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Abstract.

The explicit expressions for the Green functions of quantum systems with nonstationary Hamiltonians, which are general quadratic forms with respect of coordinates and momenta, are obtained in the mixed (Wigner's) representation.

As an example, Green functions in mixed representation for an oscillator of a variable frequency being acted upon nonstationary external force and for an electron moving in a variable electromagnetic field are considered. We found the explicit formulas for equilibrium Wigner functions for most general quadratic systems with the Hamiltonians not depending on time, and for their significant particular cases - a multidimensional anisotropic oscillator and a particle in uniform electrical and magnetic fields. The explicit formulas for average meanings of arbitrary products of the coordinates and momenta in the equilibrium thermodynamical state for systems with arbitrary quadratic Hamiltonians are found.

Introduction.

The Green function (a kernel of the evolution operator) contains all information about the dynamics of the quantum system. The exact expressions for Green functions of the Schrodinger time-dependent equation were found only for a few of Hamiltonians.

For example, they were obtained for the most general nonstationary multidimensional quadratic Hamiltonian and its special cases in the coordinate, momentum, coherent states and Fock representations in terms of the matrix elements of the inhomogeneous symplectic transformations [1-4]. At the same time in quantum mechanics the mixed representation which was introduced for the first time by Weyl [5] for arbitrary operators and Wigner [6] for density operators (see also [7-8]) plays an important role. The Feynman amplitude (Green functions) in the mixed representation is connected with the amplitude in the coordinate representation by the simple integral transformation. An expression for the Green function in the mixed representation of systems with the quadratic Hamiltonians was discussed in the work [9]. The wave functions for the quadratic nonstationary Hamiltonians without linear terms were found in the coordinate representation both in the form of Gaussian packets and in the form of Hermite polynomials in ref. [10].

The aim of the present paper is to obtain in the most simple way the explicit expression in the mixed representation of mat-

rix elements of the evolution operator for the nonstationary system with the quadratic Hamiltonian in terms of matrix elements of the dynamical inhomogeneous symplectic group using the results of papers [1-4]. We also obtain the Green functions in mixed representation for a nonstationary oscillator and for a charged particle in nonstationary uniform electrical and magnetic fields.

If the Hamiltonian of the system does not depend on time, then the simple replacement $t = -i\beta\hbar$, where $\beta^{-1} = kT$, T - absolute temperature, k - Boltzmann's constant, transforms the Green function into the equilibrium density matrix of the system $\hat{\rho} = e^{-\beta\hat{H}}$ (see for example [11]). Using such a replacement we shall obtain the explicit expressions for the equilibrium Wigner functions of multidimensional systems with the most general quadratic Hamiltonians. Such expressions were known earlier only for very simple systems, such as the one-dimensional harmonic oscillator [12-14], the particle and oscillator in magnetic and electrical fields [15-17]. We shall generalize the results of papers [12-17] to more complicated cases including the case of the anisotropic mass tensor.

Besides, we shall find the explicit expressions for the partition functions of general multidimensional quadratic systems and for average meanings of arbitrary products of coordinates and momenta in the state of thermodynamical equilibrium.

2. The general quadratic system.

Following ref. [2], we consider the quantum system with the Hamiltonian which is an arbitrary quadratic form of the coordinates and momenta (for the definiteness let us consider at first the coordinate representation)

$$\hat{H} = \frac{1}{2} \hat{q} B(t) \hat{q} + C(t) \hat{q}$$

$$\hat{q} = \begin{pmatrix} \hat{p} \\ \hat{x} \end{pmatrix}; \quad C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}; \quad B = \begin{vmatrix} b_1 & b_2 \\ b_3 & b_4 \end{vmatrix} \quad (2.1)$$

$$\hat{p} = (\hat{p}_1, \hat{p}_2, \dots, \hat{p}_N); \quad \hat{x} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_N); \quad [\hat{p}_i, \hat{x}_k] = i\hbar \delta_{ik}$$

N - the number of degrees of freedom.

N - dimensional vectors C_1 and C_2 , and N - dimensional matrices b_j may be arbitrary smooth enough functions of time. For this system there exist $2N$ linear integrals of the motion, for example, the operators of initial coordinates and momenta $\hat{x}_0$ and $\hat{p}_0$. The integral of the motion $\hat{Q}$ satisfies the equation

$$[i\hbar \frac{\partial}{\partial t} - \hat{H}, \hat{Q}] \psi = 0 \quad (2.2)$$

in the space of solutions of the Schrödinger equation; therefore it transforms any solution of the Schrödinger equation to another solution of the same equation.

Integrals of the motion $\hat{p}_0$ and $\hat{x}_0$ are connected with operators $\hat{p}$ and $\hat{x}$ by linear relations:

$$\begin{aligned}\hat{p}_0(t) &= \lambda_1(t)\hat{p} + \lambda_2(t)\hat{x} + \delta_1(t); \quad \hat{p}_0(0) = \hat{p} \\ \hat{x}_0(t) &= \lambda_3(t)\hat{p} + \lambda_4(t)\hat{x} + \delta_2(t); \quad \hat{x}_0(0) = \hat{x}\end{aligned} \quad (2.3)$$

where N - dimensional matrices λ_k and N - dimensional vectors δ_i form $2N$ - dimensional matrix $\Lambda(t)$ and $2N$ - dimensional vector $\Delta(t)$, satisfying the system of the first order ordinary differential equations

$$\dot{\Lambda} = \Lambda \Sigma B; \quad \Lambda(0) = \Sigma_{2N}; \quad \Lambda = \begin{vmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{vmatrix} \quad (2.4)$$

$$\dot{\Delta}(t) = \Lambda \Sigma C; \quad \Delta(0) = 0; \quad \Delta = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}; \quad \Sigma = \begin{vmatrix} 0 & E_N \\ -E_N & 0 \end{vmatrix}$$

E_N - the unit $N \times N$ - matrix.

