisfying the commutation relation $[\hat{f}(t_1), \hat{f}(t_2)] = C$ - number, one should measure the following "optimal observable" (in Heisenberg's picture):

$$\hat{R}(t) = \frac{\int_0^t \int_0^t S'(t_1) K^{-1}(t_1, t_2) \hat{y}(t_2) dt_1 dt_2}{\int_0^t \int_0^t S'(t_1) K^{-1}(t_1, t_2) S'(t_2) dt_1 dt_2} \quad (29)$$

Here $K(t_1, t_2) = \frac{1}{2} \langle \hat{f}(t_1) \hat{f}(t_2) + \hat{f}(t_2) \hat{f}(t_1) \rangle$; $\hat{y} = \hat{S}' + \hat{f}$ and $K^{-1}(t_1, t_2)$ is the kernel of the operator inverse to the operator with the kernel $K(t_1, t_2)$. It can be shown under certain assumptions concerning the spectrum width of a signal $S'(t)$ that in the case of a harmonic oscillator the optimal variable (29) is reduced to the difference link operator (24). The details will be given elsewhere.

Classical force and measurements of average values.

Till now we considered an extremely simplified model of a real gravitational wave antenna: a harmonic (linear) one-dimensional oscillator. Since real systems include nonlinear interactions and more than one degrees of freedom, it is very desirable to have a more general proof of the nonexistence of the quantum limitations of principle in measuring classical external forces. Such a proof indeed exists, and it is very simple, almost trivial. It is founded on Ehrenfest's theorem, according to which if the Hamiltonian of a system has the form

$$\hat{H} = \frac{1}{2m} \hat{p}^2 + \mathcal{U}(\hat{q}) - f(t)\hat{q}$$

where $\mathcal{U}(\hat{q})$ is an arbitrary potential, and $f(t)$ - an external force (with the characteristic wave length much greater than the dimensions of the system), then the average values $\langle \hat{q} \rangle$ and $\langle \partial \mathcal{U} / \partial q \rangle$ satisfy the relation

$$f(t) = m \frac{d^2}{dt^2} \langle \hat{q} \rangle + \langle \frac{\partial \mathcal{U}}{\partial q} \rangle \quad (30)$$

Since quantum mechanics does not impose any restrictions on the accuracy of measurements of average values, the classical force $f(t)$ can be measured in principle with an arbitrary accuracy required. One can easily imagine a mental experiment based on equation (30). Suppose that the time interval $(0, \tau)$ is divided into N parts. Then one should prepare an ensemble of $n \times N$ oscillators in the same initial (arbitrary) state. At the mo-

ment t_1 one should measure the coordinates of the first, second, ..., n -th oscillator; at the moment t_2 - the coordinates of the $(n+1)$ -th, ..., $(2n)$ -th oscillator, and so on. Thus the approximate average values $\langle \hat{q} \rangle$ and $\langle \partial U / \partial q \rangle$ will be obtained at each moment t_k :

$$\langle \hat{q}(t_k) \rangle \approx \frac{1}{n} \sum_{j=(k-1)n+1}^{kn} q_j; \quad \langle \frac{\partial U}{\partial q}(t_k) \rangle \approx \frac{1}{n} \sum_{j=(k-1)n+1}^{kn} \frac{\partial U}{\partial q}(q=q_j)$$

Taking a sufficiently large number n one can determine the average values $\langle \hat{q}(t) \rangle$ and $\langle \partial U / \partial q \rangle$ with an arbitrary accuracy at each moment of time. Taking a sufficiently large number N one can determine with an arbitrary accuracy the accelerations $d^2 \langle \hat{q} \rangle / dt^2$ at each moment. Therefore the measurement of an external force $f(t)$ is reduced to the measurements of coordinates of $n \times N$ oscillators; so that in the limit $n, N \rightarrow \infty$ the force can be determined with an arbitrary desirable accuracy.

Evidently, this scheme hardly can be applied to a real experiment, but from the point of view of principle it is quite legitimate. Thus the problem of quantum limitations in measurements of classical external forces does not exist in principle. Limitations can arise only due to nonadequate measurement procedures.

Conclusion.

Let us formulate briefly the basic results of the paper.

