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ON THE ORDER OF LEVELS IN CHARMONIUM-LIKE MODELS

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ABSTRACT

We determine the order of the first energy levels in potentials of the form

$$V(r) = \int_{-2}^{\infty} d\alpha \rho(\alpha) r^{\alpha}$$

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1. Introduction

The new particle spectroscopy¹⁾ can be well-described by a non-relativistic potential model²⁾. In that scheme the J/ψ and ψ' are assigned to the 1S and 2S states in a potential being the sum of a Coulomb-like part and a confining part. Since there is no unique field theoretical basis from which one may derive the form of the potential, different schemes have been proposed. Not only different confining parts were suggested³⁾ also modifications of the Coulomb part (running coupling constant) were considered.

Here we will continue our recent investigations^{4,5)} on the order of levels to cover also the more general class of models which have been proposed in the literature. After developing the general arguments in Section 2, we study in Section 3 the 2S - 1P level difference and in Section 4 the energy difference of the 2S - 1D states for potentials having different convexity properties. Denoting by E_{2S} , E_P and E_D the energies of the second S state, the first P and the first D state, we obtain "canonical" level order $E_{2S} > E_P$ especially for a superposition of the form

$$V(r) = -\frac{\alpha}{r} + \int_{-1}^{\infty} d\alpha' \rho(\alpha') r^{\alpha'}, \quad \begin{array}{ll} \rho \geq 0 & \alpha' \geq 0, \alpha' < -1 \\ \rho \leq 0 & -1 \leq \alpha' < -\delta < 0 \end{array} \quad (1)$$

The D state which has been observed only recently⁶⁾ lies above the 2S state for potentials

$$V(r) = -\frac{\alpha}{r} + \int_{-2}^2 d\alpha' \rho(\alpha') r^{\alpha'}, \quad \begin{array}{ll} \rho \geq 0 & \alpha' \geq 0 \\ \rho \leq 0 & \alpha' \leq 0 \end{array} \quad (2)$$

More general conditions covering also other cases are given in the text. Clearly, in all cases, our potentials should be less singular than $-1/r^2$ at the origin.

2. Basic Equations

The equations which we will derive relate the energy difference of two states to the variation of that quantity with respect to some coupling parameters. Our conclusions will depend only on the number of nodes and the angular momentum of the wave functions considered.

Let us denote by u the second S-wave function which has, according to standard Sturm-Liouville theory, exactly one node ($2m = n = 1$) and fulfills

$$-u'' + V(r)u = E_{2S}u \quad u(0) = 0 \quad (3)$$

and let us denote by v the nodeless ground state wave function for angular momentum ℓ

$$-v'' + \frac{\ell(\ell+1)}{r^2}v + V(r)v = Ev \quad v(0) = 0 \quad (4)$$

E will later on be specified to become either E_P or E_D . For the potential we will choose a convenient dependence on the coupling strength

$$V(r) = -\frac{\alpha}{r} + \lambda^2 v(\lambda r) \quad (5)$$

implying that the dimensions of λ and α are minus one; $\lambda = 1$ corresponds to the potential we would like to solve. If we denote by $\epsilon = E_{2S} - E$ the level difference which will concern us here:

$$\epsilon = \int_0^\infty dr (u'^2 - v'^2) - \ell(\ell+1) \int_0^\infty \frac{dr}{r^2} v^2 + \int_0^\infty dr (u^2 - v^2) V(r) \quad (6)$$

then the shift of ϵ by changing λ is given by the Feynman-Hellman theorem⁷⁾

$$\frac{d\epsilon}{d\lambda} = \int_0^\infty dr \frac{u^2 - v^2}{r} W(\lambda r), \quad W(r) = \frac{d}{dr} (r^2 v(r)) \quad (7)$$

In deriving (6) we have normalized both wave functions to 1. Our main conclusions use properties of the difference of the square of the wave functions entering in (7) which we will obtain with the help of the Wronskian relation:

Lemma 1: $u-v$ vanishes once and only once in $(0, \infty)$.

Proof: Let us choose u and v positive for small r ; since $u \sim r$, $v \sim r^{\ell+1}$ for $r \rightarrow 0$, $u-v$ is positive for r small; denote by r_0 the point where $u(r_0) = 0$, at $r = r_0$, $u - v < 0$, so $u - v$ has at least one node.