Matrix Λ is symplectical: $\Lambda \Sigma \tilde{\Lambda} = \Sigma$. This result is true both in quantum and classical mechanics since transformations (2.3) is canonical. The symplecticity condition means that $\det \Lambda = 1$, and matrices λ_k satisfy the following conditions:

$$\begin{aligned}\lambda_2 \tilde{\lambda}_1 &= \lambda_1 \tilde{\lambda}_2 & \lambda_4 \tilde{\lambda}_1 - \lambda_3 \tilde{\lambda}_2 &= E_N \\ \lambda_3 \tilde{\lambda}_4 &= \lambda_4 \tilde{\lambda}_3 & \tilde{\lambda}_4 \lambda_1 - \tilde{\lambda}_2 \lambda_3 &= E_N \\ \tilde{\lambda}_4 \lambda_2 &= \tilde{\lambda}_2 \lambda_4 & \Lambda^{-1} &= \begin{vmatrix} \tilde{\lambda}_4 & -\tilde{\lambda}_2 \\ -\tilde{\lambda}_3 & \tilde{\lambda}_1 \end{vmatrix} \\ \tilde{\lambda}_1 \lambda_3 &= \tilde{\lambda}_3 \lambda_1\end{aligned} \quad (2.5)$$

The mixed representation is defined as follows. If the kernel

of an arbitrary operator $\hat{A}$ in the coordinate representation $A(x, x') = \langle x | \hat{A} | x' \rangle$ is given then the function

$$G_A(q, p) = \int A\left(q + \frac{u}{2}, q - \frac{u}{2}\right) e^{-i \frac{pu}{\hbar}} du \quad (2.6)$$

is called the symbol (or kernel) of the operator $\hat{A}$ in the mixed representation [5-6]. The kernel of the operator $\hat{A}$ in the coordinate representation can be obtained from its symbol G_A by means of the inverse Fourier transformation. The Green function of the system considered in the coordinate representation satisfies the following system of $2N$ equations [2]

$$\begin{aligned} \hat{Q}_{\{x_2\}} G(x_2, x_1, t) &= q_{\{x_1\}}^T G(x_2, x_1, t) \\ \hat{Q} &= \begin{pmatrix} \hat{p}_0 \\ \hat{x}_0 \end{pmatrix} = \Lambda \hat{q} + \Delta \end{aligned} \quad (2.8)$$

or

$$(\Lambda \hat{q} + \Delta) G = G \hat{q} \quad (2.9)$$

The designation $\hat{I}_{\{x\}} \psi(x, y)$ means that the operator $\hat{I}$ acts on the function ψ as the function of the variable x , while the variable y is considered as a parameter. In the mixed representation the operator $\hat{q}$ is

$$\hat{q} = y - \frac{i\hbar}{2} \sum \frac{\partial}{\partial y}; \quad y = \begin{pmatrix} p \\ q \end{pmatrix}; \quad \frac{\partial}{\partial y} = \begin{pmatrix} \partial/\partial p \\ \partial/\partial q \end{pmatrix} \quad (2.10)$$

Then equation (2.9) in mixed representation takes the form

$$\left(\Lambda y - \frac{i\hbar}{2} \Lambda \Sigma \frac{\partial}{\partial y} + A\right) G(y, t) = \left(y + \frac{i\hbar}{2} \Sigma \frac{\partial}{\partial y}\right) G(y, t) \quad (2.11)$$

Equation (2.11) is easily integrated, so that the Green function $G(y, t)$ in the mixed representation has the following form

$$G(y, t) = K(t) \exp \left\{ \frac{i}{\hbar} y \Sigma (\Lambda + E_N)^{-1} (\Lambda - E_N) y + \frac{2i}{\hbar} y \Sigma (\Lambda + E_N)^{-1} \right\} \quad (2.12)$$

Evidently, the matrix of quadratic form in the exponent must be symmetrical, i.e. it must satisfy the equality:

$$\Sigma (\Lambda + E_N)^{-1} (\Lambda - E_N) = - (\tilde{\Lambda} - E_N) (\tilde{\Lambda} + E_N)^{-1} \Sigma \quad (2.13)$$

The validity of expression (2.13) is easily verified using the symplecticity condition for matrix Λ (2.5). Introducing the expression for the Green function $G(y, t)$ from (2.12) to the Schrödinger equation in the mixed representation and using the initial condition $G(y, 0) = 1$ we find for the function

$$K(t) = \exp \left\{ \int \left(-\frac{i\hbar}{4} \left[\Lambda p (\Sigma B \Sigma \Lambda) + \frac{1}{2} \ell \Sigma B \Sigma \ell \right] - \frac{1}{2} C \Sigma \ell \right) dt \right\} \quad (2.15)$$

where

$$A = \frac{i}{\hbar} \Sigma (\Lambda + E_N)^{-1} (\Lambda - E_N); \quad \ell = \frac{2i}{\hbar} \Sigma (\Lambda + E_N)^{-1} \Delta$$

Before calculating expressions for the Green function of concrete quantum system, let us consider an interesting particular case of the general quadratic system when the Hamiltonian of the system does not depend on time. We suppose that the matrix $B(t) = \text{const}$, and the vector $C(t) = 0$. Then equation (2.4) for the matrix A is easily integrated, and we find

$$A = \exp(\Sigma B t) \quad (2.16)$$

so that the matrix A has the form

$$A = \frac{i}{\hbar} \Sigma \text{th} \left(\frac{\Sigma B}{2} t \right) \quad (2.17)$$

and

$$K(t) = \exp \left\{ \int \frac{-i}{4} \left[\text{sp} \left(\Sigma B \text{th} \frac{\Sigma B}{2} t \right) \right] dt \right\} \quad (2.18)$$

Using the identity

$$\frac{d}{dt} \ln \det \mu = \text{sp} (\dot{\mu} \mu^{-1}) \quad (2.19)$$

we find the expression for the time-dependent function

$$K(t) = \left[\det \text{ch} \left(\frac{\Sigma B}{2} t \right) \right]^{-1/2} \quad (2.20)$$

Putting these values for the matrix A and the functi-

on $K(t)$ in expression (2.12) we can obtain the Green function of the given system in the mixed representation

$$G(y, t) = [\det \operatorname{ch}(\frac{\Sigma B}{2} t)]^{-1/2} \exp\left\{\frac{i}{\hbar} y \Sigma \operatorname{th}(\frac{\Sigma B}{2} t) y\right\} \quad (2.21)$$

The substitution $t = -i\hbar\beta$, transforms $G(y, t)$ to the equilibrium Wigner function of the system $W(y; \beta) = G(y, -i\hbar\beta)$

$$W(y; \beta) = [\det \cos(\frac{\hbar\beta\Sigma B}{2})]^{-1/2} \exp\left\{\frac{1}{\hbar} y \Sigma \operatorname{tg}(\frac{\hbar\beta\Sigma B}{2}) y\right\} \quad (2.22)$$

The partition function of the system is

$$Z(\beta) = \frac{1}{(2\pi\hbar)^N} \int W(y; \beta) dy = 2^{-N} [\det \sin(\frac{\hbar\beta\Sigma B}{2})]^{-1/2} \quad (2.23)$$

Evidently, this formula is correct only in the case $\det B \neq 0$, i.e. when the motion is finite.

