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ON-SHELL THREE-BODY MULTIPLE SCATTERING EQUATIONS

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A B S T R A C T

A physically intuitive method for constructing exact on-shell multiple scattering equations is outlined, and some examples of such equations are considered in some detail. In particular, we obtain as a special case the same on-shell equation that was recently derived using a three-body generalization of the two-body Kowalski Noyes method, and another special case is identified as an improved version of Brayshaw's boundary condition formalism (BCF) equation. The rôle of on-shell versus off-shell two-body input is discussed.

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1. - INTRODUCTION

The modern development of three-body quantum mechanics started when the three-body scattering problem was reformulated as a multiple scattering problem. In addition to the all-important mathematical advantages of this reformulation, it provides a fairly straightforward physical picture of a three-body process as a succession of two-body collisions. Since these two-body subprocesses in general occur in the neighbourhood of the third particle, the initial and final two-body relative momenta are not restricted to be on the energy-shell defined by the subsystem energy. Although the subprocesses therefore can take place under conditions far from those of a physical two-body scattering, this picture of three-body processes is widely used.

A different picture of the three-body process is usually connected with the use of separable interactions and quasi-particle ideas. While in the previous picture the particles were freely propagating between the collisions, the emphasis is in this case on the propagation of two-body clusters. An initial two-body cluster (e.g., a bound state) and the third particle scatter into a new cluster state (e.g., another bound pair and a free particle, a one-particle exchange process) which propagates as a cluster state until there is another scattering, etc. The success of this picture is based on the finding that in many cases it is sufficient to represent two-body continuum states by a small number of cluster states.

Instead of using approximate two-body cluster states, one can also treat the two-body continuum in all generality. In such a case a free particle and an interacting pair, described by a two-body scattering state, will emerge after each collision. This state of the three particles propagates until there is a new collision, from which a similar state in another two-body channel emerges, etc.

In the present paper we are going to exploit this last picture of the three-body scattering process. The basic idea will be that for short range forces there is a good probability that the wave emerging from a particular subsystem scattering is already asymptotic, i.e., a free wave, when it hits the next scattering potential. In other words, one subprocess is often completed before the next one starts.

Since the off-shell complications of the three-body scattering process are related to rescatterings before the previously scattered wave is asymptotic (before the previous scattering is completed) one can expect that the implementation of the above idea will lead to an on-shell theory for the three-body process.

It is, of course, not our intention to drop the contributions from off-shell rescatterings. Rather, an effective on-shell scattering model will be constructed, in which the actual on-shell rescattering term is modified so as to account also for the effect of off-shell rescatterings in an (in principle) exact manner. It will be shown that the required modification can be parametrized in terms of a real function. This is a unitary constraint, which means that also with an approximate evaluation of this real function the corresponding three-body scattering amplitudes will satisfy the unitarity equations exactly. It will also be argued that the solution of the three-body problem will (under certain circumstances) only depend weakly on the detailed structure of this parameter function. The important problem of how to construct this off-shell modification is discussed in detail for the special case considered in Section 5.

The relationship between the present on-shell formulation of the three-body scattering theory and other such formulations is discussed in Sections 5 and 6. We find in particular that the generalized Kowalski-Noyes method of Ref. 1), and (an improved version of) the BCF of Brayshaw^{2),3)} are obtained as special cases of the present formalism.

2. - NOTATION

For simplicity of presentation, we will treat a (non-relativistic) three-particle system with pair interactions in s waves only. The particles will be taken as distinguishable. Only general $1+1+1 \rightarrow 1+1+1$ scattering amplitudes will appear explicitly, with the understanding that other amplitudes (e.g., for a $2+1 \rightarrow 2+1$ process) can be obtained from these by taking residues at appropriate poles.

In this case the Faddeev three-body equations take the form

$$|\Psi_{\beta}^{+}\rangle = (1 - G_{\beta}(E+i0)V_{\beta})|p_1^0 q_1^0\rangle \delta_{\beta 1} - G_{\beta}(E+i0)V_{\beta} \sum_{\gamma} \bar{\delta}_{\beta\gamma} |\Psi_{\gamma}^{+}\rangle \quad (2.1)$$

where $|\Psi\rangle = \sum_{\beta} |\Psi_{\beta}^{+}\rangle$ is the full three-body state vector, $G_{\beta}(E+i0)$ is the two-body resolvent operator in three-body space $G_{\beta}(E+i0) = (H_0 + V_{\beta} - E - i0)^{-1}$ and $\bar{\delta}_{\beta\gamma} = 1 - \delta_{\beta\gamma}$. The initial state has arbitrarily been assigned to channel "1".

The components T^{β} of the $1+1+1 \rightarrow 1+1+1$ transition operator $T = \sum_{\beta} T^{\beta}$ are related to $|\Psi_{\beta}^{+}\rangle$ through

$$|\Psi_{\beta}^{+}\rangle = |p_1^0 q_1^0\rangle \delta_{\beta 1} - G_0(E+i0) T^{\beta}(E+i0) |p_1^0 q_1^0\rangle \quad (2.2)$$

Using $\vec{q}_{\beta}$ as the relative momentum for the β pair (if $\beta=1$, particles 2 and 3 form the β pair) and $\vec{p}_{\beta}$ as the relative momentum of particle β with respect to this pair, the total energy of the system is $\tilde{p}_{\beta}^2 + \tilde{q}_{\beta}^2$, any choice of β . Here $\tilde{p}_{\beta}^2 = p_{\beta}^2/2\mu_{\beta}$, $\tilde{q}_{\beta}^2 = q_{\beta}^2/2\mu_{\beta}$, $\mu_{\beta} = m_{\gamma}m_{\alpha}/(m_{\gamma} + m_{\alpha})$ and $\mu_{\beta} = m_{\beta}(m_{\gamma} + m_{\alpha})/(m_{\beta} + m_{\gamma} + m_{\alpha})$. In particular, in Eqs. (2.1) and (2.2), $E = \tilde{p}_1^2 + \tilde{q}_1^2$, where p_1^0 and q_1^0 are the initial state momenta. The relations between the different choices of three-body momenta are easily read off from the momentum triangle of Fig. 1.

For s waves, the free particle operator $G_0(E+i0) = (H_0 - E - i0)^{-1}$ has the co-ordinate space representation

$$G_0(r_{\beta}, r'_{\beta}; r_{\beta}^i, r_{\beta}^f; E+i0) = \frac{2}{\pi} \int_0^{\infty} p dp j_0(pr_{\beta}) G_0(r_{\beta}, r'_{\beta}; \tilde{\kappa}^2) j_0(pr_{\beta}^i) \quad (2.3)$$

where $G_0(r_{\beta}, r'_{\beta}; \tilde{\kappa}^2)$ is the two-body Green's function

$$G_0(r_{\beta}, r'_{\beta}; \tilde{\kappa}^2) = 2\mu_{\beta} \kappa h_0^{+}(\mu_{\beta} r_{\beta}^>) j_0(\mu_{\beta} r_{\beta}^i) \quad (2.4)$$