The Wronskian relation following from (3) and (4) reads

$$vu' - uv' = - \int_0^r dr \varphi(r), \quad \varphi(r) = u \cdot v \cdot \left(\frac{\ell(\ell+1)}{r^2} + E \right) \quad (8)$$

For $\epsilon \geq 0$, $\varphi(r)$ vanishes once in $(0, \infty)$, for $\epsilon < 0$ twice. Using (8) for $r = r_0$, we get

$$u(r_0)v'(r_0) = \int_{r_0}^{\infty} dr \varphi(r) < 0 \quad (9)$$

so that $\varphi(r)$ cannot be positive everywhere in (r_0, ∞) . For $\epsilon \geq 0$, $\varphi(r)$ is always negative in (r_0, ∞) ; for $\epsilon < 0$, $\varphi(r)$ will be positive for large r , and has to be negative near r_0 . In any case, $\varphi(r) > 0$ in $(0, r_0)$; so applying (8) to r_1 where $u(r_1) = v(r_1)$

$$v(r_1)(u'(r_1) - v'(r_1)) = - \int_0^{r_1} dr \varphi(r) < 0 \quad (10)$$

showing that there is only one zero of $u - v$.

Lemma 2: If $\epsilon \geq 0$ $u+v$ vanishes at most once in $r_0 < r < \infty$. Putting $u(r_2) = -v(r_2)$ we get again from (8)

$$v(r_2)(u'(r_2) + v'(r_2)) = \int_{r_2}^{\infty} dr \varphi(r) \quad (11)$$

Since $r_2 > r_0$, $\epsilon \geq 0$, $u' + v'$ is negative at r_2 implying that there is at most one point where $u + v$ vanishes.

Before applying these Lemmas, let us remark that the energy difference $\epsilon(\lambda, \alpha)$, although being a function of two variables, has to have the special form

$$\epsilon(\lambda, \alpha) = \lambda^2 \bar{\epsilon}\left(\frac{\lambda}{\alpha}\right) \quad (12)$$

This follows from applying a scale transformation to (3) and (4). By differentiating (12), we observe that

$$2\epsilon = \alpha \frac{d\epsilon}{d\alpha} + \lambda \frac{d\epsilon}{d\lambda} \quad (13)$$

or using (7) for variations with respect to α

$$2\epsilon = -\alpha \int_0^{\infty} dr \frac{u^2 - v^2}{r} + \lambda \frac{d\epsilon}{d\lambda} \quad (14)$$

Equations (14) and (7) are basic to our following discussion. The steps from (12) to (14) are equivalent to using the Virial theorem.

3. The Level Difference $\epsilon_{SP} = E_{2s} - E_P$

Theorem A: Let the potential be given as a sum of the form

$$V(r) = -\frac{\alpha}{r} + V_1(r) - V_2(r) + V_3(r) \quad V_i > 0 \quad (15)$$

and assume the convexity properties hold

$$W_i = \frac{d}{dr}(r^2 V_i) \quad , \quad \left(\frac{d}{dr}\right)^2 W_{1,3} > 0 \quad , \quad \left(\frac{d}{dr}\right)^2 W_2 < 0 \quad , \quad (16)$$

together with the conditions

$$W_1(0) = W_2(0) = W_3(\infty) = 0, \quad \lim_{r \rightarrow \infty} r^{1+\delta} V_3 = c < \infty, \quad (17)$$

for some $0 < \delta < 1$. Then the level difference $\epsilon_{SP} > 0$.

Proof: Without loss of generality we can put $\alpha = 2$ and start with the potential

$$V(r) = -\frac{\lambda + \mu}{r} + \lambda^2 V_1(\lambda r) - \lambda^2 V_2(\lambda r) + \mu^2 V_3(\mu r) \quad (18)$$

for $\lambda = \mu = 1$ we obtain again (15). Our first argument concerns the limit $\mu \rightarrow 0$ for which (18) goes over into

$$V(r) = -\frac{\alpha}{r} + \lambda^2 V_1(\lambda r) - \lambda^2 V_2(\lambda r) \quad (19)$$

with $\alpha = \lambda$. The convexity properties (16) imply that for $\epsilon_{SP} > 0$ there exists a constant $A > 0$ such that

