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ALMOST OPTIMALITY OF AN AXIOMATIC BOUND FOR $\pi^0\pi^0$ SCATTERING^{*)}

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ABSTRACT

We produce a $\pi^0\pi^0 \rightarrow \pi^0\pi^0$ amplitude $F(s,t)$ which fulfils all the analyticity, crossing and unitarity conditions used in the derivation of rigorous absolute bounds, and such that $F(4/3, 4/3) = 2.62$. When compared to the axiomatic result of Lopez and Mennessier, $-8.2 < F(4/3, 4/3) < 2.75$, this example shows that no further significant improvement of the upper bound can be expected.

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1. INTRODUCTION

The first absolute bounds on pion scattering amplitudes as rigorous consequences of the axioms of local field theory were obtained by Martin in 1964¹⁾. During more than 10 years, considerable effort has been devoted to the improvement of these bounds¹⁾⁻⁶⁾ merging into quite a substantial outcome. Considering, for instance, the time evolution of the upper and lower bounds on the $\pi^0\pi^0 \rightarrow \pi^0\pi^0$ amplitude $F(s,t)$ at the symmetry point $s = t = 4/3$ (Fig. 1), one sees that roughly one order of magnitude has finally been gained. The best present bounds^{*)}

$$- 8.2 < F(4/3, 4/3) < 2.75 \quad (1.1)$$

have been derived by a rather powerful method⁶⁾, and some recent attempts to still improve them have failed, suggesting that some ultimate limit might have been reached. Yet, the bounds (1.1) are outside the range of phenomenologically reasonable values by a factor of about 3 to 10, and have no constraining power in model building. It is therefore highly desirable to know whether the bounds (1.1) (and similar ones, e.g., on scattering lengths) turn out to be close to the best possible ones within the axiomatic framework. Of course, this can be answered only by constructing amplitudes which meet all the axiomatic requirements and almost saturate inequalities (1.1). In general, one can foresee such a construction being quite arduous, due to the difficulty in unitarizing "large" amplitudes with correct analyticity properties (e.g., via some iteration scheme "à la Atkinson"). One has to realize, however, that the elastic unitarity is not used in the derivation of axiomatic bounds, which makes the construction above much easier. Indeed, we have found that the upper bound (1.1) can be almost saturated by (surprisingly simple!) explicit amplitudes. The aim of the present paper is to show how this is achieved. In Section 2 we select a class of crossing symmetric functions whose proper analytic structure in s and t makes them good candidates for admissible amplitudes. The proof is completed in Section 3, where we show that the inelastic unitarity condition is fulfilled at all energies for appropriate choices of the free parameters. An optimization with respect to these parameters then leads to our best result, $F(4/3, 4/3) = 2.62$, the significance of which is briefly discussed in Section 4.

*) We use the normalization $F = \sqrt{s/(s-4)} \sum_{\ell} (2\ell+1) \exp(i\delta_{\ell}) \sin\delta_{\ell} P_{\ell}(z)$.

2. CHOICE OF AMPLITUDES

Clearly enough, in order to maximize the scattering amplitude at points inside the Mandelstam triangle, one has to select functions $F(s,t)$ as large as possible (compatible with unitarity) in the physical neighbourhood of the threshold $s = 4$. The most efficient way of doing this is to enforce the (dominant) S wave to have a threshold "bound state". In fact, such an abnormal behaviour already shows up in the derivation of the upper bound (1.1): the method of Lopez and Mennessier⁶⁾ involves the construction of a set of "approximate" partial waves^{*)} whose i) S wave exhibits a threshold bound state, ii) higher wave contribution to $F(4/3, 4/3)$ is very small.

These considerations lead us to temporarily forget the $\ell \geq 2$ waves and to introduce, in a first step, an appropriate family of amplitudes of the form:

$$F^C(s,t) = \rho \left[C(s) + C(t) + C(u) \right] \quad , \quad (\rho > 0) \quad , \quad (2.1')$$

where

$$C(s) = \frac{1}{\sqrt{4-s}} \sum_{n=1}^N \frac{\alpha_n p_n}{p_n + \sqrt{4-s}} \quad , \quad (p_n, \alpha_n > 0 ; \sum \alpha_n^2 = 1) \quad . \quad (2.1'')$$