Now we suppose that the matrix $B(t) = \text{const}$ and the vector $C(t) = \text{const}$. Solving equations (2.4) with respect to the matrix Λ and the vector Δ , we find

$$\begin{aligned} \Lambda &= \exp(\Sigma B t) \\ \Delta &= (\Sigma B)^{-1} [\exp(\Sigma B t) - 1] \Sigma C \end{aligned} \quad (2.24)$$

Substituting these values to the expressions (2.12) and (2.15) we have:

$$C_p(y, t) = [\det \operatorname{ch}(\frac{\mathbb{Z}B}{2}t)]^{-1/2} \exp\left\{\frac{i}{\hbar} (y + B^{-1}C) \times \right. \\ \left. \times (\mathbb{Z}t \operatorname{th} \frac{\mathbb{Z}B}{2}t)(y + B^{-1}C) + \frac{it}{2\hbar} C B^{-1}C\right\} \quad (2.25)$$

The substitution $t = -i\hbar\beta$ reduce it to the equilibrium Wigner function

$$W(y; \beta) = [\det \operatorname{coth}(\frac{\hbar\beta\mathbb{Z}B}{2})]^{-1/2} \exp\left\{\frac{1}{\hbar} (y + B^{-1}C) \times \right. \\ \left. \times [\mathbb{Z}t \operatorname{th}(\frac{\hbar\beta\mathbb{Z}B}{2})](y + B^{-1}C) + \frac{\beta}{2} C B^{-1}C\right\} \quad (2.26)$$

The partition function of the system has the form

$$\mathcal{Z}(\beta) = \mathcal{Z}_0(\beta) \exp\left(\frac{\beta}{2} C B^{-1}C\right) \quad (2.27)$$

where $\mathcal{Z}_0(\beta)$ - the partition function in the case if $C=0$, which was given by formula (2.23) (we suppose, that $\det B \neq 0$).

Formulas for the matrix $\exp(\mathbb{Z}Bt)$ in terms of the roots of the characteristic equation of matrix $\mathbb{Z}B$ are given in the appendix 1. Formulas for finding elements of the inverse block matrix of the type $(\Lambda + E_N)^{-1}$ (Frobenius formulas) are given in the appendix 2. The explicit formulas for the Green function in terms N - dimensional matrices $\lambda_j(t)$: matrix elements of matrix $\Lambda(t)$ are given in the appendix 3.

In some important cases it is possible to express the Wigner function not in terms of the functions of the $2N \times 2N$

matrix B , but in terms of the functions of $N \times N$ - dimensional matrices b_j . For example, it could be done if all matrices b_1, b_2, b_3, b_4 commute between themselves. Let us immediately give the final expression for the equilibrium Wigner function, supposing for simplicity that the vector $\vec{C}_1 = 0$

$$W(q, p; \beta) = 2^N [\det D_+(\beta)]^{-1/2} \exp \left\{ -\vec{r} \frac{b_1 \sinh(\gamma_0 \beta \hbar)}{\hbar \gamma_0 D_+(\beta)} \vec{r} - \right. \\ \left. - \vec{r} \frac{\gamma_0 D_-(\beta)}{\hbar b_1 \sinh(\gamma_0 \beta \hbar)} \vec{r} + \frac{\beta}{2} \vec{C}_2 \frac{b_4}{\gamma_0^2 + \Omega^2} \vec{C}_2 \right\} \quad (2.28)$$

(Since all matrices in the given relation commute with each other, we write simply $1/A$ for the inverse matrix A^{-1}). Here we have introduced the following notations:

$$\Omega = \frac{1}{2} (\tilde{b}_2 - b_2) = -\tilde{\Omega}; \quad \gamma_2 = \frac{1}{2} (\tilde{b}_2 + b_2); \quad \gamma_0^2 = b_4 b_1 - \gamma_2^2; \\ D_{\pm}(\beta) = \cosh(\gamma_0 \beta \hbar) \pm \cosh(\beta \hbar \Omega); \quad \vec{x} = \vec{q} + \frac{b_2}{\gamma_0^2 + \Omega^2} \vec{C}_2; \quad (2.29) \\ \vec{v} = \vec{p} + b_1^{-1} \left[\gamma_2 + \gamma_0 \frac{\sinh(\beta \hbar \Omega)}{\sinh(\gamma_0 \beta \hbar)} \right] \vec{q} + (\gamma_0^2 + \Omega^2) \left[1 - \gamma_0 \frac{\sinh(\beta \hbar \Omega)}{\sinh(\gamma_0 \beta \hbar)} \right] \vec{C}_2$$

Generally speaking, the matrix equation $\vec{x}^2 = \gamma_0^2$ has many solutions. However this does not lead to any ambiguity, because the expansion of the given Wigner function in a power series with respect of γ_0 contains only even degrees of matrix γ_0 . In reality, according to the Hamilton-Cayley theorem [18], all functions of matrices are polynomials of matrices γ_0^2 and Ω of the degrees not higher than $(N-1)$.

Note that in the classical limit $\hbar \rightarrow 0$ the vector $\vec{c}$ transforms to the usual kinetic momentum: $\vec{c} \rightarrow \vec{p} + \vec{b}_1^{-1} \vec{b}_2 \vec{q}$. For the partition function we obtain the following formula:

$$\mathcal{Z}(\beta) = 2^{-N} [\det D_-(\beta)]^{-1/2} \exp \left\{ \frac{\beta}{2} \vec{c}_2 \frac{\vec{b}_1}{\gamma_0^2 + \alpha^2} \vec{c}_2 \right\} \quad (2.30)$$