$$\int_0^\infty dr \frac{u^2 - v^2}{r} [A + W_1 - W_2 - Br] > 0 \quad (20)$$

A and B have to be adjusted so that $-A + Br$ crosses $W_1 - W_2$ at the two zeros of $u^2 - v^2$. Conditions (16) make sure that each line intersects

the function $W_1 - W_2$ at most twice. Due to our normalization conditions for u and v , (20) gives an inequality between two "moments":

$$\int_0^{\infty} dr \frac{u^2 - v^2}{r} [A + W_1 - W_2] > 0 \quad (21)$$

Using (21) and our basic equations (7) and (14) we prove that

$$E_{SP} \geq 0 \Rightarrow \frac{dE_{SP}}{d\lambda} > 0 \quad (22)$$

giving monotonicity once $\epsilon_{SP} \geq 0$. To exclude the crossing situation we use perturbation theory for small λ :

$$(\Delta E_{SP})_{pert.} = \int_0^{\infty} dr (u_0^2 - v_0^2) [\lambda^2 V_1(\lambda r) - \lambda^2 V_2(\lambda r)] \quad (23)$$

where u_0 and v_0 denotes the unperturbed wave function. Since

$$\int_0^1 d\lambda W_{1,2}(\lambda r) = r \int_0^1 d\lambda \frac{d}{d\lambda} \lambda^2 V_{1,2}(\lambda r) = r V_{1,2}(r) \quad (24)$$

we conclude that $rV_{1,2}$ has the same convexity property as $W_{1,2}$. Using (21) for $u_0^2 - v_0^2$ and for $V_1 - V_2$ we obtain

$$(\Delta E_{SP})_{pert.} > 0 \quad (25)$$

Using continuity proves that $\epsilon_{SP} > 0$ for all λ . Next we take the limit $\mu \rightarrow \infty$ in Eq. (18). This limit corresponds to $\lambda \rightarrow 0$, so that we have to deal with

$$V(r) = -\frac{\sigma}{r} + \mu^2 V_3(\mu r) \quad (26)$$

for $\sigma = \mu$. For μ small we use arguments leading from (23) to (25) and obtain $(\Delta \epsilon_{SP})_{pert.} > 0$. For μ large we prove in Appendix A that the lowest levels are determined by the harmonic oscillator approximation and conclude $\epsilon_{SP}(\mu) > 0$ for $\mu \rightarrow \infty$. Consider now the (σ, μ) plane. We know $\epsilon_{SP} > 0$ for $\mu \rightarrow 0$ which corresponds to $\sigma \rightarrow \infty$, similarly $\epsilon_{SP} > 0$ for $\mu \rightarrow \infty$ which corresponds to $\sigma \rightarrow 0$. So only islands of zeros are

allowed. By approaching such an island, we encounter the situation where

$$\int_0^{\infty} dr \frac{u^2 - v^2}{r} = \int_0^{\infty} dr \frac{u^2 - v^2}{r} W_3(r) = 0 \quad (27)$$

Using (20) for W_3 , we conclude that $u = v$ which is absurd. So we know $\epsilon_{SP} > 0$ for (26). We apply the same argument once more to the (λ, μ) plane for the potential (18) and conclude $\epsilon_{SP} > 0$ for all λ and μ .

Theorem B: The potential $V = -\alpha/r + \bar{V}$ with

$$W(r) = \frac{d}{dr} (r^2 \bar{V}), \quad \left(\frac{d}{dr}\right)^2 W < 0, \quad W(\infty) = 0 \quad (28)$$

gives the non-canonical level order $E_{SP} < 0$.

Proof: Again we find constants $A > 0$ and $B > 0$ implying that

$$\int_0^{\infty} dr \frac{u^2 - v^2}{r} [A + W - Br] < 0 \quad (29)$$

using again (7) and (14) for the case in hand, we obtain at a crossing point ($\epsilon_{SP} = 0$) a negative slope $d\epsilon_{SP}/d\lambda < 0$. Using (29) once more for the perturbation expression and continuity proves theorem B.