Here, $r_{\beta}^> = \max(r_{\beta}, r_{\beta}^i)$, $r_{\beta}^< = \min(r_{\beta}, r_{\beta}^i)$, $\kappa = \sqrt{2\mu(E - \tilde{p}^2)}$ and $\tilde{\kappa}^2 = \frac{1}{2\mu} \kappa^2$. If $\tilde{p}^2 > E$, we have $\text{Im } \kappa > 0$, so that $h_0^{+}(\mu r) = (e^{i\mu r}/\mu r)$ is exponentially decreasing when $p \rightarrow \infty$. Similarly, the operator $G_{\beta}(E+i0)$ has the representation

$$G_p(r_p, y_p; r'_p, y'_p; E + i0) = \frac{2}{\pi} \int_0^\infty p^2 dp \, j_0(py_p) G_p(r_p, r'_p; \tilde{\kappa}^2) j_0(py'_p) \quad (2.5)$$

where

$$G_p(r_p, r'_p; \tilde{\kappa}^2) = 2\mu_p \kappa \, f_0^+(\kappa, r_p^>) \, \psi_0^+(\kappa, r_p^<) \quad (2.6)$$

In Eq. (2.6), ψ_0^+ is the usual outgoing wave scattering solution to the s wave Schrödinger equation, and f_0^+ is the solution to the same equation that is uniquely determined by the condition $f_0^+(\kappa, r) \rightarrow h_0^+(\kappa r)$, $r \rightarrow \infty$ (the Jost solution). The Green's function in (2.6) is for a large class of potentials an analytic function of κ for $\text{Im } \kappa \geq 0$, except for two-body bound state poles on the positive imaginary axis. Note that these poles give important contributions to the integration in Eq. (2.5) [according to the "+i0" prescription, for p real and $\tilde{p}^2 > E$ we have $\tilde{\kappa} = i(\tilde{p}^2 - E - i0)^{1/2}$, with $\text{Re } \tilde{\kappa} > 0$].

We finally recall that the two-body half-off-shell transition amplitude is related to ψ_0^+ through

$$t_p(\kappa, q; \tilde{\kappa}^2 + i0) = \frac{2}{\pi} \int_0^\infty r^2 dr \, t_p(\kappa, r; \tilde{\kappa}^2 + i0) \, j_0(qr) \quad (2.7)$$

$$t_p(\kappa, r; \tilde{\kappa}^2 + i0) = \psi_0^+(\kappa, r) V_p(r)$$

3. - ON-SHELL AND OFF-SHELL RESCATTERINGS

With the notation of the previous section we can now write down the Faddeev equations (2.1) in all details

$$\begin{aligned} \Psi_{\beta}^{+}(r_{\beta}, y_{\beta}) = & j_0(p_1^0 y_1) \left[j_0(q_1^0 r_1) - \int_0^{\infty} r_1'^2 dr_1' G_1(r_1, r_1'; \tilde{q}_1^0{}^2) V_1(r_1') j_0(q_1^0 r_1') \right] \delta_{\beta 1} \\ & - \frac{2}{\pi} \int_0^{\infty} p_{\beta}^2 dp_{\beta} \int_0^{\infty} r_{\beta}^2 dr_{\beta} \int_0^{\infty} y_{\beta}^2 dy_{\beta} j_0(p_{\beta} y_{\beta}) G_{\beta}(r_{\beta}, r_{\beta}'; y_{\beta}^0{}^2) V_{\beta}(r_{\beta}') j_0(p_{\beta} y_{\beta}') \\ & \quad - \frac{1}{2} \int dx \sum_{\gamma \neq \beta} \Psi_{\gamma}^{+}(r_{\gamma}, y_{\gamma}) \end{aligned} \quad (3.1)$$

In the last factor, r_{γ} and y_{γ} are to be expressed in terms of r_{β}' , y_{β}' and $x = \hat{r}_{\beta}' \cdot \hat{y}_{\beta}'$ (see the co-ordinate triangle of Fig. 2),

$$r_{\gamma} = \sqrt{(\mu_{\beta}/m_{\alpha})^2 r_{\beta}'^2 + y_{\beta}'^2 + 2 \mu_{\beta}/m_{\alpha} r_{\beta}' y_{\beta}' x}$$

and

$$y_{\gamma} = \sqrt{(\mu_{\beta}/m_{\alpha})^2 r_{\beta}'^2 + (\mu_{\gamma}/m_{\alpha})^2 y_{\beta}'^2 - 2 \frac{\mu_{\beta} \mu_{\gamma}}{m_{\alpha}} r_{\beta}' y_{\beta}' x}$$

The integration over x projects out s waves in the β channel. In what follows we will emphasize the multiple scattering character of the Faddeev equations. The initial plane wave $j_0(p_1^0 y_1) j_0(q_1^0 r_1)$ propagates not as (the s wave component of) a plane wave but as a scattering wave (incoming + scattered wave) in the initial channel 1. Therefore, if the interactions V_2 and V_3 were absent, the driving term of Eq. (3.1) (i.e., the first two terms on the right-hand side) would already incorporate the full solution to the scattering problem. In general, this wave hits one of the potentials V_{β} , $\beta \neq 1$, and the Green's function G_{β} (instead of just G_0) guarantees that the resulting contribution to the scattered part of Ψ_{β}^{+} corresponds to a complete subsystem scattering, not just a Born term contribution. All such rescattering contributions to Ψ_{β}^{+} are contained in the Faddeev equations, as can be seen upon iteration of this equation.

Consider now the contributions to the rescattering term in Eq. (3.1) in somewhat greater detail. A considerable simplification of this term would be obtained if one two-body subprocess was always completed before the next one started, since in such a case $f_0^+(\kappa, r^>)$ in the rescattering term [see Eq. (2.6)] could be replaced by its asymptotic form $h_0^+(\kappa, r)$. Furthermore, $\psi_0^+(\kappa, r^<)$ would always be equal to $\psi_0^+(\kappa, r')$ since r would always be larger than r' . For short range potentials, we now argue that this is indeed often the case. To make this point more explicit, consider the range of variation of the variables r_Y and y_Y of $\psi_Y^+(r_Y, y_Y)$. Keeping r_β' fixed, the integrations over y_β' and x in (3.1) correspond to integration over a strip in the $r_Y y_Y$ plane bounded by the lines $r_Y = \pm 2r_\beta' + 2y_Y$ and $r_Y = 2r_\beta' - 2y_Y$ (equal mass case, see Fig. 3). With the further assumption that the potentials are short range (range $\approx R$), the integration over r_β' in Eq. (3.1) is essentially cut-off at the value R , and the entire region from which ψ_Y^+ contributes is bounded by the two lines $r_Y = \pm 2R + 2y_Y$ (Fig. 3). Clearly, most of this region is outside the range of forces in the γ channel (i.e., corresponds to $r_Y \gtrsim R$). The significance of this observation becomes even more evident if we note that in general [from (2.2)] $\psi_Y^+(r_Y, y_Y)$ is related to the off-shell scattering amplitude through

$$\begin{aligned} \psi_Y^+(r_Y, y_Y) &= j_0(p_Y y_Y) j_0(q_Y r_Y) \delta_{Y1} \\ &- \int_0^\infty p_Y^2 dp_Y \int_0^\infty q_Y^2 dq_Y j_0(p_Y y_Y) j_0(q_Y r_Y) \frac{T_Y^\gamma(q_Y, p_Y)}{\tilde{p}_Y^2 + \tilde{q}_Y^2 - E - i\epsilon} \end{aligned} \quad (3.2)$$