It is correct if $\det D_-(\beta) \neq 0$. If the matrix is equal to zero we obtain more simple expressions (they relates to the particular cases of the problem of a multidimensional anisotropic oscillator when matrices $\vec{b}_1$ and $\vec{b}_4$ commute; particularly, these formulas are valid if the matrix $\vec{b}_1$ (tensor of inverse masses) is proportional to the unit matrix):

$$\begin{aligned} W(q, p; \beta) = & [\det ch(\frac{1}{2} \gamma_0 \beta \hbar)]^{-1} \exp \left\{ -\vec{p} \frac{\vec{b}_1}{\hbar \gamma_0} \text{th}(\frac{1}{2} \beta \hbar \gamma_0) \vec{p} - \right. \\ & \left. - \vec{x} \frac{\vec{b}_1}{\hbar \vec{b}_4} \text{th}(\frac{1}{2} \beta \hbar \gamma_0) \vec{x} + \frac{\beta}{2} \vec{c}_2 \vec{b}_4^{-1} \vec{c}_2 \right\}; \det \vec{b}_4 \neq 0; \det \vec{b}_1 \neq 0 \quad (2.31) \\ \gamma_0^2 = & \vec{b}_4 \vec{b}_1; \quad \vec{x} = \vec{q} + \vec{b}_4^{-1} \vec{c}_2 \end{aligned}$$

$$\mathcal{Z}(\beta) = 2^{-N} [\det ch(\frac{1}{2} \beta \hbar \gamma_0)]^{-1} \exp \left\{ \frac{\beta}{2} \vec{c}_2 \vec{b}_4^{-1} \vec{c}_2 \right\} \quad (2.32)$$

The simple expression for the Wigner function can be also found in the case of Hamiltonian

$$\hat{H} = \frac{1}{2} (\hat{P} - \alpha \hat{X}) \vec{b}_1 (\hat{P} - \alpha \hat{X}) + \vec{c}_2 \hat{X}; \det \vec{b}_1 \neq 0 \quad (2.33)$$

(α -- $N \times N$ - matrix). If $N = 3$ it is the Hamiltonian of the charged particle in the electromagnetic field; the vector αX being the vector potential of the magnetic field. All matrices $\lambda_j(t)$ may be found in the explicit forms even if matrices α and β_2 do not commute. In this case matrices β_2 and β_4 are expressed in terms of α and β_1

$$\beta_2 = -\beta_1 \alpha; \quad \beta_4 = \tilde{\alpha} \beta_1 \alpha = \tilde{\beta}_2 \beta_1^{-1} \beta_2$$

It is not difficult to verify that matrices λ_j , $j = 1, 2, 3, 4$ and vectors δ_1 have the following forms

$$\begin{aligned} \lambda_1(t) &= E_N + \tilde{\beta}_2 \zeta(t); & \lambda_2(t) &= \tilde{\beta}_2 \zeta(t) \beta_1^{-1} \beta_2; \\ \lambda_3(t) &= -\beta_1 \zeta(t); & \lambda_4(t) &= E_N - \beta_1 \zeta(t) \beta_1^{-1} \beta_2; \\ \delta_1(t) &= [t E_N + \tilde{\beta}_2 X(t)] \tilde{\zeta}_2; & \delta_2(t) &= -\beta_1 X(t) \tilde{\zeta}_2 \end{aligned} \quad (2.34)$$

According to Hamilton-Cailey's theory [18], $\zeta(t)$ and $X(t)$ are polynomials of the degree $(N-1)$ of matrix $\mathcal{N}$, so that matrices $\zeta(t)$ and $X(t)$ exist in the case of degenerate matrix $\mathcal{N}$.

In the general case the expression for the Wigner function is very cumbersome, therefore we give the result only for the case when $\tilde{\zeta}_2 = 0$

$$W(q, p; \beta) = \left\{ \det [i\theta_1 \sigma(-i\beta \hbar) Q(\beta)] \right\}^{-1/2} \times \exp \left\{ -\frac{1}{2} (p - \alpha q) Q^{-1}(\beta) (p - \alpha q) \right\};$$

$$Q(\beta) = \frac{1}{2} \hbar \cotg \left(\frac{1}{2} \beta \hbar \hbar \right) \theta_1^{-1}$$
(2.36)

If $\beta \rightarrow 0$, $\sigma(-i\beta \hbar) \rightarrow -i\beta \hbar$, then the Wigner function becomes the classical equilibrium distribution function. Note, that the coordinate q and the generalized momentum p enter into the Wigner function only in the form of the kinetic momentum $p - \alpha q$. This result was obtained earlier by Levinson, but only in the special case of the 3-dimensional motion of a particle in a uniform magnetic field.

If matrices θ_1 and θ_2 commute, we can obtain the formula (2.36) from (2.28) with the aid of the limit passage $\gamma_0^2 + \alpha^2 \rightarrow 0$. Note, that in all cases when the matrix θ_1 equals to zero, the Wigner function is factorized. This means that it equals to the product of a function of coordinates only by a function of momenta only, in agreement with the result of ref. [22].

3. One - dimensional nonstationary oscillator.

Let us consider the quantum system described by the Hamiltonian

$$\hat{H} = \left(\frac{\hat{p}^2}{2m} + \frac{m\omega^2(t)}{2} \hat{x}^2 \right)$$
(3.1)

$\omega(t)$ - an arbitrary continuous function of time. For simplicity we suppose $m = 1$.

Integrals of the motion of this system are connected with operators of coordinates and momenta by relations:

$$\begin{aligned}\hat{P}_0 &= \epsilon_1(t) \hat{P} - \dot{\epsilon}_1(t) \hat{X} \\ \hat{X}_0 &= -\epsilon_2(t) \hat{P} + \dot{\epsilon}_2(t) \hat{X}\end{aligned}\quad (3.2)$$

where $\epsilon_1(t)$ and $\epsilon_2(t)$ are particular solutions of the equation of a classical harmonic oscillator

$$\ddot{\epsilon} + \omega^2(t) \epsilon = 0$$

under conditions

$$\begin{aligned}\epsilon_1(0) &= 1; & \dot{\epsilon}_1(0) &= 0; \\ \epsilon_2(0) &= 0; & \dot{\epsilon}_2(0) &= 1\end{aligned}\quad (3.3)$$

It follows from (3.2), that

$$\begin{aligned}\lambda_1 &= \epsilon_1; & \lambda_3 &= -\epsilon_1 \\ \lambda_2 &= -\dot{\epsilon}_1; & \lambda_4 &= \dot{\epsilon}_1\end{aligned}$$

Comparing the relation (3.1) with (2.1) we see, that

$$C = 0, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & \omega^2(t) \end{pmatrix}$$