Such a form has the correct analytic structure (including crossing) and exhibits the desired threshold behaviour due to the factor $(4-s)^{-1/2}$ contained in $C(s)$. The remaining sum in $C(s)$ is merely a convenient choice of parametrization, and defines our class of functions $F^C(s,t)$. Notice that the expression (2.1) is nothing but a specific Cini-Fubini approximation⁷⁾ of the amplitude: it has the Mandelstam analyticity with only single spectral functions. Hence the absorptive parts are independent of the transfer variable and the $\ell \geq 2$ partial waves:

$$f_\ell^C(s) = \sqrt{\frac{s-4}{s}} \frac{1}{2} \int_{-1}^1 dz P_\ell(z) F^C(s,t) \quad , \quad (z = 1 + \frac{2t}{s-4}) \quad (2.2)$$

have no imaginary parts but non-vanishing real parts (thus violating unitarity). As for the S wave $f_0^C(s) = r_0^C(s) + i a_0^C(s)$, its explicit form is readily found:

*) "approximate" in the sense that they do not sum up into an acceptable amplitude.

$$\begin{cases} a_0^C(s) = \frac{\rho}{\sqrt{s}} \sum_{n=1}^N \frac{\alpha_n p_n^2}{s-4+p_n^2} \\ r_0^C(s) = \rho \sqrt{\frac{s-4}{s}} \sum_{n=1}^N \alpha_n p_n \left(\frac{4}{s-4} \text{Log} \frac{\sqrt{s}+p_n}{2+p_n} - \frac{1}{s-4+p_n^2} \right) \end{cases} \quad (2.3)$$

Now, the important point is that the unitarity of the S wave, $|f_0(s)|^2 \leq a_0(s)$ $\forall s \geq 4$, is certainly obeyed by the expressions (2.3) when ρ is small enough: the only trouble could arise from the asymptotic region $s \rightarrow \infty$, and a simple inspection of the asymptotic behaviours in Eq. (2.3) shows that it makes no problem indeed. Then, in order to test quantitatively the efficiency of our ansatz (2.1), it is worth performing a first maximization of $F^C(4/3, 4/3)$ with respect to the parameters ρ , p_n and α_n under the constraint $[a_0^C(s)]^2 + [r_0^C(s)]^2 \leq a_0^C(s)$, $s \geq 4$. For $N = 2$, one obtains $\text{Max } F^C(4/3, 4/3) = 2.38$, and also finds that the corresponding $\ell \geq 2$ waves (purely real!) are very small.

This is quite encouraging and prompts us to try to cure the defects of the amplitude (1.1), by perturbing it in such a way that the $\ell \geq 2$ partial waves acquire a positive imaginary part without the nice features of $F^C(s, t)$ be destroyed. A natural way of doing this is to combine $F^C(s, t)$ with a "small" piece $\epsilon F^M(s, t)$ having non-vanishing (positive) double spectral functions:

$$F(s, t) = (1 - \epsilon) F^C(s, t) + \epsilon F^M(s, t) \quad (2.4)$$

and to readjust the parameters of $F^C(s, t)$. In this manner, we can hope to recover a fully unitary amplitude (eventually, the adjunction of $F^M(s, t)$ will also turn out to improve the maximum value of $F(4/3, 4/3)$!). For $F^M(s, t)$, we shall make the simple, no parameter, choice:

$$F^M(s, t) = \rho \left[M(s) M(t) + M(t) M(u) + M(u) M(s) \right], \quad M(s) = \frac{1}{\sqrt{4-s}} \quad (2.5)$$

This amplitude has a Mandelstam representation with a factorized double spectral function $\rho(s', t') = \rho[(s'-4)(t'-4)]^{-\frac{1}{2}}$. Of course, such a spectral function, the support of which covers the whole region $s' \geq 4$, $t' \geq 4$, is not physically

acceptable. As far as the axiomatic analyticity is concerned, it implies that the complete amplitude (2.4) enjoys the required properties, except for its absorptive part $A(s,t)$ which is holomorphic in t within too small ellipses. But this is irrelevant here, inasmuch as the full analyticity of $A(s,t)$ is not taken into account in the derivation of the bounds (1.1) (we shall come back to this point in Section 4). Also the somewhat unpleasant divergence of $M(s)$ at $s = 4$ can be easily removed without altering the final results, e.g. by taking $M(s) = 1/(\eta + \sqrt{4-s})$ with $\eta > 0$ arbitrarily small.

Hence, we can consider functions of the form (2.4) with F^C and F^M defined by Eqs. (2.1) and (2.5) as amplitudes well adapted to our purpose. The inelastic unitarity condition remains to be imposed.