However, outside the range of forces in the γ channel $\psi_Y^+(r_Y, y_Y)$ is completely determined by the on-shell scattering amplitude $T_Y^\gamma(\kappa_Y, p_Y)$, $\tilde{\kappa}_Y^2 = E - \tilde{p}_Y^2$, [from (2.6) and (3.1)]

$$\begin{aligned} \psi_Y^+(r_Y, y_Y) &= j_0(p_Y y_Y) j_0(q_Y r_Y) \delta_{Y1} \\ &- \int_0^\infty p_Y^2 dp_Y j_0(p_Y y_Y) \pi \mu_Y \kappa_Y h_0^+(\kappa_Y r_Y) T_Y^\gamma(\kappa_Y, p_Y) \end{aligned} \quad (3.3)$$

where $r_\gamma \gg R$, and where

$$T^{\beta}(\kappa_{\beta}, p_{\beta}) = \frac{1}{p_{\beta}^2} \delta(p_1 - \tilde{p}_1) t_1(q_1^0, q_1^0; \tilde{q}_1^0 + i0) \delta_{\beta 1} \\ + \left(\frac{2}{\pi}\right)^2 \int_0^{\infty} r_{\beta}^2 dr_{\beta} \int_0^{\infty} y_{\beta}^2 dy_{\beta} f_0(p_{\beta} y_{\beta}) t_{\beta}(\kappa_{\beta}, r_{\beta}; \tilde{\kappa}_{\beta}^2 + i0) \frac{1}{2} \int_{-1}^1 dk \sum_{\gamma \neq \beta} \psi_{\gamma}^{\dagger}(r_{\gamma}, y_{\gamma}) \quad (3.4)$$

Therefore, in most parts of the region of integration, the rescattering contributions in the Faddeev equations (3.1) are uniquely related to the on-shell three-body transition amplitude $T^{\gamma}(\kappa_{\gamma}, p_{\gamma})$. Off-shell rescattering contributions, on the other hand, originate only from the region where $r_{\gamma} \lesssim R$, i.e., when a non-asymptotic wave in the γ channel hits the $V_{\beta}(r_{\beta}^{\dagger})$ potential. These observations suggest that the treatment of three-body processes would be simplified if the essential on-shell character of these processes could be exploited. In the next section we are going to outline a procedure for doing just this.

Before ending this section there is need for a digression on the meaning of Eqs. (3.3) and (3.4) when $\tilde{p}^2 > E$, i.e., when κ is imaginary. The contributions to the wave function in (3.3) from this region of integration are of short range, and therefore not expected to be very important (except for the contribution from the two-body bound state pole). On the other hand, it should be noticed that the exponential decline of $h_0^+(\kappa, r)$ in (3.3) when $\kappa \rightarrow i0$ is to some extent compensated by the exponential growth of the function $\psi_0^+(\kappa, r)$ [Eq. (2.6)] which enters as a factor in $T^{\gamma}(\kappa_{\gamma}, p_{\gamma})$, and that this product of exponentials has to be handled with some care. For a strictly finite range potential, $V(r) \equiv 0$ for $r > R$, $\psi_0^+(\kappa, r)$ grows no faster than $\exp(-i\kappa R)/\kappa R$. There is in this case, therefore, no problem to redefine T by this factor in such a way that the compensating exponential factors remain together. For more general potentials we will be content with the following simple modification of this procedure. Since we are discussing contributions to a region ($r \gtrsim R$) where the representation (3.3) is not very good anyway, also an approximate version of Eq. (2.6) would be quite adequate for the derivation of an alternative to (3.3) when $\kappa \rightarrow i0$. The redefinition suggested above can again be used if we arbitrarily cut off the potential at $r = R$. Another alternative would be to replace the potential by a two-body boundary condition model. In such a case we have, instead of (2.6),

$$G_{\beta} V_{\beta} \Big|_{\text{sc}} = 2\mu_{\beta} \kappa \ h_0^{+}(\kappa r') \frac{N(\kappa, r)}{D(\kappa)} \quad (3.5)$$

for $r' > R$, where $N(\kappa, r)$ and $D(\kappa)$ are defined in Eqs. (6.3) and (6.5) (for details of the boundary condition model we refer to Section 6 and the reference given there). If in this case the $D(\kappa)^{-1}$ factor is not included in the (modified) T^V amplitude but rather kept together with $h_0^{+}(\kappa, r')$, we again have a representation where the partly cancelling exponentials are explicitly kept together and therefore easily handled.

It must also be discussed at which point $\kappa = i|\kappa_c|$ we should shift from (3.3) to some alternative representation. The representation (3.3) really becomes inadequate only when κ reaches the two-body potential (left hand) cut. Recalling in addition that an accurate representation is desired at the two-body bound state pole, a convenient choice for κ might therefore lie somewhere between this pole and the onset of the cut (however, other choices might under special circumstances be at least as convenient).

To conclude this digression we repeat that the scattered wave part of (3.3) should in what follows only be taken literally for $0 < \tilde{p}^2 < E + |\tilde{\kappa}_c|^2$, and should be replaced by an expression involving a redefined amplitude [with a corresponding modification of (3.4)] when $\tilde{p}^2 > E + |\tilde{\kappa}_c|^2$, e.g., as discussed above. However, for simplicity of presentation, we will in what follows mostly use (3.3) and (3.4) as is, with the understanding that when $\tilde{p}^2 > E + |\tilde{\kappa}_c|^2$, i.e., $\text{Im } \kappa > |\kappa_c|$ some modifications are required.

4. - THE ON-SHELL THREE-BODY THEORY

The observations in the previous section will now be used to derive equations for the on-shell three-body scattering amplitude components. It will be convenient to separate the on-shell and off-shell rescattering regions, and for this purpose we define a function $J'_\beta(r_\gamma, r_\alpha)$ (α, β, γ different) such that $J'_\beta \approx 1$ when r_γ and $r_\alpha \gtrsim R$, with $J'_\beta \approx 0$ otherwise. In addition, when J'_β restricts the range of the x integration in (3.4) we include a factor in J'_β that makes this integration normalized on the new range [typically, $\frac{1}{2} \int_{-1}^1 dx \dots \rightarrow \frac{1}{2} \int_{-\bar{x}}^{\bar{x}} dx J'_\beta \dots$, where $J'_\beta = \bar{x}^{-1} \theta(r_\gamma - R) \theta(r_\alpha - R)$, see Fig. 3. For equal mass particles and for $r_\beta < 2R$, $\bar{x} = 1$ when $y_\beta > R + \frac{1}{2}r_\beta$, and $\bar{x} = (y_\beta^2 + \frac{1}{4}r_\beta^2 - R^2)/r_\beta y_\beta$ when $R + \frac{1}{2}r_\beta > y_\beta > \sqrt{R^2 - \frac{1}{4}r_\beta^2}$. There is no contribution to the integral when $y_\beta < \sqrt{R^2 - \frac{1}{4}r_\beta^2}$. Introducing this separation in (3.4), we write this equation [combined with (3.3)] on the form

$$T^\beta(k_\beta, p_\beta) = T_0^\beta(k_\beta, p_\beta) - \sum_\gamma \int_0^\infty p_\gamma^2 dp_\gamma K_{\beta\gamma}(p_\beta, p_\gamma) T^\gamma(k_\gamma, p_\gamma) + T_R^\beta(k_\beta, p_\beta) \quad (4.1)$$

The driving term in (4.1) is defined by the expression

$$T_0^\beta(k_\beta, p_\beta) = \frac{1}{p_\beta^2} \delta(p_i - p_i^0) t_1(q_i^0, q_i^0; \tilde{q}_i^0 + i0) \delta_{\beta 1} \\ + \left(\frac{2}{\pi}\right)^2 \int_0^\infty r_\beta^2 dr_\beta \int_0^\infty y_\beta^2 dy_\beta j_0(p_\beta y_\beta) t_\beta(y_\beta, r_\beta; \tilde{y}_\beta^2 + i0) \frac{1}{2} \int_{-1}^1 dx \theta_\beta(r_\beta, x) j_0(\tilde{q}_i^0, r) j_0(p_i^0, y_i) \overline{\delta}_{\beta 1}$$