Putting these values for the matrix B , vector C and for matrices λ in (2.12) and (2.15) we obtain the Green function in the mixed representation for the one-dimensional quantum oscillator in the absence of an external force:

$$G(q, p; t) = \exp \left\{ \int \left[\frac{-1}{4 \det(\Lambda + E)} (\dot{w}^2(t) a_1 + a_4) \right] dt \right\} \times \quad (3.4)$$

$$\times \exp \left\{ \frac{i}{\hbar \det(\Lambda + E)} [p a_2 p + p a_3 q + q a_3 p + q a_4 q] \right\}$$

where a_1 , a_2 , a_3 , a_4 are as follows

$$a_1 = \epsilon_2 (\epsilon_1 - 1) - \epsilon_2 (\epsilon_1 + 1)$$

$$a_2 = (\epsilon_1 + 1)(\dot{\epsilon}_2 - 1) - \epsilon_2 \dot{\epsilon}_1$$

$$a_3 = -(\dot{\epsilon}_2 + 1)(\epsilon_1 - 1) + \dot{\epsilon}_1 \epsilon_2$$

$$a_4 = \dot{\epsilon}_1 (\dot{\epsilon}_2 + 1) - \dot{\epsilon}_1 (\dot{\epsilon}_2 - 1)$$

If the Hamiltonian does not depend on time, i.e. $\omega(t) = \text{const} = \omega$, then the Green function of the Harmonic oscillator in the mixed representation will be

$$G(q, p; t) = \left(\cos \frac{\omega t}{2} \right)^{-1} \exp \left\{ \frac{-i}{\hbar \omega} \left[t q \left(\frac{\omega^2}{2} \right) (p^2 + \omega^2 q^2) \right] \right\} \quad (3.5)$$

4. Charged particle in a variable magnetic field.

Let us consider the particle with the charge e and the mass m in the classical magnetic field with the potential

$$A(\vec{r}, t) = \frac{1}{2} [\vec{H}(t), \vec{r}]$$

(4.1)

where $\vec{H}(t)$ - the strength of the magnetic field. We choose the axis $\vec{x}$ along the magnetic field, i.e. $A_x = 0$. The Hamiltonian of such a system is

$$\hat{H} = \frac{1}{2} \left[(\hat{p}_x - \frac{e}{c} A_x)^2 + (\hat{p}_y - \frac{e}{c} A_y)^2 \right] \quad (4.2)$$

Integrals of the motion of this system are connected with operators of coordinates and momenta by the relations of the form (2.3), where matrices λ_k have the form:

$$\lambda_1 = \epsilon_1(t) A; \quad \lambda_3 = -\epsilon_2(t) A;$$

$$\lambda_2 = -\dot{\epsilon}_1(t) A; \quad \lambda_4 = \dot{\epsilon}_2(t) A;$$

$$A = \begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix}; \quad \theta = \frac{1}{2} \int^t \omega(\tau) d\tau;$$

$$\omega(t) = \frac{e H(t)}{2mc}$$

where $\epsilon_1(t)$ and $\epsilon_2(t)$ - particular solutions of the equation $\ddot{\epsilon}(t) + \frac{\omega^2(t)}{4} \epsilon(t) = 0$ under conditions (3.3).

Then the Green function of this system in the mixed representation can be written as follows,

$$G(q, p; t) = 4 (\dot{\epsilon}_2 + 2 \cos \theta + \epsilon_2)^{-1} \times \\ \times \exp \left\{ -\frac{i}{\hbar} \left[\frac{1}{\epsilon_2} (q_1^2 + q_2^2) (\epsilon_1^2 + \epsilon_2^2 - 2 \cos \theta) + 2 \int (\dot{\epsilon}_2 - \epsilon_1) q_2 + \right. \right. \quad (4.3)$$

$$+ 2 \sin \theta q_2] P_2 + [(\dot{E}_2 - E_2) q_2 - 2 \sin \theta q_2] P_2 + 2 E_2 (P_1^2 + P_2^2) \Big] (\dot{E}_2 + 2 \cos \theta + E_2)^{-1} \Big\}$$

In particular case $H = \text{const}$ this function becomes (in this case $E_1 = \cos \frac{\omega t}{2}$; $E_2 = \frac{2}{\omega} \sin \frac{\omega t}{2}$; $\theta = \frac{1}{2} \omega t$)

$$G(q, p; t) = \left(\cos \frac{\omega t}{2} \right)^{-1} \exp \left\{ \frac{-i}{\hbar m \omega} \tan \left(\frac{\omega t}{2} \right) \left(\vec{p} - \frac{e}{c} \vec{A} \right)^2 \right\} \quad (4.4)$$

Now let us consider a particle with a charge e and an arbitrary quadratic dispersion law $E(p) = \frac{1}{2} (p_x^2/m_x + p_y^2/m_y + p_z^2/m_z)$, which moves in the homogeneous constant magnetic field H and electrical field E . The matrix β_x is diagonal: $\beta_x = \text{diag} (m_x^{-1}, m_y^{-1}, m_z^{-1})$. The matrix α is antisymmetrical (if the vector potential chosen as $A = \frac{1}{2} [H \times q]$). The vector C_1 equals zero and $C_2 = eE$. Using results of previous parts and the appendix A3 we can find the following formula for the equilibrium Wigner function

$$W(q, p, \beta) = \left[\text{ch} \left(\frac{1}{2} \hbar \omega \beta \right) \right]^{-1} \exp \left\{ \vec{q} \cdot \vec{p}(\beta) \cdot \vec{q} + \beta e \vec{E} \cdot \vec{q} + \right. \\ \left. + \frac{1}{2} \beta e^2 \left[\gamma(\beta) \vec{E} \beta_x \vec{E} - \frac{1}{\omega^2} \left(\gamma(\beta) - \frac{1}{12} \beta^2 \hbar^2 \right) \frac{e^2 (\vec{E} \cdot \vec{H})^2}{c^2 m_x m_y m_z} \right] \right\};$$

$$\omega^2 = \frac{e^2}{c^2} \left[\frac{H_x^2}{m_y m_z} + \frac{H_y^2}{m_x m_z} + \frac{H_z^2}{m_x m_y} \right] \quad (4.5)$$

$$\gamma(\beta) = \frac{1}{\omega^2} \left[\frac{1}{2} \hbar \omega \beta \operatorname{cth} \left(\frac{1}{2} \hbar \omega \beta \right) - 1 \right];$$

$$\vec{v} = \vec{p} - \frac{e}{2c} [\vec{H} \times \vec{q}] + \frac{e^2 \gamma(\beta)}{c} [(\vec{E}_z \cdot \vec{E}) \times \vec{H}]$$

