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INSTANTONS IN SUPERSYMMETRIC QUANTUM MECHANICS

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A B S T R A C T

The rôle of classical solutions, i.e. instantons, in a simple supersymmetric theory, supersymmetric quantum mechanics, is discussed. It is explained how instantons can be responsible for supersymmetry breaking. It turns out that an interesting reformulation of the theory exists, in terms of only one bosonic field that describes the instantons. Except for the complicated (non-local) boundary conditions, the path integral becomes that of a free field theory.

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1. INTRODUCTION

The rôle of classical solutions in supersymmetric field theories is a very intriguing one. A few phenomena in supersymmetric theories which bear witness of this are the following.

- The generation of central charges in the supersymmetry algebra from topological charges connected with solitons and monopoles, as in the two-dimensional Wess-Zumino model and the four-dimensional $N = 4$ supersymmetric Yang-Mills theory¹⁾.
- The breaking of supersymmetry. Instanton-effects can lead to supersymmetry breaking, because instantons may connect fermionic states in one topological sector of the theory to bosonic states in another one^{2),3)}.
- The quantum corrections to the mass of the particles corresponding to classical extended objects seem to vanish in various cases, indicating the almost unchanged persistence of these objects in the quantum theory. This takes place, e.g., for the solitons, in the three-dimensional CP^{N-1} model⁴⁾.
- Finally, a more speculative possibility is that the well-known vanishing of the β function of $N = 4$ Yang-Mills theory results from the theory describing a gas of non-interacting monopoles. Indications of a similar behaviour in the much simpler model of supersymmetric quantum mechanics are discussed below.

In order to study such phenomena related to classical solutions, it is convenient to study first a simple model for which I take supersymmetric quantum mechanics. This is equivalent to a (0+1) dimensional relativistic field theory. Depending on the choice of the potential, such a model may admit instanton solutions, i.e., finite action solutions to the imaginary time field equations.

In the following, I will first present the model, discuss its supersymmetry and the existence of instanton solutions. Subsequently I will exhibit the rôle of instantons in the breaking of supersymmetry. Finally, I will show that it is possible to reformulate the theory in terms of one bosonic field, corresponding to the instantons of the original theory, such that the path integral becomes a pure Gaussian. However, this field satisfies unusual boundary conditions.

2. N = 2 SUPERSYMMETRIC QUANTUM MECHANICS

Supersymmetric quantum mechanics of a particle with one bosonic and one fermionic degree of freedom, q and ψ , is defined by the Lagrangian^{2),3)}:

$$\mathcal{L} = \frac{1}{2} \left(\frac{\partial}{\partial t} q \right)^2 - \frac{1}{2} V'(q)^2 + \frac{1}{2} \bar{\psi} (i\sigma_2 \frac{\partial}{\partial t} + V''(q)) \psi . \quad (2.1)$$

Here V' and V'' are the derivatives with respect to q of a function $V(q)$ called the superpotential. The two real components $\psi_{1,2}$ of the anticommuting fermion variable together form a one-component complex spinor:

$$\psi_{\pm} = \frac{1}{\sqrt{2}} (\psi_1 \pm i\psi_2) . \quad (2.2)$$

Equivalently, the theory may be described in terms of the Hamiltonian operator:

$$\hat{H} = \frac{1}{2} \hat{p}^2 + \frac{1}{2} V'(\hat{q})^2 - \frac{1}{2} V''(\hat{q}) [\hat{\psi}_+, \hat{\psi}_-] \quad (2.3)$$

supplemented by the canonical (anti-) commutation relations:

$$[\hat{p}, \hat{q}] = -i, \quad \{\hat{\psi}_+, \hat{\psi}_-\} = 1, \quad \hat{\psi}_{\pm}^2 = 0 . \quad (2.4)$$

This commutator algebra can, as is well known, be realized by the operator representations

$$\begin{aligned} \hat{p} &= -i \frac{\partial}{\partial q}, & \text{and} & & \hat{\psi}_+ &= \frac{\partial}{\partial \zeta}, \\ \hat{q} &= q, & & & \hat{\psi}_- &= \zeta, \end{aligned} \quad (2.5)$$

where ζ is an anticommuting c number. A second and sometimes more convenient way to represent the fermionic operators, is by the 2×2 matrix representation

$$\hat{\psi}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{\psi}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (2.6)$$

