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ON THE SELF-DUALITY OF SOLUTIONS
OF THE YANG-MILLS EQUATIONS

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A B S T R A C T

We prove that for a pure $SU(2)$ Yang-Mills theory, the vanishing of the energy momentum tensor is equivalent to the (anti) self-duality of the field tensor, if the equations of motion are satisfied.

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There has been much interest recently in classical solutions of Yang-Mills theories. All known solutions of the Euclidean equations of motion which give a finite action are either self-dual or anti-self-dual ¹⁾. It has even been conjectured that all solutions are necessarily of this type. This would imply that the energy momentum tensor, $\Theta_{\mu\nu}$, vanishes, which is relevant to the vacuum structure of the theory. To our knowledge, no proof of this conjecture exists, even in the simple case of vanishing $\Theta_{\mu\nu}$ ^{*)}.

In this paper we shall prove that for a pure SU(2) Yang-Mills theory, only (anti) self-dual solutions of the equations of motion contribute to $\Theta_{\mu\nu} = 0$.

The Yang-Mills Lagrangian in the Euclidean metric is

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a. \quad (1)$$

We introduce the matrices

$$A_\mu = A_\mu^a \tau_a, \quad F_{\mu\nu} = F_{\mu\nu}^a \tau_a, \quad (2)$$

where the τ^a are the Pauli matrices :

$$[\tau_a, \tau_b] = i \epsilon_{abc} \tau_c. \quad (3)$$

Then :

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu] \quad (4)$$

and the equations of motion are :

$$\partial_\mu F_{\mu\nu} = -ig [A_\mu, F_{\mu\nu}] \quad (5)$$

^{*)} In a recent paper by Polyakov ²⁾ it is claimed, without published proof, that $\Theta_{\mu\nu} = 0$ implies the (anti) self-duality of the field tensor.

The dual tensor

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta} \quad (6)$$

satisfies the equations of motion trivially (Bianchi identity). The energy momentum tensor can be written as

$$\theta_{\mu\nu} = \frac{1}{4} \text{Tr} \left[(F - \tilde{F})_{\mu\lambda} (F + \tilde{F})_{\nu\lambda} + \mu \leftrightarrow \nu \right]. \quad (7)$$

From the form of $\theta_{\mu\nu}$ it is obvious that for any group $SU(N)$, $F_{\mu\nu} = \pm \tilde{F}_{\mu\nu}$ implies $\theta_{\mu\nu} = 0$. We shall show that the reciprocal statement holds for the $SU(2)$ gauge group. First we introduce

$$\begin{aligned} A_{\mu\nu} &= (F - \tilde{F})_{\mu\nu} \\ S_{\mu\nu} &= (F + \tilde{F})_{\mu\nu} \end{aligned} \quad (8)$$

so that $\tilde{A} = -A$ and $\tilde{S} = +S$. Clearly,

$$\theta_{\mu\nu} = \frac{1}{4} \text{Tr} \left[A_{\mu\lambda} S_{\nu\lambda} + \mu \leftrightarrow \nu \right] \quad (9)$$

and if we calculate the trace we find :

$$\theta_{\mu\nu} = \frac{1}{4} \left[A_{\mu\lambda}^a S_{\nu\lambda}^a + \mu \leftrightarrow \nu \right]. \quad (10)$$

We now take six independent components A_{1k}, S_{1k} , where $k=2, 3$ or 4 . Because of the (anti) self-duality properties of (A) S all other components can be expressed in terms of these six. If we write $\theta_{\mu\nu}$ in terms of these components, we find that $\theta_{\mu\nu} = 0$ is equivalent to

$$\sum_{a=1}^3 A_{1\lambda}^a S_{1\mu}^a = 0 \quad (11)$$

where λ and μ independently take the values 2, 3 or 4. These are nine orthogonality conditions for the six vectors $\vec{A}_{1\lambda}, \vec{S}_{1\mu}$. To satisfy these conditions, we must have one of four situations :

- (i) $\vec{A}_{1\lambda} = 0, \lambda = 2, 3, 4$, but then $F = +\tilde{F}$,
- (ii) $\vec{S}_{1\lambda} = 0, \lambda = 2, 3, 4$, but then $F = -\tilde{F}$,
- (iii) the three vectors $\vec{A}_{1\lambda}$ are colinear, and the vectors $\vec{S}_{1\mu}$ are in the plane perpendicular to the direction of $\vec{A}_{1\lambda}$,
- (iv) the three vectors $\vec{S}_{1\lambda}$ are colinear, and the vectors $\vec{A}_{1\mu}$ are in the plane perpendicular to the directions of $\vec{S}_{1\lambda}$.

We shall work out case (iii).

Then we write :

$$\begin{aligned} A_{\mu\nu} &= a_{\mu\nu} (\hat{n} \cdot \vec{e}) \\ S_{\mu\nu} &= \vec{S}_{\mu\nu} \cdot \vec{e} \end{aligned} \quad (12)$$

with

$$\hat{n} \cdot \vec{S}_{\mu\nu} = 0, \quad \hat{n} \cdot \hat{n} = 1.$$

F and $\tilde{F}$ are as follows :

$$\begin{aligned} F_{\mu\nu} &= \frac{1}{2} [a_{\mu\nu} (\hat{n} \cdot \vec{e}) + \vec{S}_{\mu\nu} \cdot \vec{e}], \\ \tilde{F}_{\mu\nu} &= \frac{1}{2} [-a_{\mu\nu} (\hat{n} \cdot \vec{e}) + \vec{S}_{\mu\nu} \cdot \vec{e}]. \end{aligned} \quad (13)$$