3. CHECK OF UNITARITY AND OPTIMIZATION

In the physical region $s \geq 4$ the partial waves $f_\ell(s) = r_\ell(s) + ia_\ell(s)$ of the amplitude (2.4) take the form

$$\begin{cases} a_\ell(s) = \frac{p}{\sqrt{s}} \left[(1-\epsilon) \sum_{n=1}^N \frac{\alpha_n p_n^2}{s-4+p_n^2} J_{\ell 0} + \epsilon \sqrt{\frac{2}{s+4}} I_\ell(s) \right], \\ r_\ell(s) = p \sqrt{\frac{s-4}{s}} \left\{ (1-\epsilon) \sum_{n=1}^N \left[-\frac{\alpha_n p_n}{s-4+p_n^2} J_{\ell 0} + \frac{2\alpha_n}{s+4} J_\ell(s, p_n) \right] + \frac{\epsilon}{s+4} K_\ell(s) \right\}, \end{cases} \quad (3.1)$$

where

$$\begin{cases} I_\ell(s) \equiv \int_{-1}^1 dz \frac{P_\ell(z)}{\sqrt{1 - \frac{s-4}{s+4} z}} = \frac{2\sqrt{2}}{2\ell+1} \frac{\sqrt{s+4}}{\sqrt{s+2}} \left(\frac{\sqrt{s-2}}{\sqrt{s+2}} \right)^\ell, \\ J_\ell(s) \equiv \int_{-1}^1 dz P_\ell(z) \frac{p}{1 - \frac{s-4}{s+4} z + p \sqrt{\frac{2}{s+4}} \sqrt{1 - \frac{s-4}{s+4} z}}, \\ K_\ell(s) \equiv \int_{-1}^1 dz \frac{P_\ell(z)}{\sqrt{1 - \left(\frac{s-4}{s+4} z \right)^2}}. \end{cases} \quad (3.2)$$

The parameters ρ , ε , p_n and α_n must be restricted in such a way that:

$$a_\ell^2(s) + r_\ell^2(s) \leq a_\ell(s) \quad , \quad s \geq 4 \quad , \quad \ell = 0, 2, 4, \dots \quad (3.3)$$

But one has first to make sure that this is possible at all, i.e., that the behaviour of the amplitudes (3.1) when $s \rightarrow 4$, $s \rightarrow \infty$ or (and) $\ell \rightarrow \infty$ is compatible with (3.3). If this holds, then clearly conditions (3.3) can be imposed for all s and ℓ by choosing ρ small enough (notice that the $a_\ell(s)$'s do not vanish except at $s = 4$ for $\ell \geq 2$). We shall show that the inequalities (3.3) are satisfied in a large (parameter dependent) region of the (s, ℓ) plane. This follows from the fact that the real parts (3.1) are bounded in terms of the imaginary parts. Actually, the expressions (3.2) can be recast in a Froissart-Gribov form, owing to the proper analyticity of the integrands. For example:

$$I_\ell(s) = \frac{1}{\pi} \int_{\frac{s+4}{s-4}}^{\infty} dz \frac{Q_\ell(z)}{\sqrt{\frac{s-4}{s+4} z - 1}} \quad , \quad (3.4)$$

$$J_\ell(s, p) = \frac{1}{\pi} \int_{\frac{s+4}{s-4}}^{\infty} dz \frac{Q_\ell(z)}{\sqrt{\frac{s-4}{s+4} z - 1}} \frac{p^2 \sqrt{\frac{2}{s+4}}}{p^2 \frac{2}{s+4} + \left(\frac{s-4}{s+4} z - 1 \right)} \quad . \quad (3.5)$$

Comparing these two expressions and using the positivity of $Q_\ell(z)$ for $z > 1$, we immediately see that for all s and ℓ :

$$0 \leq J_\ell(s, p) \leq \sqrt{\frac{s+4}{2}} I_\ell(s) \quad . \quad (3.6)$$

One obtains similarly

$$0 \leq K_\ell(s) \leq \sqrt{2} I_\ell(s) \quad . \quad (3.7)$$

Inserting these inequalities into Eqs. (3.1) we get the desired bounds, valid for $\ell \geq 2$:

$$0 \leq r_\ell(s) \leq \rho \sqrt{\frac{2(s-4)}{s(s+4)}} \left[(1-\epsilon) \sum_n \alpha_n + \frac{\epsilon}{\sqrt{s+4}} \right] I_\ell(s) \quad (3.8)$$