The theory defined here possesses an $N = 2$ supersymmetry generated by two charges $\hat{Q}_\pm$:

$$\hat{Q}_\pm = (\hat{p} \pm i V'(\hat{q})) \hat{\psi}_\pm, \quad (2.7)$$

which satisfy the algebraic relations

$$\{\hat{Q}_+, \hat{Q}_-\} = 2\hat{H}, \quad [\hat{H}, \hat{Q}_\pm] = 0, \quad \hat{Q}_\pm^2 = 0. \quad (2.8)$$

This is the Poincaré supersymmetry algebra in $(0+1)$ dimensions. It is easy to verify that the $\hat{Q}_\pm$ transform the bosonic and fermionic variables into each other. Besides the supercharges, there is one other conserved quantity in the theory, which is the fermion number

$$\hat{n} = \hat{\psi}_+ \hat{\psi}_-, \quad [\hat{H}, \hat{n}] = 0. \quad (2.9)$$

This operator has eigenvalues 0 and 1, as is easy to see in the matrix representation (2.6). Of course, this is nothing but the expression of the Pauli principle in this model.

3. THE SUPERPOTENTIAL AND INSTANTONS

The dynamics of the discussed theories depends on one specific function of the bosonic variable q , the superpotential $V(q)$. The interesting properties of the model turn out to depend only on the general form of this potential: its number of minima and maxima and its behaviour at infinite q . The two most interesting cases for illustrative purposes are the supersymmetric harmonic oscillator and supersymmetric q^4 theory (Fig. 1). In the first case, the superpotential and its derivatives are:

Fig. 1. The potential of q^4 theory

$$V(q) = \frac{1}{2} \omega q^2, \quad \frac{1}{2} V'(q)^2 = \frac{1}{2} \omega^2 q^2, \quad V''(q) = \omega. \quad (3.1)$$

There is no coupling between the bosons and the fermions, supersymmetry is unbroken, and the spectrum consists of two disjoint sets of harmonic oscillator states, distinguished by their fermionic quantum number $(0,1)$ and a relative shift ω only.

In the second case the superpotential is cubic:

$$V(q) = \frac{\lambda}{3} q^3 - \mu^2 q, \quad \frac{1}{2} V'(q)^2 = \frac{\lambda^2}{2} \left(q^2 - \frac{\mu^2}{\lambda} \right)^2, \quad (3.2)$$

$$V''(q) = 2\lambda q.$$

The bosonic potential is of the familiar double-well type, with two classical minima at $q_{\pm} = \pm(\mu/\sqrt{\lambda})$, while there is a Yukawa-type coupling of q to the fermions. As is well known, in theories of this type the spectrum of states is decisively influenced by the possibility of tunnelling of the particle between the minima of the potential. In the classical theory such a behaviour is signalled by the existence of instanton solutions to the imaginary time field equations^{5),6)}. Integrating the field equations once, and looking for zero-fermion solutions, we find that the instantons satisfy

$$\begin{cases} \frac{\partial}{\partial \tau} q_c \pm V'(q_c) = 0, \\ \psi_c = 0. \end{cases} \quad \tau = it = \mp \int^{q(t)} \frac{dq}{V'(q)} \quad (3.3)$$

Fig. 2. Instanton solution of q^4 theory

As is clear from Fig. 2, this solution interpolates between the two classical minima of the bosonic potential, as expected.

4. SUPERSYMMETRY BREAKING

Before continuing, let me briefly discuss the connection between the matrix representation and the anticommuting c number representation of the fermions. It is given by the following:

- in the matrix representation, the wave function has two components:

$$\Psi(q) = \begin{pmatrix} \psi_1(q) \\ \psi_2(q) \end{pmatrix}.$$

The upper- and lower-component describe one- and zero-fermion states, respectively;

- in the anticommuting c number representation, the wave function depends on the two variables (q, ζ) ; expanding in terms of the anticommuting variable ζ gives:

$$\Psi(q, \zeta) = a_1(q) + \zeta a_2(q).$$

In this representation it is a_1 (a_2) which corresponds to the one- (zero-) fermion component, and hence we may identify it with ψ_1 (ψ_2) above.