(4.2)

and the kernel is defined by

$$K_{\beta\gamma}(p_\beta, p'_\gamma) =$$

$$= \left(\frac{2}{\pi}\right)^2 \int_0^\infty r_\beta^2 dr_\beta \int_0^\infty y_\beta^2 dy_\beta j_0(p_\beta y_\beta) t_\beta(\kappa_\beta, r_\beta; \tilde{\kappa}_\beta^2 + i0) \frac{1}{2} \int_{-1}^1 dx \rho_\beta(r_\beta, r_\alpha) j_0(p'_\gamma y_\gamma) \gamma_{\mu\gamma} \kappa'_\gamma h_0^+(\kappa'_\gamma r_\gamma) \bar{\delta}_{\beta\gamma}$$

(4.3)

with suitable modifications when $\tilde{p}_\beta^2$ and/or $\tilde{p}_\gamma^2 > E + |\tilde{\kappa}_c|^2$ (see end of Section 3). The term $T_R^\beta(\kappa_\beta, p_\beta)$ accounts for all contributions (mostly off-shell rescatterings) which are not already accounted for in T_0^β or $K_{\beta\gamma}$.

In (4.2) and (4.3) as in (3.1), r_γ, y_γ and r_α are to be expressed in terms of r_β, y_β and x . As always, κ_β is related to p_β through $\tilde{\kappa}_\beta^2 = E - p_\beta^2$ and is therefore not an independent variable. When κ'_γ is real, i.e., when $0 \leq \tilde{p}_\gamma^2 \leq E$, the expressions (4.2) and (4.3) involve an infinite range integration (in y_β) of oscillating functions. A more convenient representation for T_0^β and $K_{\beta\gamma}$ is obtained if $\mathcal{P}_\beta$ is replaced by $1 - (1 - \mathcal{P}_\beta)$ and the contributions from the two terms are evaluated separately. For example, the contribution to $K_{\beta\gamma}$ from the first term is simply [see Appendix I ; $q = |\vec{p}_\gamma' + (\mu_\beta/m_\alpha)\vec{p}_\beta|$ and $z = \hat{p}_\beta \cdot \hat{p}_\gamma'$]

$$\frac{1}{2} \int_{-1}^1 dz \frac{t_\beta(\kappa_\beta, q; \tilde{\kappa}_\beta^2 + i0)}{\frac{p_\beta^2}{2\mu_\beta} + \frac{p_\gamma^2}{2\mu_\gamma} + \frac{p_\beta p_\gamma'}{\mu_\alpha} z - E - i0}$$

(4.4)

while in the second term the previously infinite range y_β integration is now cut off at $\lesssim 2R$. A similar result holds for T_0^β . It can be noted that the contribution (4.4) has in general a singularity due to the vanishing of the denominator (unless $E < 0$), and that the singular point coincides with the on-shell point $q = \kappa_\beta$ (see Appendix I). This singular part of (4.4)

(there is another singular contribution from the two-body bound state pole in t) therefore only depends on the on-shell two-body t matrix. This result indicates that the on-shell three-body rescattering predominantly proceeds via two-body bound states and, whenever this is kinematically possible, via on-shell two-body scattering. In such a case, the importance of the on-shell versus the off-shell two-body input is obviously enhanced.

Returning to the discussion of Eq. (4.1), it is clearly suggested by the physics involved that the off-shell rescattering contributions represented by T_R^B should be treated on the same footing as the on-shell rescattering contributions. We therefore introduce a kernel function $R_{\beta\gamma}$ such that

$$T_R^B(k_\beta, p_\beta) = - \sum_\gamma \int_0^\infty p_\gamma^2 dp_\gamma' R_{\beta\gamma}(p_\beta, p_\gamma') T^B(k_\gamma, p_\gamma') \quad (4.5)$$

The final form for the on-shell reduction of the Faddeev equations is therefore

$$T^B(k_\beta, p_\beta) = T_0^B(k_\beta, p_\beta) - \sum_\gamma \int_0^\infty p_\gamma^2 dp_\gamma' [K_{\beta\gamma}(p_\beta, p_\gamma') + R_{\beta\gamma}(p_\beta, p_\gamma')] T^B(k_\gamma, p_\gamma') \quad (4.6)$$

Here, T_0^B is (almost) the single scattering part of T^B while the $K_{\beta\gamma}$ term represents on-shell and the $R_{\beta\gamma}$ term off-shell rescattering contributions.

The on-shell equation (4.6) is completely equivalent to the three-body Faddeev equations only if it is supplemented with a suitable procedure for finding $R_{\beta\gamma}$. It is shown in Appendix II that the unitarity conditions on T^B restricts $R_{\beta\gamma}$ to be of the form

$$R_{\beta\gamma}(p_\beta, p_\gamma') = \mathcal{L}_\beta^{-1}(k_\beta) C_{\beta\gamma}(p_\beta, p_\gamma') \quad (4.7)$$

where $C_{\beta\gamma}$ is a real function and where $\mathcal{L}_\beta$ is the two-body Jost-function ^{*)}

^{*)} Recall that $\mathcal{L}(k) = |\mathcal{L}(k)| \exp(-i\delta(k))$, where $\delta(k)$ is the two-body phase shift; Eq. (4.7) therefore expresses the fact that $R_{\beta\gamma}(p_\beta, p_\gamma') = |R_{\beta\gamma}(p_\beta, p_\gamma')| \exp(i\delta(k_\beta))$ when k_β is real. It is useful to extract $\mathcal{L}_\beta^{-1}$ [rather than simply $\exp(i\delta)$] since this factor faithfully represents some important features of the analytic structure of the kernel of the Faddeev equations, in particular the two-body bound state poles. One could also include an additional real factor and separate out an on-shell two-body t matrix from $R_{\beta\gamma}$.

Since the function $C_{\beta\gamma}$ represents the full complexity of the off-shell three-body scatterings, it can only be obtained from the solution of an off-shell, i.e., two-variable integral equation. However, the reality of $C_{\beta\gamma}$ signals that the kernel of this equation will be less singular than the kernel of the original Faddeev equations; this is also clear from the fact that the unitarity singularities are generated in (4.6) and should be absent from $C_{\beta\gamma}$. Although a detailed evaluation of $C_{\beta\gamma}$ is in general quite complicated it is therefore at least expected to be much more manageable than the straightforward solution of the Faddeev equations themselves.

On the other hand, any real but approximate $C_{\beta\gamma}$ will produce a three-body scattering amplitude that satisfies the unitarity constraints. It is also well-known that there is only little sensitivity in the three-body problem to (two-body) off-shell variations, and since such variations mostly affect $C_{\beta\gamma}$ one can conclude that already quite crude estimates of $C_{\beta\gamma}$ might produce accurate three-body scattering amplitudes. This conclusion is supported by the numerical results obtained with Brayshaw's boundary condition formalism (BCF) ^{2),3)}, in particular since in Section 6 we are going to show that the BCF is (closely related to) a special case of the on-shell scattering formalism developed here.

The problem of how to actually construct $C_{\beta\gamma}$ will in this paper only be pursued for the special case considered in Section 5.