The main result is that the problem of quantum limitations in measuring a classical force does not exist in principle due to Ehrenfest's theorem: measuring average values of coordinates and certain functions of coordinates one can determine an external force with any desirable accuracy.

However, since in a real gravity wave experiment one cannot use an ensemble consisting of a great number of the same antennas, one has to find such observables whose dispersions can be made, in principle, equal to zero at all moments of time: then the external force can be determined exactly with the aid of a single device measuring the observable concerned. We have found the general mathematical condition distinguishing the observables admitting a continuous measurement from all others and discussed the physical sense of this condition.

We have shown that quantum integrals of motion - operators of the initial coordinates and momenta belong to the class of observables suitable for a continuous measurement in the case of a harmonic oscillator. Strictly speaking, the measurement of integrals of motion should be called the "quasi-non-demolition measurement", since it requires the appropriate choice of the initial state of the quantum system (or the creation of such a state in the process of the first measurement which destroys, generally speaking, the previous state of the

system).

The nonexistence of quantum limitations for the sensitivity of the gravity wave antennas was shown earlier in another way in refs. [3-4]. However, the consequence of that papers was the idea that to achieve a high sensitivity one should change radically the measuring procedures (against the traditional methods used in antennas of the first generation) to satisfy the requirements of "quantum nondemolition measurements"; for example, various schemes of "stroboscopes" or devices for measuring the operators $\hat{Q}_0(t)$ and $\hat{P}_0(t)$ were proposed.

Our analysis clarifies and supplements the results of refs. [3-5]. The new important result obtained in the present paper consists in discovering a new class of observables, not coinciding with integrals of motion but satisfying the requirements of nondemolition measurements. The measurement of these observables (the simplest example of them is the difference link) can be called "strictly nondemolition measurement" since it can be carried out exactly and continuously irrespectively of the state of the quantum system. We think that such observables can occur more suitable for the employment in a real gravity wave experiment since they can be measured with the aid of the same methods that are usually applied in classical experiments, for example, the spectral methods or, more generally, the methods of quantum theory of optimal filtration (as it was proposed, e.g., in ref. [6]).

What important problems are still to be solved?

1. We neglected the fluctuations arising due to the interaction of an oscillator with the external world (thermal noise). This problem is considered in detail in our subsequent paper.
2. It would be very desirable to find the observables admitting nondemolition measurements in the case of a nonlinear oscillator or a system of coupled oscillators.

Besides, it is interesting to compare our approach to the problem of nondemolition measurements with another approach, based on the criterion of the maximum information derived from the measurement [23].

3. We considered the model of an antenna with constant parameters. The classical theory of an antenna with time-dependent parameters (using the phenomenon of the parametric resonance) was given in ref. [24]. It can be easily generalized to the quantum case, using the solutions of the Schrödinger equation for a harmonic oscillator with time-dependent parameters (see ref. [21] and references therein). This will be done in another paper.

4. We would like to emphasize once more, that we considered the problem of detecting weak forces at the level of principles. Therefore the question whether the operators like $\hat{Q}_0(t)$ and $\hat{q}(t; \tau)$ do correspond to physical observables which can be measured in a real experiment is, strictly speaking, still open, as well as a more fundamental problem of the validity of basic postulates of quantum mechanics relating to the measurement process, especially the postulate on the possibility of instant nondisturbing measurement. Unless this fundamental

problem (including the problem of the interpretation of the time-energy uncertainty relation) is solved in a convincing and rigorous manner, all reasonings on quantum continuous non-demolition measurements are doomed to be only plausible but not well-founded assumptions.

Some results of this paper were published in ref. [25] .

Acknowledgments.

The authors are grateful to V.B.Braginsky, V.L.Ginsburg, M.A.Markov, and W.G.Unruh for helpful discussions.