After having established this simple connection, I next write the Hamiltonian in matrix notation as:

$$\hat{H} = -\frac{1}{2} \frac{\partial}{\partial q^2} \mathbb{1} + U(q) \quad (4.1)$$

Fig. 3. Graphical representation of the potential $U(q)$

where the potential $U(q)$ has the matrix form:

$$U(q) = \frac{1}{2} V'(q)^2 \mathbb{1} + V''(q) \sigma_3 \quad (4.2)$$

as indicated in Fig. 3. Note that the Hamiltonian is diagonal with respect to fermion number. The effect of the fermions is to lift the degeneracy between the two wells of the bosonic potential. The split in energy between the wells is $\Delta\epsilon = V''(q_+)$. However, except for a reflection, no difference exists between the one- and zero-fermion potentials, hence the energy spectrum of the $n = 0, 1$ states will be degenerate. Therefore one might expect that it would be possible to change the fermion number in the ground state of the system without adding any energy to it. This would correspond to the existence of a zero-mass fermion in a field theory.

There is only one objection to this: the ground states of the $n = 0, 1$ spectra are localized in different wells of the potential. In field theory language: they correspond to different vacuum-expectation values of the field q . Hence, as long as the particle is in the lowest energy state, fermion number cannot be changed, unless the particle simultaneously tunnels to the other well of the potential. It is exactly at this point that the instantons become crucial: they mediate the tunnelling, and hence assist in the possibility of creating zero-mass fermions. At the same time they make the transformation of one vacuum state into the other possible one, whence the vacua are no longer invariant under supersymmetry. As a result, supersymmetry is broken, and the zero-mass fermions actually become the Goldstone fermions corresponding to this broken supersymmetry.

As is well known, in a broken supersymmetry phase the ground state energy of the system must be different from zero. Thus for the consistency of our explanation, the instantons must make a finite contribution to the ground state energy. On the other hand, this

additional energy may not spoil the picture of states localized in the lowest well of the potential, which results from the energy splitting between the two originally degenerate minima. This instanton contribution to the energy has been calculated to be³⁾:

$$\epsilon_0 = \frac{V''(q_+)}{\pi} e^{-2\Delta V} \ll \Delta\epsilon \quad (4.3)$$

This fulfils all the requirements.

5. REFORMULATION OF THE THEORY

After this discussion of the rôle of instantons in supersymmetry breaking, I turn now to an entirely different aspect of the supersymmetric models under consideration. The object is to rewrite the path integral of these theories in a very simple way by integrating over the fermionic variables, and then making a transformation of the bosonic variables such that its Jacobian cancels the fermion determinant.

Take the path integral

$$K(q, \zeta | q', \zeta') = \int_{(q', \zeta')}^{(q, \zeta)} Dq D\psi e^{i \int_t^t \mathcal{L}(q, \psi) dt} \quad (5.1)$$

where the functional integral extends over all trajectories between the points (q', ζ') , (q, ζ) of superspace. It was shown in Ref. 3) that the fermionic integral can be completely performed, leading to the result (in imaginary time):

$$K_E(q, \zeta | q', \zeta') = \int_{q'}^q Dq \left\{ \zeta e^{\int \mathcal{L}_E^{(-)} d\tau} - \zeta' e^{\int \mathcal{L}_E^{(+)} d\tau} \right\} \quad (5.2)$$

where $\mathcal{L}_E^{(\pm)}$ is the zero/one-fermion part of the Euclidean Lagrangian:

$$\mathcal{L}_E^{(\pm)} = -\frac{1}{2} \left(\frac{\partial}{\partial \tau} q \right)^2 - \frac{1}{2} V'(q)^2 \pm \frac{1}{2} V''(q) \quad (5.3)$$