Although we have so far tried to emphasize the equivalence of the present formulation with the conventional Faddeev equations an alternative philosophy (borrowed from Brayshaw ^{2),3)}) could also be adopted. After all, in most applications we do not know very well what the two-body off-shell input really is. Therefore, instead of calculating the contributions to $C_{\beta\gamma}$ in some potential model, one could consider $C_{\beta\gamma}$ as a phenomenological input to the three-body theory and parametrize this function directly. With this point of view, Eq. (4.6) is the complete three-body theory, and what remains is to settle for a standard two-body off-shell continuation to be used in T_0^β and $K_{\beta\gamma}$, and use potential models to find out what would be a reasonable parametrization of $C_{\beta\gamma}$.

The results of this section can now be summarized. The three-body Faddeev equations have been reduced to the on-shell form (4.6) in which a separation of on-shell ($K_{\beta\gamma}$) and off-shell ($R_{\beta\gamma}$) rescattering contributions have been made. The single-scattering term T_0^β and the function

$K_{\beta\gamma}$ are explicit expressions involving only two-body entities. The part $R_{\beta\gamma}$ of the kernel has the important representation (4.7) in terms of a function $C_{\beta\gamma}$ which from unitarity is real. The function $C_{\beta\gamma}$ can either be determined from an off-shell three-body equation, or can be treated as a phenomenological input to the equations. In any case, as long as $C_{\beta\gamma}$ is taken to be real, the solutions to the one-variable on-shell equation (4.6) will give results consistent with unitarity.

We finally mention that the method developed here might appear similar to the K matrix approach to the three-body problem⁴⁾. In that case one first has to solve an off-shell equation for the real three-body K matrix, and then an on-shell one-variable three-body Heitler-type equation for the scattering amplitude with the K matrix in the kernel. However, this is mostly a similarity in form, and the physical content is in fact quite different.

5. - THE CASE $\mathcal{P}=1$ AND THE GENERALIZED KN METHOD

A simple special case of the on-shell formalism outlined in the previous section is obtained if $\mathcal{P}$ is chosen equal to one everywhere. It is not entirely within the spirit of our approach to do so since this choice tends to counteract the attempt to separate on-shell from off-shell rescattering contributions. More precisely, the $R_{\beta\gamma}$ part of the kernel in Eq. (4.6) is in this case asked to play a somewhat larger rôle than was originally intended, since it will also have to compensate for the use of the asymptotic form (3.3) of the three-body wave function inside the range of forces, i.e., where it does not approximate the true wave function very well. On the other hand, the formal simplicity of this choice is certainly appealing. Thus if $\mathcal{P}=1$, $K_{\beta\gamma}$ is simply given by (4.4) with no additional contribution, and there is a similar simplification for T_o^β . The function $R_{\beta\gamma}$ still has a representation of the form (4.7) with $C_{\beta\gamma}$ real, and one can in this case without much effort construct the off-shell equation from which $C_{\beta\gamma}$ is to be determined.

It is not difficult to verify that the special case contemplated in this section is in fact identical to the three-body generalization of the (two-body) Kowalski-Noyes (KN) method that was recently developed in Ref. 1). The main tool used in this reference is the generalized three-body half-off-shell function $F_{\beta\gamma}$ which generates a half-off-shell scattering amplitude component from the on-shell components,

$$T^{\beta}(q_{\beta}, p_{\beta}) = \sum_{\gamma} \int_0^{\infty} p_{\gamma}^2 dp_{\gamma}' F_{\beta\gamma}(q_{\beta}, p_{\beta}; p_{\gamma}') T^{\gamma}(k_{\gamma}', p_{\gamma}') \quad (5.1)$$

This function is real and satisfies a non-singular, off-shell and therefore two-variable integral equation. The on-shell amplitude is obtained from $F_{\beta\gamma}$ as the solution of an integral equation which is identical to Eq. (4.6) in this paper (with $\mathcal{P}=1$). The relation between the $R_{\beta\gamma}$ of (4.6) and the half-off-shell function can therefore be established. In a practical calculation one would probably not solve the equations for the half-off-shell function directly, but rather the related equations for the combination

$$\hat{C}_{\beta\gamma}(q_{\beta}, p_{\beta}; p_{\gamma}') = \sum_{\lambda \neq \beta} \frac{1}{2} \int_{-1}^1 dx (\tilde{p}_{\lambda}^2 + \tilde{q}_{\lambda}^2 - E - i\delta)^{-1} [F_{\lambda\gamma}(q_{\lambda}, p_{\lambda}; p_{\gamma}') - F_{\lambda\gamma}^{\circ}(q_{\lambda}, p_{\lambda}; p_{\gamma}')] \quad (5.2)$$

where $F_{\lambda\gamma}^{\circ}(q_{\lambda}, p_{\lambda}; p_{\gamma}') = \delta_{\lambda\gamma} (1/p_{\gamma}^2) \delta(p_{\gamma} - p_{\gamma}')$ coincides with $F_{\lambda\gamma}(q_{\lambda}, p_{\lambda}; p_{\gamma}')$ on the energy shell (as before, q_{λ} and p_{λ} are understood to be functions of q_{β} , p_{β} and $x = \hat{q}_{\beta} \cdot \hat{p}_{\beta}$). The expression for $R_{\beta\gamma}$ in terms of this function is simply

$$R_{\beta\gamma}(p_{\beta}, p_{\gamma}') = \int_0^{\infty} q^2 dq t_{\beta}(u_{\beta}, q; \tilde{x}_{\beta}^2) \hat{C}_{\beta\gamma}(q, p_{\beta}; p_{\gamma}') \quad (5.3)$$

Note that $\hat{C}_{\beta\gamma}$ is a real function, so that $R_{\beta\gamma}$ is a real function times the simple phase factor $\exp(i\delta(u_{\beta}))$, as expected from the general discussion in Section 4.

From the point of view of the present paper it is reassuring that the reduction of the three-body equations developed here using physical guide-lines can be made to coincide with a method that was developed for mathematical convenience and simplicity. It is also very satisfactory to see that for this case the analysis of the present paper can be supplemented by

explicit equations from which the function $R_{p\gamma}$ in Eq. (4.6) can be determined from first principles, i.e., from a given set of pair interactions. On the other hand, the present work also provides a simple physical interpretation for the more formal reduction of the three-body equations developed in Ref. 1).

We finally recall that Noyes ⁵⁾ was the first to advocate the use of the two-body Kowalski-Noyes representation in the three-body context, and that he also indicated the rôle it would play, namely to separate the three-body problem into an interior and an exterior part.

6. - COMPARISON WITH BRAYSHAW'S BCF

In order to further illustrate the general on-shell formalism developed in Section 4 we will now consider in more detail the particular choice for the cut-off function introduced in passing already in Section 4,

$$\theta_p(r_\gamma, r_\alpha) = \bar{x}^{-1} \theta(r_\gamma - R) \theta(r_\alpha - R) \quad (6.1)$$

More importantly, we will also choose the half-off-shell two-body t matrices that appear in T_0^β and in $K_{p\gamma}$ according to the two-body boundary condition model.

This case will also serve to illustrate the relationship between our on-shell formalism and Brayshaw's boundary condition formalism (BCF) ^{2),3)}.