Remark

Potentials of the form $V = -\alpha/r + \lambda/r^\delta$ with $0 < \delta < 1$, $\lambda > 0$, are not in any class treated so far. In Appendix B we prove that one obtains a negative level difference. Let us note that the question of sign of the level difference in this case is meaningful only for some small values of λ ; otherwise the potential will not have bound states at all.

Next we turn to the energy difference $\epsilon_{SD} = E_{2S} - E_D$. For the last two examples, giving $\epsilon_{SP} < 0$, we conclude immediately $\epsilon_{SD} < 0$, since the positive angular momentum operator has to shift all levels up.

4. The Level Difference $\epsilon_{SD} = E_{2S} - E_D$

Let us deal here with potentials of the form

$$V(r) = -\frac{\alpha}{r} + \lambda^2 V_1(\lambda r) - \lambda^2 V_2(\lambda r) \quad V_i > 0 \quad (30)$$

and impose the conditions

$$W_i = \frac{d}{dr} (r^2 V_i) \quad \begin{aligned} \left(\frac{d}{dr}\right)^2 W_1 &> 0, \quad \left(\frac{1}{r} W_1\right)(0) = 0, \quad W_2(0) = 0, \\ \left(\frac{d}{dr}\right)^2 W_2 &< 0, \quad \left(\frac{1}{r} W_2\right)(\infty) = 0, \end{aligned} \quad (31)$$

In the limit $\lambda \rightarrow 0$, we know $\epsilon_{SD} < 0$ from the hydrogen levels; so we study again the crossing situation. Using (31) we find $A > 0$ such that

$$\int_0^\infty dr \frac{u^2 - v^2}{r} [A + W_1 - W_2 - Br] > 0 \quad (32)$$

holds for $\epsilon_{SD} \geq 0$, and finally with the help of (7) and (14), we conclude

$$\epsilon_{SD} \leq k \frac{d\epsilon_{SD}}{d\lambda}, \quad k = \frac{1}{2} \left(\lambda + \frac{\alpha}{A} \right) > 0, \quad (33)$$

From (33) we learn that at a crossing-point the slope has to be positive and if $\epsilon_{SD} > 0$, it is a monotonous function of λ ; if $\epsilon_{SD} \leq 0$ for $\lambda \rightarrow \infty$, we know that no crossing occurred, while $\epsilon_{SD} > 0$ for large λ implies that there occurs exactly one level crossing. To evaluate ϵ_{SD} for $\lambda \rightarrow \infty$ we remark again that ϵ_{SD} has the following dependence on α and λ

$$\epsilon_{SD}(\alpha, \lambda) = \lambda^2 \overline{\epsilon_{SD}}\left(\frac{\alpha}{\lambda}\right) \quad (34)$$

so that we have to deal with $\alpha \rightarrow 0$ and with

$$h = -\frac{d^2}{dr^2} + \lambda^2 V_1(\lambda r) - \lambda^2 V_2(\lambda r) + \omega^2 r^2 \quad (35)$$

for $\omega = 0$. We added in (35) a harmonic oscillator potential since we know the 2S - D level difference Δ for $\lambda = 0$ is zero. $\omega \rightarrow 0$ corresponds again to $\lambda \rightarrow \infty$. For Δ we obtain in a similar manner as before our fundamental equations

$$2\Delta = \omega^2 \int_0^\infty dr r^2 (u^2 - w^2) + \lambda \frac{d\Delta}{d\lambda} \quad (36)$$

$$\frac{d\Delta}{d\lambda} = \int_0^{\infty} dr \frac{u^2 - w^2}{r} (W_1 - W_2)(\lambda r) \quad (37)$$

where u and w denote here the 2S and the D wave function for the Hamiltonian h .

Theorem C: Let $V(r)$ be given as in Eqs. (30) and (31) and assume that

$$W_i > 0, \left(\frac{d}{dr^2}\right)^2 \left(\frac{W_i(r)}{r}\right) < 0, \left(\frac{d}{dr^2}\right)^2 \left(\frac{W_2(r)}{r}\right) > 0, \quad (38)$$

then the level difference ϵ_{SD} is negative.

