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GAUGE UNAMBIGUOUS QUANTIZATION OF YANG-MILLS THEORY
AROUND CLASSICAL SOLUTIONS

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A B S T R A C T

We quantize the Yang-Mills theory around an arbitrary classical solution. The correct treatment of the zero mode which is generated by the breaking of gauge invariance eliminates the infinite gauge volume and leads to an unambiguous expression for the generating functional. This has the usual background gauge form independently of the gauge in which the classical solution has been chosen. This shows that the eventual multiplicity of classical solutions satisfying certain gauge conditions, recently found by Gribov, is an irrelevant problem for the quantization.

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We discuss in this letter the quantization of Yang-Mills theory around classical solutions, giving a specific emphasis to the gauge arbitrariness. We shall see that the prescription is perfectly well-defined : after choosing a classical solution, no gauge arbitrariness remains in the theory. The correct treatment of the zero mode gives rise to a background gauge term relative to the classical solution around which we expand the theory, irrespectively of the gauge in which the classical solution is chosen. In particular, we show that the multiplicity of classical solutions satisfying an arbitrary gauge condition, recently pointed out by Gribov ¹⁾, is an irrelevant problem for quantizing non-Abelian Yang-Mills theories.

We are interested in developing the theory around a classical solution A_μ^{cl} . Let us write the Yang-Mills field as :

$$A_\mu^i = A_\mu^{i\text{cl}} + Q_\mu^i \quad (1)$$

Under a gauge transformation these fields transform like :

$$\begin{aligned} \hat{A}_\mu &\rightarrow {}^R \hat{A}_\mu = U \hat{A}_\mu U^{-1} + \frac{i}{g} U \partial_\mu U^{-1} \\ \hat{A}_\mu^{\text{cl}} &\rightarrow {}^R \hat{A}_\mu^{\text{cl}} = U \hat{A}_\mu^{\text{cl}} U^{-1} + \frac{i}{g} U \partial_\mu U^{-1} \\ \hat{Q}_\mu &\rightarrow {}^R \hat{Q}_\mu = U \hat{Q}_\mu U^{-1} \end{aligned} \quad (2)$$

where

$$\begin{aligned} U &= \exp i \Omega^j \frac{\lambda_j}{2} \\ \hat{A}_\mu &= A_\mu^j \frac{\lambda_j}{2} \end{aligned} \quad (3)$$

the λ_j denoting the generators of the gauge group. We shall call orbit the space spanned by ${}^\Omega \hat{A}_\mu^{\text{cl}}$, varying Ω in the homotopy class of the identity. This restriction on Ω will be understood from now on. The orbit is specified by the Pontryagin number of A_μ^{cl} . Let us denote the action by

$$S(A_\mu^{\text{cl}} + Q_\mu) = \int F_{\lambda\nu}(A_\mu^{\text{cl}} + Q_\mu) F_{\lambda\nu}(A_\mu^{\text{cl}} + Q_\mu) \quad (4)$$

where the space-time integral is tacitely understood.

The generating functional can be written as a path integral :

$$Z(j_\mu) = \frac{\int \mathcal{D}Q_\mu \exp \{ S(A_\mu^{\text{cl}} + Q_\mu) + j_\mu Q_\mu \}}{\int \mathcal{D}Q_\mu \exp S(A_\mu^{\text{cl}} + Q_\mu)} \quad (5)$$

Under gauge transformations j_μ transforms under the adjoint representation, like Q_μ in Eq. (2).

The expansion of the action $S(A_\mu^{\text{cl}} + Q_\mu)$ will give rise to a quadratic form in Q_μ which, due to the gauge invariance under (2), admits zero eigenvalues. In fact, a classical solution A_μ^{cl} satisfies :

$$\left. \frac{\delta \mathcal{L}(A_\mu)}{\delta A_\mu} \right|_{A_\mu = A_\mu^{\text{cl}}} = 0 \quad (6)$$

and, due to gauge invariance, the same equation holds for $\Omega_{A_\mu^{\text{cl}}}$. Differentiating (6) with respect to the gauge parameters we obtain

$$\left. \frac{\delta^2 \mathcal{L}(A_\mu)}{\delta A_\mu \delta A_\nu} \frac{\delta A_\nu}{\delta \Omega} \right|_{A_\mu = A_\mu^{\text{cl}}} = 0 \quad (7)$$

which proves that the $\delta A_\mu^{\text{cl}} / \delta \Omega$ are zero eigenvectors for the quadratic kernel $(\delta^2 S(A_\mu) / \delta A_\mu^2) |_{A_\mu = A_\mu^{\text{cl}}}$. A_μ^{cl} and $\delta \Omega$ being labelled by the adjoint representation, the zero modes - which can be labelled also by a space-time variable z - can be written as $(q_0^j)_i$:

$$(q_0^{j,z})_i \equiv \frac{\delta A_\mu^{i,\text{cl}}(x)}{\delta \Omega_j(z)} = \left[-\frac{\delta_{ij}}{g} \partial_\mu + f_{jik} A_\mu^{k,\text{cl}}(x) \right] \delta(x-z) \quad (8)$$

where the f_{ijk} are the structure constants of the gauge group. We have

$$\int dx (q_0^{j,z})_i \phi^i(x) = D^{ji}(A_\mu^{\text{cl}}) \phi^j(z) ; \quad D^{ji}(A_\mu^{\text{cl}}) \equiv \frac{\delta_{ij}}{g} \partial_\mu + f_{jik} A_\mu^{k,\text{cl}} \quad (9)$$

Let us remark at this point that other zero modes will appear if the classical solution breaks other invariances as translation or dilatation. They have obviously to be treated in the usual manner, but since they are irrelevant for our purpose, we shall not deal with them explicitly.

The general method for dealing with a zero mode consists in expanding the field Q_μ on the basis of the eigen functions q_n of the quadratic kernel. Making explicit the gauge arbitrariness of the classical solution, we write :

$$A_\mu = \bar{A}_\mu^d + \xi_0^{(R)} q_0(\bar{A}_\mu^d) + \sum_{n=1}^{\infty} \xi_n^{(R)} q_n(\bar{A}_\mu^d) \quad (10)$$

We shall now change as usual the set of independent variables from (q_0, q_n) to (Ω, q_n) . This is achieved by the Faddeev-Popov trick of introducing a determinant Δ through the condition :

$$\Delta \int \mathcal{D}\Omega \, \delta(N \xi_0^{(R)}) = 1 \quad ; \quad N = \mathcal{D}(\bar{A}_\mu^d) \mathcal{D}(\bar{A}_\mu^d) \quad (11)$$

Δ is the Jacobian corresponding to the change of variables. The restriction to small quantum excitations q_n orthogonal to the zero mode - restriction implicitly assumed in considering the quadratic form in Q_μ - guarantees that the determinant is regular. This amounts to say that the change of variables we performed is justified, or, in other words, that the equation $\xi_0^{(R)} = 0$ admits a unique solution $\Omega = \tilde{\Omega}$. From (11)

$$\Delta = \det \left. \frac{\delta N \xi_0^{(R)}}{\delta \Omega} \right|_{\Omega = \tilde{\Omega}} \quad (12)$$

and, using (10),

$$\left. \frac{\delta N \xi_0^{(R)}}{\delta \Omega} \right|_{\Omega = \tilde{\Omega}} = \left[\frac{\delta \bar{A}_\mu^d}{\delta \Omega} \frac{\delta \bar{A}_\mu^d}{\delta \Omega} + \sum_{n=1}^{\infty} \xi_n^{(\tilde{\Omega})} \frac{\delta q_n(\bar{A}_\mu^d)}{\delta \Omega} \frac{\delta \bar{A}_\mu^d}{\delta \Omega} \right]$$

$$\delta(N \xi_0^{(R)}) = \delta \left(\mathcal{D}(\bar{A}_\mu^d) [A_\mu - \bar{A}_\mu^d] \right) \quad (13)$$

Inserting (11) into (5) and using expressions (13), we get :

$$Z(\mu) = \frac{\int d\Omega \int \mathcal{D}Q_\mu \delta(D(A_\mu^{\text{cl}})Q_\mu) \text{Det}[D(A_\mu^{\text{cl}})D(A_\mu^{\text{cl}}+Q_\mu)] \exp[S(A_\mu^{\text{cl}}+Q_\mu)+\int \mu Q_\mu]}{\int d\Omega \int \mathcal{D}Q_\mu \delta(D(A_\mu^{\text{cl}})Q_\mu) \text{Det}[D(A_\mu^{\text{cl}})D(A_\mu^{\text{cl}}+Q_\mu)] \exp S(A_\mu^{\text{cl}}+Q_\mu)} \quad (14)$$