In the two-body boundary condition model the outgoing wave scattering solution $\psi^+(\kappa, r)$ to the Schrödinger equation is taken as

$$\psi^+(\kappa, r) = \frac{i\mu(\kappa r + \delta)}{\kappa r} \exp(i\delta), \quad r > c \quad (6.2)$$

where c is the boundary condition radius. For simplicity, c will here be chosen equal to the radius R in the cut-off function. The phase shift and the on-shell t matrix are uniquely determined, for instance, by the logarithmic derivative $f(\tilde{\kappa}^2)$ of $r\psi^+(\kappa, r)$ at $r=R$, e.g., $t(\tilde{\kappa}^2) = N(\kappa)/D(\kappa)$, where

$$N(k) = f(\tilde{k}^2) j_0(kR) - \cos(kR)$$

$$D(k) = \pi \mu k \left(f(\tilde{k}^2) - i k R \right) \frac{1}{k R} \exp(i k R) \quad (6.3)$$

For use in a three-body context, the above model has to be supplemented with an expression for the product of the two-body wave function and the potential, i.e., for the half-off-shell t matrix [Eq. (2.7)]. Following Brayshaw we take $t(k, p; \tilde{k}^2) = N(k, p)/D(k)$, where

$$N(k, p) = f(\tilde{k}^2) j_0(pR) - \cos(pR) \quad (6.4)$$

or equivalently $t(k, r; \tilde{k}^2) = N(k, r)/D(k)$, where

$$N(k, r) = \frac{\pi}{2} \left(f(\tilde{k}^2) \frac{1}{R^2} \delta(r-R) + \frac{1}{r} \delta'(r-R) \right) \quad (6.5)$$

It should be noted that in the half-off-shell t matrix, $t(k, p; \tilde{k}^2) = t(k, k; \tilde{k}^2) f_k(p)$, it is only the half-off-shell factor that is affected by the boundary condition model assumption, $f_k(p) = N(k, p)/N(k)$, while the on-shell t is left untouched. We now get ($t(\tilde{k}^2) \equiv t(k, k; \tilde{k}^2)$)

$$\begin{aligned} T_0^\beta(k_\beta, p_\beta) &= \frac{1}{p_\beta^2} \delta(p_\beta - p_\beta^0) t_1(\tilde{q}_1^2) \delta_{\beta 1} \\ &+ \left(\frac{2}{\pi}\right)^2 \frac{t_\beta(\tilde{k}_\beta^2)}{N(k_\beta)} \int_0^2 y_\beta dy_\beta j_0(p_\beta y_\beta) \frac{1}{2} \int_{-1}^1 dx \int_0^2 r_\beta dr_\beta N(k_\beta, r_\beta) \rho_\beta(r_\beta, r_\alpha) j_0(q_1 r_1) j_0(p_1 y_1) \bar{\delta}_{\beta 1} \end{aligned} \quad (6.6)$$

and

$$K_{\beta\gamma}(p_\beta, p'_\gamma) =$$

$$= \left(\frac{2}{\pi}\right)^2 \frac{t_\beta(\tilde{\kappa}_\beta^2)}{N(\kappa_\beta)} \int_0^\infty y_\beta^2 dy_\beta \int_0^\infty y_\gamma^2 dy_\gamma \int_0^\infty dx \int_0^\infty r_\beta dr_\beta N(\kappa_\beta, r_\beta) \mathcal{O}_\beta(r_\beta, r_\gamma) \int_0^\infty y_\gamma^2 dy_\gamma \pi_{\mu_\beta \mu'_\gamma} h_0^+(\kappa'_\gamma r_\gamma) \overline{\delta_{\beta\gamma}}$$

(6.7)

The main advantage of the boundary condition model off-shell continuation is that one of the integrations involved in the evaluation of Eqs. (6.6) and (6.7) becomes trivial thanks to the δ functions in $N(\kappa, r)$. In the expressions (6.6) and (6.7) the cut-off function is supposed to be effective only after the integration over r_β has been carried out [i.e., the derivative implied by the δ' function in $N(\kappa, r)$ is to be carried out before the cut-off is applied].

It is not a priori clear that a model choice for the off-shell continuation can be made without changing the physics of the problem. However, from the unitarity proof in Appendix II it should be clear that if only the same choice is made in T_0^β and $K_{\beta\gamma}$, all effects of using a model off-shell continuation instead of the original one can be absorbed in the function $C_{\beta\gamma}$. Therefore, in this case $C_{\beta\gamma}$ not only parametrizes the influence of off-shell rescatterings but also the effects of using a model (instead of the true) two-body off-shell continuation in the direct and on-shell rescattering terms.

With the functions T_0^β and $K_{\beta\gamma}$ specialized according to Eqs. (6.6) and (6.7), the on-shell equation (4.6) is very similar to the Brayshaw BCF equation [Eq. (4.112) of Ref. 3]. However, the agreement is not complete, and one reason for that is fairly trivial. To simplify things further, Brayshaw employs a modified off-shell continuation, $N(\kappa, r) \rightarrow N^c(r)$, where $N^c(r)$ is obtained by replacing $f(\tilde{\kappa}^2)$ in Eq. (6.5) by a constant f^c , e.g., $f^c = \lim_{p \rightarrow \infty} f(\tilde{\kappa}^2)$ (recall that $\tilde{\kappa}^2 = E - p^2$). For $p^2 > E + |\tilde{\kappa}_c|^2$ (see end of Section 3), Brayshaw in addition replaces the on-shell $t(\tilde{\kappa}^2)$ by an expression of the type (6.3) but with $f(\tilde{\kappa}^2)$ being a meromorphic function that is real for $\tilde{\kappa}^2 < 0$, and thereby obtains a satisfactory formulation also in this region. These differences only illustrate

the point we made earlier, namely that there is a freedom in the choice of two-body off-shell continuation in the on-shell rescattering term. There is no approximation involved in this new choice, but it will, of course, correspond to a new function $C_{\beta\gamma}$.

A less transparent difference has the following origin. Starting from boundary condition considerations, Brayshaw is led to an equation similar to (4.6) but with a slightly different driving term. More precisely, while the first term in Eq. (6.6) corresponds to taking $y_0 = 0$ in the expression

$$t_1(\vec{q}_1^2) \frac{2}{\pi} \int_{y_0}^{\infty} y^2 dy f_0(p, y) f_0(p, y) \delta_{p_1}, \quad (6.8)$$

Brayshaw is left with a y_0 that in general is different from zero.

This is unfortunate, not only because this term is more complicated but also because the unitarity condition on $C_{\beta\gamma}$ will be affected. Repeating the discussion in Appendix II, one can verify that $C_{\beta\gamma}$ can no longer be a purely real function, but is required to have a well-defined imaginary part. Brayshaw supplies this imaginary part with an arbitrarily chosen real part, and the sum is added as an explicit contribution to $K_{\beta\gamma}$. As a consequence, the BCF equations both have a more complicated driving term and a more complicated kernel than our equations. Furthermore, while in our case the three-body t matrix trivially reduces to the two-body t matrix when two of the interactions are switched off, this is not the case in the BCF equations.

We therefore consider the equations obtained in this section as an improved version of Brayshaw's BCF equations. They embody exactly the same physical assumptions but are simpler both conceptually and in form.

From the point of view developed here, it is obvious that the boundary condition aspects of the BCF are really minor. The only rôle played by the boundary condition formalism is in fact to provide a (convenient) standard off-shell continuation of the input two-body t matrix. The physics involved in the model is essentially independent of this choice.

7. - DISCUSSION

It has been a long standing hope that the study of the three-nucleon system could provide some information on those aspects of the two-body nucleon-nucleon force that is not directly accessible in two-nucleon experiments (off-shell effects). Unfortunately, it has, over the last years, become increasingly evident that very few three-nucleon observables show any sensitivity to such effects. It is now also generally accepted that this is not a particular feature of the nucleon-nucleon force but that it is a general feature of three-body quantum mechanical systems. That this should be so is not evident neither from the Schrödinger nor from the Faddeev formulation of the three-body problem, as it is only after extensive numerical work that this conclusion has been reached.