The elements of the symmetrical matrix 3x3 $\eta(\beta)$ are

$$\eta_{xx} = \frac{\operatorname{th} \left(\frac{1}{2} \hbar \omega \beta \right)}{m_x m_y m_z \hbar \omega} \left(m_y m_z + \gamma(\beta) \frac{e^2}{c^2} H_x^2 \right) \quad (4.6)$$

$$\eta_{xy} = \frac{\operatorname{th} \left(\frac{1}{2} \hbar \omega \beta \right)}{m_x m_y m_z \hbar \omega} \gamma(\beta) \frac{e^2}{c^2} H_x H_y$$

Other elements η_{jk} are obtained from η_{xx} and η_{xy} by means of evident permutations of indices. If the electrical field is absent, the partition functions equals

$$\mathcal{Z}(\beta) = \left(\frac{m_x m_y m_z}{(2\pi)^3 \beta} \right)^{3/2} \frac{\omega V}{2\hbar^2 \operatorname{sh} \left(\frac{1}{2} \hbar \omega \beta \right)} \quad (4.7)$$

where V - the volume of the vessel containing the particle. This formula was obtained for the first time in ref. [19] by summing over energy levels.

5. Average meanings.

In conclusion we want to give formulas for average meanings of coordinates and momenta of the system with quadratic time-independent Hamiltonian.

To obtain these formulas we calculate first the average meaning of the function e^{yX} , where X - an arbitrary auxiliary vector, and $y = \begin{pmatrix} p \\ q \end{pmatrix}$.

From the definition we have:

$$\langle e^{yX} \rangle = \frac{1}{(2\pi\hbar)^N} \int W(y; \beta) e^{yX} dy$$

Let us analyse the case when C equal zero. Using $W(y; \beta)$ from relation (2.22), we have

$$\langle e^{yX} \rangle = 2^{-N} \left[\det \sinh \left(\frac{\hbar \beta \Sigma B}{2} \right) \right]^{-1/2} \exp \left\{ X \left[\frac{-i}{\hbar} \Sigma \operatorname{tg} \left(\frac{\hbar \beta \Sigma B}{2} \right) \right] X / 4 \right\} \quad (5.1)$$

Expanding it in a power series with respect of X we find

$$\langle e^{yX} \rangle = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=1}^n X_{a_1} X_{a_2} \dots X_{a_n} \langle y_{a_1} y_{a_2} \dots y_{a_n} \rangle \quad (5.2)$$

Let us expand the exponent of the right-hand part of eq.(5.1), introducing the designation

$$\left[-\frac{4}{\hbar} \Sigma \operatorname{tg} \left(\frac{\hbar \beta \Sigma B}{2} \right) \right]^{-1} = A$$

(5.3)

$$e^{X_a A_{a\beta} X_\beta} = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{j=1}^{2m} X_{a_1} X_{a_2} \dots X_{a_j} A_{a_1 a_2} A_{a_3 a_4} \dots A_{a_{j-1} a_j}$$

Equating coefficients of equal degrees of X in expressions (5.2) and (5.3) and taking into account (5.1) we

find

$$\langle y_{d_1} y_{d_2} \dots y_{d_m} \rangle = 2^{-N} \left[\det \sin \left(\frac{\hbar B \pm B}{2} \right) \right]^{-1/2} \times$$

$$\times [C_{d_1, d_2, \dots, d_m}]^{-1} \sum_{\substack{d_1, d_2, \dots, d_m \\ \beta_1, \beta_2, \dots, \beta_m}} A_{\beta_1 \beta_2} A_{\beta_2 \beta_3} \dots A_{\beta_{m-1} \beta_m} \quad (5.4)$$

$\sum_{\substack{d_1, d_2, \dots, d_m \\ \beta_1, \beta_2, \dots, \beta_m}}$ means the sum over all possible pair combinations of elements $d_1, d_2, \dots, d_m$. Since the substitution $\beta_1 \leftrightarrow \beta_2$ etc. does not change anything we divide this sum by the number of different pair combinations of elements $d_1, d_2, \dots, d_m$, which we designate by the symbol $C_{d_1, d_2, \dots, d_m}^2$.

In the general case ($c \neq 0$) we use for $W(y; \beta)$ the expression (2.26); then

$$\langle e^{yx} \rangle = 2^{-N} \left[\det \sin \left(\frac{\hbar B \pm B}{2} \right) \right]^{-1/2} \exp \left\{ \frac{\beta}{2} C B^{-1} C + \right.$$

$$+ \frac{1}{\hbar} C B^{-1} \sum \text{tg} \left(\frac{\hbar B \pm B}{2} \right) B^{-1} C \left. \right\} \exp \left\{ -\frac{1}{4} \left[\frac{2}{\hbar} \sum \text{tg} \left(\frac{\hbar B \pm B}{2} \right) B^{-1} C + \right. \right.$$

$$\left. \left. + X \right] \left[\frac{1}{\hbar} \sum \text{tg} \left(\frac{\hbar B \pm B}{2} \right) \right]^{-1} \left[\frac{2}{\hbar} \sum \text{tg} \left(\frac{\hbar B \pm B}{2} \right) B^{-1} C + X \right] \right\} \quad (5.5)$$

We write the expansion of the function e^{yx} in the power series in the following form:

$$\langle e^{yx} \rangle = \left\langle \sum_{h=0}^{\infty} \frac{(y_d x_d)^h}{h!} \right\rangle = \sum_{h_1, \dots, h_m} \frac{x_1^{h_1} x_2^{h_2} \dots x_m^{h_m}}{h_1! h_2! \dots h_m!} \langle y_1^{h_1} y_2^{h_2} \dots y_m^{h_m} \rangle \quad (5.6)$$