Now we define the new variable

$$X^\pm(\tau) = q(\tau) \mp \int_\tau^{\tau_0} V'(q) d\tau \quad (5.4)$$

where τ_0 is a fixed endpoint of integration. The Jacobian of this transformation may be found by discretizing time, calculating the resulting finite dimensional determinant, and only then taking the continuum limit. This way, the following result is established:

$$\det \left(\frac{\partial X^\pm}{\partial q} \right) = e^{\pm \int_{\tau}^{\tau_0} V''(q) d\tau} . \quad (5.5)$$

Moreover, the action in terms of this new variable, calculated from (5.3), is simply:

$$S_E^{(\pm)} = \pm \Delta V - \frac{1}{2} \int_{\tau}^{\tau_0} \left(\frac{\partial}{\partial \tau} X^\pm \right)^2 d\tau \quad (5.6)$$

whence we find for the path integral:

$$\begin{aligned} K_E(q\zeta\tau | q'\zeta'\tau') &= e^{-\Delta V_{\zeta}} \int DX^- e^{-\frac{1}{2} \int (\dot{X}^-)^2 d\tau} \\ &\quad - e^{\Delta V_{\zeta'}} \int DX^+ e^{-\frac{1}{2} \int (\dot{X}^+)^2 d\tau} \end{aligned} \quad (5.7)$$

The boundary conditions on the field $X^{(\pm)}$ are given by:

$$\begin{aligned} X^\pm(\tau_0) &= q(\tau_0) \\ X^\pm(\tau) &= q(\tau) \mp \int_{\tau}^{\tau_0} V'(q) d\tau . \end{aligned} \quad (5.8)$$

The result (5.7) is deceptively simple: it looks like a free field theory. That it is not follows from the boundary conditions (5.8). Namely, the transformation (5.4) does not map points of q space into points of X space. Rather it maps paths in q space into points in X space, and is thus very non-local. As to the nature of these paths we observe that for a constant value of $X^\pm$ we have

$$\frac{\partial}{\partial \tau} X^\pm = 0 \quad \leftrightarrow \quad \frac{\partial}{\partial \tau} q \mp V'(q) = 0 . \quad (5.9)$$

Hence constant values of $X^\pm$ correspond to the instanton paths in q space!

This remarkable result leads us to the following conclusion: if we interpret the path integral (5.7) with the boundary conditions that $X^\pm$ is fixed at τ, τ_0 , then the theory is a theory of free instantons and anti-instantons. These boundary conditions translated into q space mean that one has to specify the instanton on which the particle moves at τ, τ_0 . However, the actual boundary conditions from which we started fixed the values of q at τ, τ_0 , not those of $X^\pm$.

At the same time, and equivalently, the wave function $\Psi(q, \zeta, \tau)$ is transformed into a non-local functional $\Psi(X(\tau))$ of the path $X^\pm(\tau)$. There does not exist a unitary transformation from the "q representation" to the "X representation" of the wave functions.

What is the importance of this result to other supersymmetric theories? It seems there is good reason to believe that it is of more general validity; namely, it was shown in Ref. 7) that, for any rigidly supersymmetric theory, one can find a transformation like (5.4) order by order in perturbation theory, such that the path integral becomes Gaussian. It would be interesting to know whether such a transformation can always be linked to classical solutions.

In this context, it can be remarked that there are strong indications for our result to be valid for the solitons in the (1+1) dimensional complex Wess-Zumino model. As was mentioned in the Introduction, one might attempt to relate the corresponding result for the monopoles in $N = 4$ Yang-Mills theory to the other properties of that model. However, at present, this remains mere speculation.

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REFERENCES

1. D. Olive and E. Witten, Phys.Letters 78B (1978) 97.
2. E. Witten, Princeton Preprint (1981).
3. P. Salomonson and J.W. van Holten, CERN Preprint TH.3148 (1981).
4. S. Rouhani, Nuclear Phys. B182 (1981) 462.
5. A.M. Polyakov, Nuclear Phys. B120 (1977) 429.
6. G. 't Hooft, Phys.Rev. D14 (1976) 343.
7. H. Nicolai, Phys.Letters 89B (1980) 341 ; Nuclear Phys. B176 (1980) 419.