Proof: In a similar way as before (38) implies the existence of a positive constant $D > 0$ such that

$$\int_0^{\infty} dr (u^2 - w^2) \left[C - \frac{W_1}{r} + \frac{W_2}{r} + Dr^2 \right] > 0 \quad (39)$$

Using the normalization condition for u and w , we obtain from (36) and (37) the inequality

$$2\Delta \geq \left(\frac{\omega^2}{D} + \lambda\right) \cdot \frac{d\Delta}{d\lambda} \quad (40)$$

which implies $d\Delta/d\lambda \leq 0$ for $\Delta = 0$. The perturbation theoretical argument shows again that $\Delta < 0$ for all λ implying $\epsilon_{SD} < 0$ for all λ .

Corollary: $V(r) = -(\alpha/r) + \bar{V}(r)$ with

$$W(r) \geq 0, \quad W = \frac{d}{dr} (r^2 \bar{V}), \quad \left(\frac{1}{r} W\right)(0) = 0, \quad \left(\frac{d}{dr}\right)^2 W(r) > 0 \quad (41)$$

$$\left(\frac{d}{dr^2}\right)^2 \left(\frac{W(r)}{r}\right) > 0$$

implies that there exists exactly one level crossing.

Proof: The only change in the proof given above concerns inequalities (39) and (40), which are reversed; therefore, we obtain $\Delta > 0$ for all λ implying ϵ_{SD} positive for large λ , so that exactly one level crossing has to occur.

5. Discussion

Let us illustrate our conditions by special examples. For potentials of the form

$$V(r) = -\frac{\alpha}{r} + \lambda r^\delta \quad (42)$$

we summarize the level order in Figs. 1 and 2. For superpositions of powers of r , we obtain from Theorems 1 and 3 that potentials belonging to the class given in Eq. (1) imply $\epsilon_{SP} > 0$ and these given by Eq. (2) imply $\epsilon_{SD} > 0$. On the other hand

$$V(r) = -\int_{-2}^{-1} d\alpha' \rho(\alpha') r^{\alpha'}, \quad \rho > 0 \quad (43)$$

implies $\epsilon_{SP} < 0$ and

$$V(r) = -\frac{\alpha}{r} + \int_2^{\infty} d\alpha' \rho(\alpha') r^{\alpha'}, \quad \rho > 0 \quad (44)$$

implies one level crossing for ϵ_{SD} . As a special example, a potential coming from asymptotic freedom⁸⁾ ideas like

$$V(r) = -\alpha \left(\frac{1}{r} - 1 \right) \frac{1}{\ln \frac{1}{r}} + c + kr \quad (45)$$

belongs to the class of Eqs. (1) and (2).

Since our derivation in Section 2 uses only the number of nodes of the wave functions and the fact that the angular momentum of the two wave functions u and v are ordered $\ell_u < \ell_v$ we immediately can apply the same procedure to the second P level and the first D level. So a potential like

$$V(r) = -\int_{-1}^0 d\alpha' \rho(\alpha') r^{\alpha'} + \int_0^2 d\alpha' \sigma(\alpha') r^{\alpha'}, \quad \begin{matrix} \rho \geq 0 \\ \sigma \geq 0 \end{matrix} \quad (46)$$

has the canonical level order:

$$E_{1S} < E_{2P} < E_{2S} < E_{3D} < E_{3P} \quad (47)$$

where we have used the hydrogen notation. Let us mention also that we found for simple potentials numerically also $E_{3P} < E_{3S}$ which cannot be proved with our methods, since the 3S wave function already has two zeros within $(0, \infty)$.

Finally, let us comment on a point of mathematical rigour. To use the argument, that the perturbation series determines the slope for small coupling constants, we have to assume that the perturbation expansion gives at least an asymptotic expansion to the true answer, as it is the case, for instance, for the an-harmonic oscillator ⁹⁾.

Appendix A

Here we will determine the lowest part of the spectrum of the operator

$$-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} - \frac{\alpha}{r} + \lambda^2 V(\lambda r) \quad (A1)$$

defined on $L^2 [0, \infty), dr]$ for large coupling constant λ . It is assumed that $V(r)$ approaches $c/r^{1+\epsilon}$ $0 < \epsilon < 1$ in that limit. Let us therefore apply the unitary scale transformation

$$r = \mu x, \quad \lambda^2 = \frac{\alpha}{\mu} = \left(\frac{\lambda}{\mu}\right)^{1-\epsilon} c \quad (A2)$$

and study the Hamiltonian

$$h_\ell = -\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{x^2} + \lambda^2 V(x) \quad v(x) = -\frac{\gamma}{x} + \frac{\gamma}{x^{1+\epsilon}} \quad (A3)$$

in the strong coupling limit $\lambda \rightarrow \infty$.