with

$$a_\ell(s) = \rho \epsilon \sqrt{\frac{2}{s(s+4)}} I_\ell(s) \quad (3.9)$$

Equations (3.8) and (3.9) allow us to write down a sufficient condition for ensuring (3.3)

$$\frac{2\rho}{s+4} \left\{ \frac{\epsilon^2}{s} + \frac{s-4}{s} \left[(1-\epsilon) \sum_n \alpha_n + \frac{\epsilon}{\sqrt{s+4}} \right]^2 \right\} I_\ell(s) \leq \epsilon \sqrt{\frac{2}{s(s+4)}} \quad (3.10)$$

Finally, using the obvious inequality [from the first Eq. (3.2)]

$$I_\ell(s) \leq \frac{2\sqrt{2}}{2\ell+1} \sqrt{s+4} \frac{(s-4)^2}{(\sqrt{s+2})^5} \quad , \quad (3.11)$$

and making some elementary majorizations, we can transform the condition (3.10) into:

$$2\ell+1 \geq \frac{4\rho}{\epsilon} \left\{ \frac{\epsilon^2}{4} + \left[(1-\epsilon) \sum_n \alpha_n + \frac{\epsilon}{2\sqrt{2}} \right]^2 \right\} \left(\frac{s-4}{s} \right)^2 \quad , \quad (\ell \geq 2) \quad (3.12)$$

Hence unitarity is guaranteed in the region of the (s, ℓ) plane defined by Eq. (3.12). As expected, this region covers all points $s \geq 4$, $\ell \geq 2$ when ρ is sufficiently small. For larger values of ρ , Eq. (3.12) tells us that only a finite number $(0 \leq \ell \leq \ell_{\max})$ of partial waves remain to be checked, and that the threshold region $s \rightarrow 4$ is safe for all $\ell \geq 0$ (the threshold behaviour of

the S-wave is compatible with unitarity by construction). Thus, we are left with the only limit $s \rightarrow \infty$ at fixed ℓ ($\leq \ell_{\max}$). This is readily settled in view of the asymptotic behaviour of $f_\ell(s)$, as extracted from Eq. (3.1):

$$\begin{cases} a_\ell(s) \underset{s \rightarrow \infty}{\simeq} \frac{4p\varepsilon}{2\ell+1} \frac{1}{s} \\ r_\ell(s) \underset{s \rightarrow \infty}{\simeq} 2p(1-\varepsilon) \sum_n \alpha_n p_n \frac{\log s}{s} \end{cases} \quad (\ell \geq 0) \quad (3.13)$$

Eq. (3.13) implies the existence of $s_{\max}$ such that condition (3.3) is fulfilled for $s > s_{\max}$ ($\ell \leq \ell_{\max}$).

To summarize, we have shown that i) there is a range of values of the parameters for which all the partial waves are confined within the unitarity circle, ii) in practice, given a set of such values, the unitarity has to be checked numerically only in a finite region $4 \leq s \leq s_{\max}$, $0 \leq \ell \leq \ell_{\max}$.

We are now entitled to perform numerical evaluations on the computer. Practically, one finds that the actual limitation on the values of $F(4/3, 4/3)$ are entirely due to the S and D waves. Therefore, one can proceed as follows.

i) Fix N and maximize the quantity

$$F(4/3, 4/3) = p \left[\frac{3}{2} \sqrt{\frac{3}{2}} (1-\varepsilon) \sum_{n=1}^N \frac{\alpha_n p_n}{p_n + 2\sqrt{\frac{3}{2}}} + \frac{9\varepsilon}{8} \right] \quad (3.14)$$

under the constraint (3.3) for $\ell = 0$ and $\ell = 2$ (which is relatively easy because the first two partial waves have rather simple, explicit forms). This provides us with optimal values of ρ , ε , p_n and α_n , and corresponding cut-off $s_{\max}$ and $\ell_{\max}$.