Now consider the following element of $SU(2)$:

$$U = 2i (\hat{n} \cdot \vec{e}). \quad (14)$$

Clearly we have

$$U^2 = -1$$

and

$$\begin{aligned}
 U \tilde{F}_{\mu\nu} U^{-1} &= 4(\hat{n} \cdot \vec{\tau})^{\frac{1}{2}} [-a_{\mu\nu}(\hat{n} \cdot \vec{\tau}) + \vec{S}_{\mu\nu} \cdot \vec{\tau}] (\hat{n} \cdot \vec{\tau}) \\
 &= -\frac{1}{2} [a_{\mu\nu}(\hat{n} \cdot \vec{\tau}) + \vec{S}_{\mu\nu} \cdot \vec{\tau}] \\
 &= -F_{\mu\nu}.
 \end{aligned} \tag{15}$$

Starting from the other possibility [case (iv)] we find that

$$F_{\mu\nu} = U \tilde{F}_{\mu\nu} U^{-1}$$

with the same U , where $\hat{n}$ is now the direction of the vectors $\vec{S}_{1\lambda}$.

So we have found that $\Theta_{\mu\nu} = 0$ implies either $F_{\mu\nu} = \pm \tilde{F}_{\mu\nu}$ or

$$F_{\mu\nu} = \pm U \tilde{F}_{\mu\nu} U^{-1} \tag{16}$$

with $U^2 = -1$. This shows that $\Theta_{\mu\nu} = 0$ by itself does not imply the (anti) self-duality of $F_{\mu\nu}$. Using the equations of motion we shall now show that (16) implies $F = \pm \tilde{F}$.

Since U depends on x , it is useful to make a gauge transformation such that the vector $\hat{n}$ is, for every point x , in the z direction.

Diagonalizing U gives

$$U(x) = V^{-1}(x) \begin{pmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{pmatrix} V(x). \tag{17}$$

Since $U^2 = -1$, and $\text{Tr } U = 0$, we have $\lambda_1 = +i, \lambda_2 = -i$, independent of x .

So :

$$U(x) = i V^{-1}(x) \sigma_z V(x). \tag{18}$$

We make the gauge transformation V , which gives :

$$F'_{\mu\nu} = V F_{\mu\nu} V^{-1} = \pm \sigma_z \tilde{F}'_{\mu\nu} \sigma_z. \tag{19}$$

In this new gauge, we have :

$$\begin{aligned} F'_{\mu\nu} &= \pm \tilde{F}'_{\mu\nu} \\ F'^{x,y}_{\mu\nu} &= \mp \tilde{F}'^{x,y}_{\mu\nu} \end{aligned} \quad (20)$$

These conditions obviously satisfy $\Theta_{\mu\nu} = 0$, but do not correspond to the absolute minimum of the action, which is given by the topological charge. However, the equations of motion force us to have either $F'^Z_{\mu\nu} = 0$ or $F'^X_{\mu\nu} = F'^Y_{\mu\nu} = 0$, in the cases (iii) and (iv) which we consider.

Indeed, from the Bianchi identities and (19)

$$\partial_\mu \tilde{F}'_{\mu\nu} = ig [A'_\mu, \tilde{F}'_{\mu\nu}] = \mp ig [A'_\mu, \sigma_2 F'_{\mu\nu} \sigma_2]. \quad (21)$$

Using again (19), and the equations of motion (4) for F' we find also

$$\partial_\mu \tilde{F}'_{\mu\nu} = \mp \sigma_2 F'_{\mu\nu} \sigma_2 = \mp ig \sigma_2 [A'_\mu, F'_{\mu\nu}] \sigma_2. \quad (22)$$

Therefore

$$[A'_\mu, F'_{\mu\nu}] = \sigma_2 [A'_\mu, \sigma_2 F'_{\mu\nu} \sigma_2] \sigma_2 = [\sigma_2 A'_\mu \sigma_2, F'_{\mu\nu}],$$

so that

$$[F'_{\mu\nu}, A'_\mu - \sigma_2 A'_\mu \sigma_2] = 0 \quad (23)$$

Now if $A'_\mu - \sigma_2 A'_\mu \sigma_2 = 0$, in which case A'_μ has the form $A'^Z_\mu \sigma_2$, we find that $F'^X_{\mu\nu} = F'^Y_{\mu\nu} = 0$. If $A'_\mu - \sigma_2 A'_\mu \sigma_2 \neq 0$, the component $F'^Z_{\mu\nu}$ must vanish, since $\det F'^Z = 0$ implies $F'^Z = 0$ if (20) holds.

In either case we have $F'_{\mu\nu} = \pm \tilde{F}'_{\mu\nu}$, because of the conditions (20). Since the result is gauge invariant all solutions of the Euclidean equations of motion that satisfy $\Theta_{\mu\nu} = 0$ are either self-dual or anti-self-dual.

This also holds for a $U(1)$ gauge group, because of the conditions (11) which reduce to $A_{1\lambda} S_{1\mu} = 0$. In the case of a larger gauge group $SU(N)$, $N > 2$, conditions (11) do not seem to be sufficiently restrictive for the same method of work, because of the dimension of the space in which $\vec{S}_{1\lambda}$ and $\vec{A}_{1\lambda}$ are represented $\mathbb{R}^8$ for $SU(3)$. On the other hand an explicit solution which is not (anti) self-dual can be constructed for the $SU(2) \times SU(2)$ part of $SU(4)$ by taking a self-dual configuration for one $SU(2)$, and an anti-self-dual one for the other $SU(2)$ component³⁾. Such a counter example also exists for the $SU(2) \times U(1)$ part of $SU(3)$.

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