References

1. V.B.Braginsky, Physical Experiments with Test Bodies (Nauka, Moscow, 1970) - in Russian;
V.B.Braginsky and A.R.Manukin, Measurement of Weak Forces in Physical Experiments (Nauka, Moscow, 1974) [English translation - Univ. of Chicago Press, Chicago, 1977].
2. V.B.Braginsky and Yu.I.Vorontsov, Uspekhi Fiz.Nauk 114, 41 (1974) [Sov.Phys. - Uspekhi 17, 644 (1975)].
3. K.S.Thorne, R.W.Drever, G.M.Caves, M.Zimmerman, V.Sandberg, Phys.Rev.Lett. 40, 667 (1978); Rev.Mod.Phys. 52, 341 (1980).
4. V.B.Braginsky, Yu.I.Vorontsov, F.Ya.Khalili, Zhurn.Eksp. Teor.Fiz. 73, 1340 (1977) [Sov.Phys. - JETP 46, 705 (1977)]; Pisma ZhETF 27, 296 (1978) [Sov.Phys. - JETP Letters 27, 277 (1978)].
5. W.G.Unruh, Phys.Rev. D17, 1180 (1978); D18, 1764 (1978); D19, 2888 (1979).
6. A.V.Gusev and V.N.Rudenko, Zhurn.Eksp.Teor.Fiz. 74, 819 (1978); 76, 1488 (1979).
7. V.Moncrief, Ann.Phys. (N.Y.) 114, 201 (1978).
8. J.W.Hollenhorst, Phys.Rev. D19, 1669 (1979).
9. M.B.Mensky, Phys.Rev. D20, 384 (1979); Zhurn.Eksp.Teor. Fiz. 77, 1326 (1979).
10. V.B.Braginsky, Zhurn.Eksp.Teor.Fiz. 53, 1436 (1967).
11. C.W.Helstrom, Quantum Detection and Estimation Theory (Academic Press, N.Y., 1976).

12. B.A.Grishanin, Radiotekhnika i Electronica 18, 789 (1973);
B.A.Grishanin and R.L.Stratonovich, Problemi Peredachi Informatsii 6, 15 (1970);
R.L.Stratonovich, IEEE 22, 303 (1974).
13. J. von Neumann, Mathematical Foundations of Quantum Mechanics, chaps. 5-6 (Princeton Univ.Press, Princeton, 1955).
14. A.Messiah, Quantum Mechanics, vol.1, chap. V (North Holland, Amsterdam, 1961).
15. E.P.Wigner, Amer.J.Phys. 31, 6 (1963).
16. Proceedings of the Int. School of Physics "Enrico Fermi", Course 49, "Foundations of Quantum Mechanics" (Academic Press, N.Y., 1971).
17. L.D.Landau and E.M.Lifshits, Quantum Mechanics (Nauka, Moscow, 1963).
18. E.P.Wigner, Zs.Physik 133, 101 (1952); H.Araki and M.M.Yanase, Phys.Rev. 120, 622 (1960).
19. V.A.Fock, Zhurn.Eksp.Teor.Fiz. 42, 1135 (1962) [Sov.Phys.-JETP 15, 784 (1962)].
20. Y.Aharonov and D.Bohm, Phys.Rev. 122, 1649 (1961); 134, B1417 (1964); G.R.Allcock, Ann.Phys. 53, 253 (1969);
Y.Aharonov and J.L.Safko, Ann.Phys. 91, 279 (1975).
21. I.A.Malkin and V.I.Man'ko, Dynamical Symmetries and Coherent States of Quantum Systems (Nauka, Moscow, 1979) - in Russian.
22. P.A.M.Dirac, Lectures on Quantum Field Theory (Belfer Graduate School of Science, Yeshiva University, N.Y., 1967).

23. V.D.Vainshtein, Izvestiya VUZ, ser. "Fizika", No.7, 211 (1979).
24. V.B.Braginsky, Ya.B.Zeldovich, V.N.Rudenko, Pisma ZhETF 10, 437 (1969); N.D.Loza and V.N.Rudenko, Problems of Gravitation and Elementary Particles 5, 176 (1974) - in Russian.
25. V.V.Dodonov, V.I.Man'ko, V.N.Rudenko, Zhurn.Eksp.Teor. Fiz. 78, 881 (1980).

Т - 09649

Подписано в печать 19 мая 1981 года

Заказ № 266. Тираж 100 экз. п/л 1.4

Отпечатано в Редакционно-издательском
отделе ФИАН СССР

Москва, В-312, Ленинский проспект, 53