Reinterpreting the δ function as a constraint on Q_μ for each Ω , the integrand in Ω is gauge invariant. Then the integration over $d\Omega$ factors out and yields the volume of the group, that cancels between the numerator and the denominator. With the usual procedure we can cast (14) into the form:

$$Z(\mu, \eta, \bar{\eta}) = \frac{\int \mathcal{D}Q_\mu \mathcal{D}c \mathcal{D}\bar{c} \exp[\Sigma(Q_\mu) + \int \mu Q_\mu + \bar{\eta}c + \eta\bar{c}]}{\int \mathcal{D}Q_\mu \mathcal{D}c \mathcal{D}\bar{c} \exp \Sigma(Q_\mu)} \quad (15)$$

c and $\bar{c}$ being the Faddeev-Popov ghosts. $\Sigma(Q_\mu)$ is the Yang-Mills action in the background gauge ^{2),3)} :

$$\Sigma(Q_\mu) = \lim_{\alpha \rightarrow 0} \left\{ S(A_\mu^{\text{cl}} + Q_\mu) - \frac{1}{2\alpha} [D(A_\mu^{\text{cl}})Q_\mu]^2 + \bar{c} D(A_\mu^{\text{cl}})D(A_\mu^{\text{cl}}+Q_\mu)c \right\} \quad (16)$$

Let us point out that the δ function appearing in (14) has nothing to do with a gauge fixing condition. Indeed, all the gauge arbitrariness is reflected in the choice of a classical solution A_μ^{cl} along a given orbit. After having chosen A_μ^{cl} , the δ function guarantees that the fluctuations Q_μ over which we integrate are orthogonal to the orbit. We notice that our discussion is completely independent of whether or not one or many A_μ^{cl} satisfy a certain gauge condition along a given orbit. Each point of the orbit is equally good and, through the correct treatment of the zero mode, gives the entire contribution to the perturbation

development around the orbit. Pictorially, the zero mode parametrizes the orbit and therefore does not give rise to any ambiguity, while fixing a gauge implies to intersect the orbit with a gauge condition. This leads to ambiguities when the intersection is not unique.

Of course, in (14) the perturbation development will depend on the point of the orbit chosen : pictorially, the steepness of the valley may change along the orbit.

Let us add that in the terminology of fiber bundles, the Yang-Mills field A_μ is a connection. Connections generate an affine space, not a vector space. In other words, the sum of two connections is not a connection, as can be trivially understood from (2), verifying that

$${}^2A_1 + {}^2A_2 \neq {}^2(A_1 + A_2) \quad (17)$$

but

$$\lambda {}^2A_1 + (1-\lambda) {}^2A_2 = {}^2(\lambda A_1 + (1-\lambda) A_2) \quad (18)$$

Then the meaning of integrating over A_μ is not well defined. On the contrary, deviations with respect to a connection on a given fiber bundle (i.e., a given homotopy class) - typically our Q_μ - do generate a vector space. Therefore, the functional integration over Q_μ is well defined, independently of the fiber bundle on which A_μ lies, that is, independently of the Pontryagin index ν of $A_\mu^{\text{cl}, \nu}$. Contributions of orbits with different ν have to be summed :

$$Z(\mu) = \sum_{\nu=0}^{\infty} Z^\nu(\mu) \quad (19)$$

As a summary, let us compare the usual scheme for quantizing gauge theories with ours. In the former, we proceed as follows. Write

$$Z(\mu) = \frac{\int \mathcal{D}A_\mu \exp[F_{\mu\nu}(A_\mu) F_{\mu\nu}(A_\mu) + \int A_\mu]}{\int \mathcal{D}A_\mu \exp F_{\mu\nu}(A_\mu) F_{\mu\nu}(A_\mu)} \quad (20)$$

Equation (20) gives rise to a degenerate quadratic form which does not allow to define a propagator. One then introduces a gauge fixing term and uses the Faddeev-Popov trick to eliminate the infinity caused by the gauge volume. In identifying classical solutions around which to expand one falls onto the Gribov problem ¹⁾. Indeed the gauge fixing condition may intersect the orbit in a non-unique way, a fact which makes ambiguous the expansion procedure. Our scheme is instead the following. Choose a classical solution $A_\mu^{cl,v}$ - in the gauge one wishes - for every Pontryagin index v . Then write (19), and for every $Z_v(j_\mu)$ the corresponding form (5). Perform the expansion around $A_\mu^{cl,v}$ by properly treating the zero mode that arises from the breaking of gauge invariance due to the specific choice of $A_\mu^{cl,v}$. This procedure eliminates the infinity due to the gauge volume and leads to an unambiguous expression of $Z_v(j_\mu)$ given by (14) or (15). This has the form usually referred to as the background gauge which is perfectly handable and has been studied in the literature ^{2),3)}.

Let us stress that in this note we do not enter in the merit of the validity of the expansion around classical solutions ; we only discuss how an expansion should be done. Neither do we discuss the expansion around functions which are not classical solutions and give rise therefore to linear terms in the expansion of the action. We have the intention to actually use the formalism introduced here, which handles correctly gauge invariance, in order to study those questions which are basic for the understanding of gauge theories.

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