ii) Check that the partial waves $4 \leq \ell \leq \ell_{\max}$ are inside the unitarity circle for $4 \leq s \leq s_{\max}$. For $N = 2$ the result is

$$\text{Max } F(4/3, 4/3) = 2.62 \quad (3.15)$$

This maximum is attained for $\rho = 1.72$, $\varepsilon = 0.319$, $p_1 = 4$, $p_2 = 14$, $\alpha_1 = 0.961$ and $\alpha_2 = 0.276$. It shows little sensitivity to the precise shape of the amplitude $F^C(s,t)$ (determined by N , p_n , α_n) and even to the relative weight (ε) of $F^M(s,t)$. For example, taking $N = 0$ gives $\text{Max } F(4/3, 4/3) = 2.58$, whereas $N = 3$ leaves (3.15) practically unchanged. The first two partial waves of the optimal amplitude are depicted in Fig. 2. One observes that $f_0(4) = i$ and that the two contributions of F^C and F^M to the S wave combine nicely to build up an almost elastic amplitude in quite an extended neighbourhood of the threshold. This is the reason for the high value of $F(4/3, 4/3)$. The D wave is already quite small. It makes a clockwise turning loop typical of all the higher waves and touches the unitarity circle for $s = 380$.

4. DISCUSSION

We have constructed a scattering amplitude $F(s,t)$ compatible with the axiomatic requirements and taking the value 2.62 at the symmetry point. This is our main result, which has to be compared with the upper bound (1.1). It definitively answers (by the affirmative) the question of the practical optimality of the existing axiomatic upper bounds on $F(s,t)$ in the unphysical region, at least within the present rules of the game.

A similar conclusion is reached in the unsubtracted case. To see this, let us consider $F^M(s,t)$ as defined in Eq. (2.5). It behaves like $s^{-\frac{1}{2}}$ when s goes to infinity at fixed t , and hence satisfies unsubtracted dispersion relations (in contradistinction to the Cini-Fubini piece F^C). Taking now $\varepsilon = 1$ (i.e., $F = F^M$) and $\rho = 2$ [i.e., $f_0(4) = i$ according to Eqs. (3.1)], we find from Eq. (3.14) that $F(4/3, 4/3) = 2.25$. On the other hand, using $\varepsilon = 1$, $\rho = 2$ in the sufficient condition (3.10) (which now applies also to the S wave), we readily see that unitarity is fulfilled for all $s \geq 4$ and $\ell \geq 0$. Therefore, the upper bound on $F(4/3, 4/3)$ cannot be better than 2.25 in the unsubtracted case*).

As for the saturating amplitudes of the form (2.1), (2.4), (2.5), needless to say that they have no physical meaning, but were just devised to achieve our particular goal. This is quite apparent on the S and D waves (Fig. 2) which

*) This seems to contradict the bound of Ref. 2) $F_{\text{unsubtr}}(4/3, 4/3) \leq 1.8$. We have recomputed the latter and found 4.75.

are far from being realistic, especially as regards their low energy behaviour. In particular, the "rectangular" support of the double spectral functions in $F^M(s,t)$ is responsible for the strong inelasticity of the partial waves $\ell \geq 2$ at threshold. In view of the fact that the Cini-Fubini piece by itself (which has no double spectral functions) gives already $F^C(4/3,4/3) = 2.38$, we think however that our result should not be appreciably affected by restoring a correct support. Moreover, the proper way of doing this would be to impose the elastic unitarity below the inelastic threshold. This would force us to treat explicitly the three isospin amplitudes in the three channels and lead to a considerably more difficult problem.

In this paper we have not touched on the question of a possible saturation of the lower bounds [like the one in Eq. (1) or for the scattering length]. The reason is that it cannot be done as simply as for the upper bounds. Actually, subtractions are first of all unavoidable since the amplitude must take negative values in the unphysical region. Then, if one uses a Mandelstam representation, one can expect that the double spectral functions will assume a rather complicated form in order to prevent an asymptotic violation of unitarity. In spite of these difficulties it would be worth trying to fill the gap also for the lower bounds. A positive outcome would close a chapter of the axiomatic results in pion-pion scattering.

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Figure captions

Fig. 1 : Upper (UB) and lower (LB) axiomatic bounds on $\pi^0\pi^0 \rightarrow \pi^0\pi^0$ amplitude $F(s,t)$ at the symmetry point $s = t = 4/3$, versus time. The numbers (1)-(6) refer to the corresponding numbers of the references. The hatched strip represents the range of values of $F(4/3,4/3)$ covered by phenomenological models.

Fig. 2 : Argand diagrams of the S and D waves of the amplitude (2.4) saturating Eq. (3.15).
a) S wave. Broken line: contribution due to $(1-\epsilon)F^C$. Dotted line: contribution due to ϵF^M .
b) D wave.

Fig. 1

Fig. 2