At this point it is clearly called for that the three-body equations are rewritten in a manner that more clearly exposes this phenomenon. In fact, such a reformulation might turn out worth while not only for the insight that might be gained, but it can also be expected to suggest more efficient calculational procedures for the three-body problem (that exploit the insensitivity to two-body off-shell effects).

We hope that the results presented in this paper have some interest from both these points of view. The idea has been to separate in as simple a manner as possible on-shell from off-shell rescattering contributions in the Faddeev equations, with the expectation that the on-shell rescatterings are dominated by on-shell two-body scattering matrices. It was found that the on-shell rescatterings could be adequately accounted for in terms of a simple on-shell multiple scattering equation, Eq. (4.6), and that this equation could accommodate also the off-shell rescatterings in a natural (though less direct) way as an "effective" contribution to the on-shell term. The on-shell rescattering term was furthermore shown to exhibit a singularity which indeed enhances the importance of the on-shell versus the off-shell two-body t matrices. The importance of this singularity has been recognized before, in particular by Noyes⁶⁾ (the eternal triangle effect) and by Amado⁷⁾ (minimal unitarity and analyticity). We consider it beyond doubt that it is this on-shell rescattering mechanism that is responsible for the non-sensitivity to two-body off-shell effects in the three-body system.

It should perhaps be pointed out that in a three-body bound state calculation the scattered waves are exponentially damped, and no proper on-shell rescattering can take place. Off-shell effects are therefore expected to be more important, and the possible usefulness of the (in this case purely formal) separation proposed here should be critically re-evaluated.

The usefulness of the general equation (4.6) for the description of scattering phenomena is further corroborated by the special cases worked out in Sections 5 and 6. In Section 5 detailed physical interpretation was to some extent traded for mathematical simplicity, and the resulting version of (4.6) turned out identical to the on-shell equation that is obtained using a simple three-body generalization¹⁾ of the (quite successful) two-body Kowalski-Noyes method. This special case is expected to be well suited for practical calculations, in particular since the discussion of off-shell rescattering effects has been carried out in all details.

In Section 6, the on-shell rescattering contributions to Eq. (4.6) were evaluated using the off-shell continuation of the two-body t matrix that is obtained in the two-body boundary condition model. This was not an approximation but only a shifting of some two-body off-shell effects (that had been kept in the three-body on-shell rescattering term) to the off-shell rescattering term. The resulting version of Eq. (4.6) was identified as an improved version of the Brayshaw BCF equation. This result confirms that the BCF equation can be directly obtained from the conventional Faddeev equation, but also indicates that its most essential feature is not the use of the boundary condition model but rather the separation of on-shell from off-shell rescatterings together with a careful treatment of the former. The BCF equation has been used by Brayshaw in actual calculations, and his results support the general considerations given here on the importance of the on-shell versus details of the off-shell rescattering term. We finally argued that the simplified version of the BCF equation obtained here represents an improvement over the original version, both conceptually and from a practical point of view. Altogether we hope that our results in Section 6 should pave the way for a more general appreciation of this type of equations.

There are several aspects of the on-shell/off-shell rescattering separation that have not been touched upon in this paper. For instance, the detailed derivation of the off-shell rescattering contributions was only carried out for the special case of Section 5. Some limiting cases other

than those of Sections 5 and 6 should also be investigated, e.g., the short range limit ^{*}) and the limit corresponding to the scattering of a projectile off two fixed, non-overlapping scatterers (the latter should be a natural limit in this formalism ; we have verified that the three-body hard core model, which is another natural limit, fits well into the framework developed here). Last but not least, it should be very worth while to try to incorporate relativistic effects ⁸⁾ in the multiple scattering equations.

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^{*}) It has been pointed out by Brayshaw ⁸⁾ that the BCF equation reduces to the Amado equation ⁷⁾ in the zero range limit.

APPENDIX I

The kernel, Eq. (4.3), of the on-shell equation derived in Section 4 has a singularity coming from the infinite range integration over y_β (if $E > 0$), and it has some interest to isolate this singularity. We first write $\mathcal{P}_\beta$ as $1 - (1 - \mathcal{P}_\beta)$ and note that the $1 - \mathcal{P}_\beta$ term does not contribute to the singularity (from the co-ordinate triangle, Fig. 2, it can be seen that both r_α and r_γ tend to infinity for all x when y_β goes to infinity at fixed $r_\beta \lesssim R$). Furthermore, the detailed functional dependence of $t_\beta(u_\beta, r_\beta; \tilde{u}_\beta^2)$ on r_β does not affect the singularity as long as t_β is negligible for $r_\beta \lesssim R$. We therefore conclude that the contribution to the singularity can be evaluated with r_β put equal to zero in y_γ and r_γ . Since this corresponds to taking $t_\beta(u, r; \tilde{u}^2) = t_\beta(\tilde{u}^2) 1/r^2 \delta(r)$ [or $t_\beta(u, p; \tilde{u}^2) = t_\beta(\tilde{u}^2)$ in momentum space], we also conclude that the singularity will only depend on the on-shell two-body t matrix.

Using the representations

$$t_\beta(u_\beta, r_\beta; \tilde{u}_\beta^2) = \int_0^\infty p^2 dp \, t_\beta(u_\beta, p; \tilde{u}_\beta^2) j_0(pr_\beta) \quad (\text{AI.1})$$

and

$$h_0^+(u_\beta, r_\beta) = \frac{2}{\pi u_\beta} \int_0^\infty \lambda^2 d\lambda \, (\lambda^2 - u_\beta^2 - i\varepsilon)^{-1} j_0(\lambda r_\beta) \quad (\text{AI.2})$$

we first identify the channel recoupling coefficient in $K_{\beta\gamma}$,

$$\begin{aligned} \left(\frac{2}{\pi}\right)^2 \int_0^\infty y_\beta^2 dy_\beta \int_0^\infty r_\beta^2 dr_\beta \, \frac{1}{2} \int_{-1}^1 dx \, j_0(p y_\beta) j_0(r_\beta) j_0(p' y_\beta) j_0(\lambda r_\beta) &= \\ &= \frac{1}{2} \int_{-1}^1 dx \, \frac{1}{\lambda^2} \delta(\lambda - \lambda_0) \delta(p - p_0) \end{aligned} \quad (\text{AI.3})$$

where

$$\lambda_0 = |\vec{p}_\beta - m_0/m_\alpha \vec{p}'|, \quad p_0 = |\vec{p}' - m_0/m_\alpha \vec{p}_\beta|$$

and $z = \hat{p}_\beta \cdot \hat{p}'_\gamma$. The singular part of $K_{\beta\gamma}$ now takes the form

$$\frac{1}{2} \int_{-1}^1 dz = t_\beta(k_\beta, p_0; k_\beta^2) \frac{2\mu_\gamma}{\lambda_0^2 - \tilde{k}_\gamma'^2 - i\varepsilon} \bar{\delta}_{\beta\gamma} \quad (\text{AI.4})$$

where

$$\frac{1}{2\mu_\gamma} (\lambda_0^2 - k_\gamma'^2) = \frac{1}{2\mu_\beta} (p_0^2 - k_\beta^2) = \frac{p_\beta^2}{2\mu_\gamma} + \frac{p_\gamma'^2}{2\mu_\beta} - \frac{p_\beta p_\gamma'}{\mu_\alpha} = -E.$$

We can here explicitly confirm the previous conclusion that the singularity of the integrand occurs when t_β is on the energy-shell, and the final expression for $K_{\beta\gamma}$ in the neighbourhood of its singular points is therefore

$$t_\beta(k_\beta^2) \frac{\mu_\alpha}{2 p_\beta p_\gamma'} \ln \left[\frac{\frac{p_\beta^2}{2\mu_\gamma} + \frac{p_\gamma'^2}{2\mu_\beta} + \frac{p_\beta p_\gamma'}{\mu_\alpha} - E - i\varepsilon}{\frac{p_\beta^2}{2\mu_\gamma} + \frac{p_\gamma'^2}{2\mu_\beta} - \frac{p_\beta p_\gamma'}{\mu_\alpha} - E - i\varepsilon} \right] \quad (\text{AI.5})$$

It should be emphasized that this structure is independent of the details of the two-body interaction, i.e., it does not discriminate between phase equivalent interactions (e.g., a local versus a separable interaction corresponding to the same phase shifts).