Using the definition of the Hermite polynomials for several variables 20

$$\exp\left\{\frac{1}{2}\vec{a}c\vec{a} + \vec{\lambda}\vec{a}\right\} = \sum_{n_1 \dots n_m} \frac{a_1^{n_1} \dots a_m^{n_m}}{n_1! n_2! \dots n_m!} H_{n_1 \dots n_m}^c(\vec{c}\vec{\lambda}) \quad (5.7)$$

and designating for simplicity

$$\frac{2}{\hbar} \sum \text{tg}\left(\frac{\hbar \beta \pm B}{2}\right) B^{-\frac{1}{2}} c = b$$

$$\frac{2}{\hbar} \sum \text{tg}\left(\frac{\hbar \beta \pm B}{2}\right) = d = \tilde{d}$$

we expand the right-hand part of the equality (5.5) in the power series:

$$\begin{aligned} & \exp\left\{-\frac{1}{2}(b+x)d^{-\frac{1}{2}}(b+x)\right\} = \\ & = \exp\left\{-\frac{1}{2}bd^{-\frac{1}{2}}\right\} \sum_{n_1 \dots n_m}^{\infty} \frac{x_1^{n_1} x_2^{n_2} \dots x_m^{n_m}}{n_1! n_2! \dots n_m!} H_{n_1 \dots n_m}(-d^{-\frac{1}{2}}b) \end{aligned} \quad (5.8)$$

Equating the coefficients of equal degrees of X in expressions (5.6) and (5.8) and taking into account (5.5), we obtain

$$\begin{aligned} \langle y_1^{n_1} y_2^{n_2} \dots y_m^{n_m} \rangle &= 2^{-N} [\det \sin\left(\frac{\hbar \beta \pm B}{2}\right)]^{-\frac{1}{2}} \exp\left\{\frac{\beta}{2} c B^{-\frac{1}{2}} c + \right. \\ & \left. + \frac{1}{\hbar} c B^{-\frac{1}{2}} \sum \text{tg}\left(\frac{\hbar \beta \pm B}{2}\right) B^{-\frac{1}{2}} c - \frac{1}{2} b d^{-\frac{1}{2}} b\right\} H_{n_1 \dots n_m}(d^{-\frac{1}{2}}b) \end{aligned} \quad (5.9)$$

6. Conclusion.

The method of integrals of the motion allows not only to calculate efficiently the equilibrium Wigner functions of quadratic systems. It allows also to calculate the evolution of the Wigner functions of systems which obey arbitrary linear equations of motion. The evolution of the Wigner functions for some special quadratic nonstationary systems was calculated in ref. [23]. The evolution of the density matrix of an arbitrary multidimensional nonstationary quadratic system in the coordinate representation was investigated in ref. [24]. In the next papers we intend to analyse the evolution of the Wigner functions of arbitrary systems with the linear equation of the motion (including dissipative systems), following outlines of ref. [25], and apply the method of integrals of the motion for obtaining the quantum corrections to the equilibrium distribution functions for nonquadratic systems (see, e.g., [26]).

Appendix I.

Let us obtain, following ref. [18], explicit formulae for the matrix $A = \exp(\Sigma B t)$ in terms of the roots of the characteristic polynomial of matrix $A = \Sigma B$:

$$\mathcal{D}(\lambda) = \det(\lambda E_{2N} - A)$$

It follows from the symmetry of matrix B and the antisymmetry of matrix Σ that $\mathcal{D}(\lambda)$ is the even function:

$$\begin{aligned} \mathcal{D}(-\lambda) &= \det(-\lambda E_{2N} - A) = \det(\lambda \Sigma^2 - \Sigma B) = \\ &= \det[\Sigma (\lambda E_{2N} + B \Sigma) \Sigma] = \det(\lambda E_{2N} - \tilde{A}) = \det(\lambda E_{2N} - A) = \mathcal{D}(\lambda) \end{aligned}$$

(We use the relation $\det \Sigma = 1$, following from the equality $\tilde{\Sigma} = -\Sigma$).

Consequently, $\mathcal{D}(\lambda)$ has the form

$$\mathcal{D}(\lambda) = \lambda^{2N} + p_2 \lambda^2 + \dots + p_{2N-2} \lambda^2 + p_{2N} \quad (\text{AI.1})$$

so that all roots of the characteristic equation $\mathcal{D}(\lambda) = 0$ may be divided into pairs: $\pm \lambda_1, \pm \lambda_2, \dots, \pm \lambda_N$. If all roots are different, then

$$\begin{aligned} \exp(A t) &= \sum_{k=1}^N \left\{ \left[\prod_{\substack{j=1 \\ j \neq k}}^N (\lambda_k^2 - \lambda_j^2) \right]^{-1} \left[L(\lambda_k^2) \operatorname{ch}(\lambda_k t) + \right. \right. \\ &\left. \left. + A L(\lambda_k^2) \frac{\operatorname{sh}(\lambda_k t)}{\lambda_k} \right] \right\} \quad (\text{A.1.2}) \end{aligned}$$

where the matrix $L(\tilde{x})$ is defined by the following relations:

$$L(\tilde{x}) = E_{2N} \tilde{x}^{N-1} + L_1 \tilde{x}^{N-2} + \dots + L_{N-2} \tilde{x} + L_{N-1} \quad (A.1.3)$$

$$L_j = A + P_2 A^{(j-1)} + P_4 A^{(j-2)} + \dots + P_j E_{2N}; \quad j = 1, 2, \dots, N-1$$

In the case of multiple roots the formulas may be found from (A.1.2) by a limit passage.

Appendix.2.

If M is the block matrix, i.e.

$$M = \begin{pmatrix} \underbrace{A}_{n \times n} & \underbrace{B}_{n \times q} \\ \underbrace{C}_{q \times n} & \underbrace{D}_{q \times q} \end{pmatrix}$$

and $|M| \neq 0$, $|A| \neq 0$, then inverse matrix M^{-1} can be found by means of the Frobenius formulas [18]

$$M^{-1} = \begin{pmatrix} A^{-1} + A^{-1} B H^{-1} C A^{-1} & -A^{-1} B H^{-1} \\ -H^{-1} C A^{-1} & H^{-1} \end{pmatrix}$$

where

$$H = D - C A^{-1} B$$

If $|D| \neq 0$, then

$$M^{-1} = \begin{pmatrix} K^{-1} & K^{-1} B D^{-1} \\ -D^{-1} C K^{-1} & D^{-1} + D^{-1} C K^{-1} B D^{-1} \end{pmatrix}$$

where

$$K = A - B D^{-1} C$$

Appendix 3.