Lemma 3: A lower bound to the energy levels of h_0 is given by

$$E_n^{LB} = -\lambda^2 V_{min} + \left(n + \frac{1}{2}\right) \tilde{\omega} \lambda + O(1) \quad (A4)$$

$$V_{min} = \frac{\epsilon}{(1+\epsilon)^{\frac{1+\epsilon}{\epsilon}}}, \quad \tilde{\omega} = \frac{(2\epsilon)^{1/2}}{(1+\epsilon)^{3/2\epsilon}} + \delta\omega$$

where $\delta\omega$ is positive but small

Proof: As a first step, let us consider $v(x)$ as an operator on $L^2[(-\infty, \infty)]$ and choose $v(x) = \infty$ for $x < 0$. To obtain a lower bound to h_0 , we use a comparison potential v_{LB} of the form

$$v(x) \geq v_{LB}(x) = -K \left[\frac{1}{ch^2 \alpha (x-x_m)} - 1 \right] - V_{min} \quad (A5)$$

Here K and α can be chosen such that the curvature of v and v_{LB} at x_m are as close as one likes; this can be shown explicitly by using an oscillator potential with

$$v \geq v_s \geq v_{LB}, \quad v_s = \begin{cases} K - V_{min} & x \geq x_s \\ \beta(x-x_m)^2 - V_{min} & x < x_s \end{cases} \quad (A6)$$

$$K = \beta(x_s - x_m)^2$$

For the potential $\Lambda^2 v_{LB}$ we know the energy levels explicitly

$$E_n = \Lambda^2 (K - V_{min}) - \alpha^2 (\lambda - n - 1)^2 \quad (A7)$$

$$\Lambda^2 K = \alpha^2 \lambda (\lambda - 1) \quad \tilde{\omega} = 2\alpha \sqrt{K}$$

so expanding E_n for large Λ gives (A4).

Lemma 4: An upper bound to the eigenvalues of

$$h = -\frac{d^2}{dr^2} + V(r) \quad V(r) = \Lambda^2 \frac{(r-r_m)^2}{4} \omega^2 + \Lambda^2 (r-r_m)^3 R(r) \quad (A8)$$

$$|R(r)| < \frac{C}{r^2} \quad r \in [0, \infty)$$

is given by

$$E_n^{UB} = \frac{\Lambda}{2} \omega + O(1) \quad (A9)$$

Proof: Using the Min-Max principle one has to choose only an appropriate trial function

$$\psi = \left(\frac{\omega \Lambda}{\pi} \right)^{1/4} e^{-\frac{\Lambda \omega}{4} (r-r_m)^2} \quad (A10)$$

For $\Lambda \rightarrow \infty$ a simple calculation gives (A9).

Clearly our potential V belongs (after a constant shift) to the class of Eq. (A8) with $\omega^2 = 2e/(1+e)^{3/e}$. Combining therefore Lemmas 3 and 4, we obtain a lower bound to the second state of h_e

$$E_{2s} > -\Lambda^2 V_{min} + \frac{3}{2} \tilde{\omega} \Lambda + O(1) \quad (A11)$$

and an upper bound on the first P and D state behaving like

$$E_{P,D} < -\Lambda^2 V_{min} + \frac{\omega}{2} \Lambda + O(1) \quad (12)$$

which proves our conjecture that for $\Lambda \rightarrow \infty$

$$E_{2s} > E_{P,D} \quad (A13)$$

Appendix B

Proposition: $V(r) = -(1/r) + (\lambda/r^\delta)$ $0 < \delta < 1$ gives $\epsilon_{SP} < 0$.

Proof: Our usual moment inequality

$$\int_0^\infty dr \frac{u^2 - v^2}{r^\delta} < A \int_0^\infty dr \frac{u^2 - v^2}{r} \quad A > 0 \quad (B1)$$

shows that $\epsilon_{SP} < 0$ for $d\epsilon_{SP}/d\lambda = 0$; furthermore, $(\Delta\epsilon)_{pert} < 0$ so that at most one crossing is allowed. To exclude that case, we obtain an upper bound on the coupling λ such that there exists no P wave bound state (λ_p^{UB}) and a lower bound on λ such that there exist, at least, two S wave bound states.