APPENDIX II

In order to investigate the unitarity constraint on $R_{\beta\gamma}$ in Eq. (4.5) it is necessary to have on-shell transition amplitudes that refer to an arbitrary initial channel γ (instead of the standard choice $\gamma=1$ used up to now). These amplitudes will be denoted $T^{\beta\gamma}(p_\beta, p'_\gamma)$ and we have $T^\beta(k_\beta, p_\beta) \equiv T^{\beta 1}(p_\beta, p_1^0)$. They satisfy the on-shell equation (4.6) with a $T_0^{\beta\gamma}(p_\beta, p'_\gamma)$ that differs from $T_0^\beta(k_\beta, p_\beta)$ only in that channel "1" has been called channel " γ ".

For simplicity, we will assume here that there are no two-body bound states.

Consider the imaginary part of Eq. (4.6),

$$\text{Im } T = \text{Im } T_0 - \text{Im } (K+R) T^* - (K+R) \text{Im } T \quad (\text{AII.1})$$

Recalling that

$$t_\beta(k_\beta, r_\beta; \tilde{k}_\beta^2 + i0) = |t_\beta(k_\beta, r_\beta; \tilde{k}_\beta^2 + i0)| \exp(i\delta(k_\beta))$$

we use Eq. (4.2) to get

$$\text{Im } T_0^{\beta 1}(p_\beta, p_1^0) = e^{i\delta(k_\beta)} \sin \delta(k_\beta) T_0^{\beta 1*}(p_\beta, p_1^0) \quad (\text{AII.2})$$

Similarly, from (4.3) we have

$$\begin{aligned} \text{Im } K_{\beta\gamma}(p_\beta, p'_\gamma) &= e^{i\delta(k_\beta)} \sin \delta(k_\beta) K_{\beta\gamma}^*(p_\beta, p'_\gamma) \\ &+ \left(T_0^{\beta\gamma}(p_\beta, p'_\gamma) \pi \mu_\gamma u'_\gamma + \frac{1}{p_\beta^2} \delta(p_\beta - p'_\beta) e^{i\tilde{v}(k_\beta)} \sin \delta(k_\beta) \delta_{\beta\gamma} \right) \Theta(E - \tilde{p}_0'^2) \end{aligned} \quad (\text{AII.3})$$

where we have used (the generalized version of) Eq. (4.3) and the fact that

$$t_{\beta}(u_{\beta}, u_{\beta}; \tilde{u}_{\beta}^2 + i0) = - (\pi \mu_{\beta} u_{\beta})^{-1} e^{i\delta(u_{\beta})} \sin \delta(u_{\beta}) \quad (\text{AII.4})$$

The $\Theta(E - \tilde{p}_{\gamma}^2)$ factor in (AII.3) appears because $u h_0^+(u, r)$ in (4.3) has an imaginary part only when u is real, i.e., when $\tilde{u}^2 = E - \tilde{p}^2 > 0$.

Using the representation (4.7) for $R_{\beta\gamma}$ (without assuming that $C_{\beta\gamma}$ is real) one finds

$$\text{Im } R_{\beta\gamma}(p_{\beta}, p'_{\gamma}) = e^{i\delta(u_{\beta})} \sin \delta(u_{\beta}) R_{\beta\gamma}^*(p_{\beta}, p'_{\gamma}) + \mathcal{L}_{\beta}^{-1}(u_{\beta}) \text{Im } C_{\beta\gamma}(p_{\beta}, p'_{\gamma}) \quad (\text{AII.5})$$

It is now straightforward to verify [using the complex conjugate of Eq. (4.6)] that all contributions to (AII.1) having an explicit $e^{i\delta} \sin \delta$ factor cancel. Equation (AII.1) therefore reduces to

$$\begin{aligned} \text{Im } T &= - T_0 \pi \mu_{\alpha} \Theta(E - \tilde{p}^2) T^* - \mathcal{L}^{-1}(\text{Im } C) T^* - (K+R) \text{Im } T \\ &= - T \pi \mu_{\alpha} \Theta(E - \tilde{p}^2) T^* - (1+K+R)^{-1} \mathcal{L}^{-1}(\text{Im } C) T^* \end{aligned} \quad (\text{AII.6})$$

If $\text{Im } C = 0$ then Eq. (AII.6) is just the unitarity condition for T ,

$$\text{Im } T^{\beta\gamma}(p_{\beta}, p_i^0) = - \sum_{\gamma} \int_0^{\sqrt{2\mu_{\beta} E}} p_{\gamma}^2 dp_{\gamma} \pi \mu_{\beta} u_{\gamma} T^{\beta\gamma}(p_{\beta}, p_{\gamma}) T^{\gamma i}(p_{\gamma}, p_i)^* \quad (\text{AII.7})$$

We therefore conclude that for any choice of C such that $\text{Im } C = 0$, the solution of Eq. (4.6) will satisfy the three-body unitarity condition (AII.7).

REFERENCES

- 1) B.R. Karlsson - to be published.
- 2) D.D. Brayshaw - Phys.Rev. D8, 952 (1973) ; in "Few-Body Problems in Nuclear and Particle Physics", eds. R.J. Slobodrian, B. Cujec and K. Ramavataram (Les Presses de l'Université Laval, Québec, 1975), p. 28 ; Phys.Rev. D13, 1024 (1976).
- 3) D.D. Brayshaw - Ch. IV in "Modern Three-Hadron Physics", ed. A.W. Thomas (Springer-Verlag, 1977).
- 4) R.T. Cahill - Nuclear Phys. A194, 599 (1972) ;
K.L. Kowalski - Phys.Rev. D5, 395 (1972) ; Phys.Rev. D10, 1271 (1974) ;
T. Sasakawa - Nuclear Phys. A203, 496 (1973).
- 5) H.P. Noyes - Phys.Rev.Letters 15, 538 (1965) ; Phys.Rev. D5, 1547 (1972).
- 6) H.P. Noyes - in "Three-Body Problem in Nuclear and Particle Physics", eds, J.S.C. McKee and P.M. Rolph (North Holland, Amsterdam, 1970), p. 2.
- 7) R.D. Amado - Phys.Rev.Letters 33, 333 (1974).
- 8) See, in this context, D.D. Brayshaw - Phys.Rev. D11, 2583 (1975).

FIGURE CAPTIONS

- Figure 1 The three-body momentum triangle (α, β, γ cyclic).
- Figure 2 The three-body co-ordinate triangle (α, β, γ cyclic).
- Figure 3 Rescattering contributions come from the solid line strip in the (r_γ, y_γ) plane. The dashed strip corresponds to r_β' fixed. The dash-dot line limits the region cut out by the projector $\Theta(r_\gamma - R)\Theta(r_\alpha - R)$ when $r_\beta = R$.

fig. 1

fig. 2

fig. 3