Besides the identities (2.5) which are satisfied by the matrix $\Lambda(t)$ and its elements for an arbitrary nonstationary (symmetrical) matrix $B(t)$, in the case when the matrix B does not depend on time certain additional identities following from the relation $\Lambda(t) = \exp(\pm Bt)$ exist:

$$\Lambda(-t) = \Lambda^{-1}(t) = \Sigma \tilde{\Lambda}(t) \Sigma^{-1}$$

$$\lambda_1(-t) = \lambda_4(t); \quad \tilde{\lambda}_{2,3}(-t) = -\lambda_{2,3}(t) \quad (A.3.1)$$

$$\delta_1(t) = \tilde{\lambda}_3(t) \delta_2(-t) - \lambda_3^{-1}(-t) \delta_2(t)$$

$$\delta_2(t) = \lambda_4(t) \tilde{\lambda}_2^{-1}(t) \delta_1(t) + \tilde{\lambda}_2^{-1}(t) \delta_1(t)$$

Let us introduce matrices $\mu_j(\beta) = \lambda_j(-i\beta t)$ and vectors $\tilde{f}_j(\beta) = \delta_j(-i\beta t)$.

The following identities are the consequences of relations (A.3.1) and equations (2.4):

$$\mu_j(-\beta) = \mu_j(\beta); \quad \tilde{f}_K^*(-\beta) = \tilde{f}_K(\beta); \quad j=1,2,3,4; \quad K=1,2 \quad (\text{A.3.2})$$

$$\mu_1^+(\beta) = \mu_4(\beta); \quad \mu_{2,3}^+(\beta) = -\mu_{2,3}(\beta)$$

(an asterisk means the complex conjugation, the cross - the hermitian conjugation). Expanding the matrix $\mu_j(\beta)$ in a power series with respect of parameter β . It is not difficult to convince in the following symmetry properties with respect to the change of the sign of matrix $\mathcal{C}_2$ (in the three-dimensional case this corresponds to the change of the sign of the magnetic field):

$$\mu_{1,4}(-\beta, \mathcal{C}_2) = \mu_{1,4}(\beta, -\mathcal{C}_2) \quad (\text{A.3.3})$$

$$\mu_{2,3}(-\beta, \mathcal{C}_2) = -\mu_{2,3}(\beta, -\mathcal{C}_2)$$

Particularly, if $\mathcal{C}_2 = 0$, the matrices μ_1 and μ_4 are even functions of the parameter β and matrices μ_2 and μ_3 - odd functions. Then it follows from (A.3.2), that for $\mathcal{C}_2 = 0$ the matrices μ_1 and μ_4 are real and μ_2 and μ_3 are imaginary.

On the base of these identities the equilibrium Wigner function may be expressed in terms of matrices $\mu_3(\beta)$ and $\mu_4(\beta)$ and vector $\tilde{f}_2(\beta)$ only:

$$W(q, p; \beta) = \{ \det [2ip(\beta) \mu_3^{-1}(\beta)] \}^{-1/2} \exp \left\{ \frac{i}{4} [\tilde{v}p(\beta) \tilde{v} - \tilde{q}a(\beta) \tilde{q}] \right\} + \quad (\text{A.3.4})$$

$$+ 2\vec{\Phi} \operatorname{Im} [\mu_3^{-1}(\beta) \vec{F}_2(\beta)] + \frac{i}{2} \vec{F}_2 \mu_3^{-1} \mu_4 \vec{F}_2 + i \int_0^\beta \vec{F}_2(\beta') \frac{d\vec{F}_2(\beta')}{d\beta'} d\beta'] \Big\};$$

$$Q(\beta) = -\{ \operatorname{Im} (\mu_3^{-1}(\beta) [E_N + \mu_4(\beta)]) \}^{-1};$$

$$Q(\beta) = \operatorname{Im} (\mu_3^{-1}(\beta) [E_N - \mu_4(\beta)]);$$

$$\vec{v} = \vec{p} + \operatorname{Re} (\mu_3^{-1}(\beta) [\mu_4(\beta) - E_N]) \vec{\Phi} + \operatorname{Re} [\mu_3^{-1}(\beta) \vec{F}_2(\beta)]$$

(The vector $\vec{F}_1(\beta)$ can be expressed in terms of $\vec{F}_2$, μ_3 and μ_4 by means of formulas (A.3.1)).

Usually the matrix b_4 is nonsingular (all masses are finite). If the matrix B is not degenerate too, then the formula

$$\Delta(t) = [\Lambda(t) - E_{2N}] B^{-1} C \quad (\text{A.3.5})$$

together with the Frobenius formulas lead to the following explicit expression for the vector $\vec{F}_2(\beta)$

$$\vec{F}_2(\beta) = \chi_1(\beta) C_1 + \chi_2(\beta) C_2;$$

$$\chi_1(\beta) = \mu_3(\beta) b_1^{-1} (E_N + b_2 F^{-1} \tilde{b}_2 b_2^{-1}) - (\mu_4(\beta) - E_N) F \tilde{b}_2 b_1^{-1}; \quad (\text{A.3.6})$$

$$\chi_2(\beta) = -\mu_3(\beta) b_1^{-1} b_2 F^{-1} + (\mu_4(\beta) - E_N) F^{-1};$$

$$F = b_4 - \tilde{b}_2 b_1^{-1} b_2; \quad \det B = \det b_1 \cdot \det F \neq 0$$

In the special case $b_2 = 0$, when $\mu_3(\beta)$ is pure imaginary and odd matrix, while $\mu_4(\beta)$ is real and even

matrix, the formula (A.3.4) can be even more simplified:

$$W(q, p; \rho) = \left\{ \det [2(\mu_V(\rho) + E_V)]^{-1/2} \exp \left\{ \frac{i}{\hbar} P(\mu_V + E_N)^{-1} M_2(\rho) P + \right. \right. \\ \left. \left. + \frac{i}{\hbar} (q + \epsilon_V^{-1} \epsilon_2) M_2^{-1} (E_N - \mu_V) (q + \epsilon_V^{-1} \epsilon_2) + \frac{1}{2} Q \epsilon_V^{-1} \epsilon_2 \right\} \right. \\ \left. \det \epsilon_V \neq 0 \right\} \quad (A.3.7)$$

$$\mathcal{Z}(\rho) = 2^{-N/2} \left\{ \det [\mu_V(\rho) - E_N] \right\}^{-1/2} \exp \left(\frac{1}{2} \epsilon_2 \epsilon_V^{-1} \epsilon_2 \right). \quad (A.3.8)$$

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