In Ref. 10) sufficient conditions for the absence of bound states in the form of moments of V are given: If v_ℓ denotes the number of bound states with angular momentum ℓ we have

$$v_\ell (2\ell+1)^{2p-1} \leq c_p \bar{I}_p \quad \forall p \geq 1 \quad (B2)$$

$$c_p = \frac{(p-1)^{p-1} \Gamma(2p)}{p^p \Gamma^2(p)}$$

$$\bar{I}_p = \int_0^\infty dr r^{2p-1} |V|_-^p$$

where $|V|_-$ denotes the absolute value of the attractive part of the potential. This implies for our case $\ell = 1$:

$$\lambda_{\ell=1} \geq \inf_p f_\delta(p) = \lambda_p^{UB} \quad f_\delta(p)^{\frac{1}{1-\delta}} = c_p \frac{\Gamma(p+1) \Gamma(\frac{p}{1-\delta})}{(1-\delta) \cdot 3^{2p-1} \Gamma(p+1+\frac{p}{1-\delta})} \quad (B3)$$

Optimization in (B3) has been done numerically; the result is given in Figure 3.

We obtain a lower bound on λ (λ_S^{LB}) such that two S-bound states exist using the Min-Max principle. As trial functions, we take

$$\psi_1 = e^{-\alpha r} \quad \psi_2 = r e^{-\beta r} \quad (B4)$$

and determine λ such that we get zero as an upper bound on the energy of the second S wave bound state:

$$h_{ij} = (\psi_i, h \psi_j) \quad \det h = 0 \quad \text{tr } h_{ij} < 0 \quad (B5)$$

Optimization in α and β has been done again numerically. As is seen in Fig. 3 for values of the exponent δ between 0.13 and 0.95, $\lambda_p^{UB} < \lambda_s^{LB}$ so we conclude that the level difference in that range is negative and finally that no level crossing occurs for smaller values of λ . To settle the question for $0 < \delta < 0.13$, we use a comparison potential. Using Eqs. (7) and (14) and the obvious bound on $A < r_1^{1-\delta}$, where r_1 denotes the first zero of $u^2 - v^2$, we obtain from (B1): (r_c denotes the value of r where $V=0$)

$$2E_{SP} \leq K \cdot \lambda \frac{dE_{SP}}{d\lambda} \quad K = 1 - \left(\frac{r_c}{r_1}\right)^{\frac{1-\delta}{2-\delta}} \quad (B6)$$

We intend to prove $K < 0$; so that $e_{SP} = 0$ implies $dE_{SP}/d\lambda < 0$; which contradict our continuity argument, therefore implying $e_{SP} < 0$.

Let us denote by r_0 the value of r where $v = 0$ and r_{min} that at which v has its unique minimum; then obviously

$$r_1 < r_0, \quad r_{min} < r_c \quad (B7)$$

so that a lower bound on $R = r_{min}/r_0$ implies an upper bound on K . To obtain this bound on R we replace $V - E_{2S}$ by a Coulomb potential, which goes through $V(r_0)$:

$$V_{Coul} = -\frac{G}{r+a}, \quad V_{Coul}(r_0) = V(r_0) \quad (B8)$$

Since V_{Coul} is less attractive for $r < r_0$, and more attractive for $r > r_0$ then $V - E_{2S}$, we obtain $R < R_{Coul}$; but the latter is given by the ratio of the zeros of the zeroth Bessel function:

$$R = \frac{r_{min}}{r_0} > \left(\frac{5.52}{3.83}\right)^2 = R_{Coul} = 2.07 \quad (B9)$$

implying $K < 0$ for $0 < \delta < 0.13$, which proves that $e_{SP} < 0$ for that range of δ . For the small region of exponent δ near one we answered the question numerically.

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Sign of $E_{2S} - E_P$ for $V = -\frac{1}{r} + \lambda r^\delta$

FIG.1

Sign of $E_{2S} - E_D$ for $V = -\frac{1}{r} + \lambda r^\delta$

FIG.2

FIG. 3