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REGGEON QUANTUM MECHANICS: A CRITICAL DISCUSSION

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A B S T R A C T

The quantum mechanical problem of Reggeon field theory in zero transverse dimensions is re-examined in order to set up a precise mathematical framework for the case $\mu = \alpha(0) - 1 > 0$. We establish a Hamiltonian formulation in a Hilbert space for $\mu < 0$ and we prove the equivalence of the related eigenvalue problem with a "radial" Schrödinger-type equation in an $L^2(0, \infty)$ space. We prove that the S matrix and the Pomeron Green's functions, at fixed rapidity Y and triple Pomeron coupling $\lambda \neq 0$, have a spectral decomposition and are analytic in μ for $-\infty < \mu < +\infty$. For $\mu > 0$, we confirm most of the qualitative results found by previous authors, and in particular the tunnelling shift $[\sim \exp(-\mu^2/2\lambda^2)]$ setting the scale for the asymptotic behaviour in Y .

In the classical limit of λ/μ small we find that the action, for $\mu > 0$, develops a singularity in Y at some value Y_0 . We give arguments to show that for $Y \ll Y_0$ the perturbative result is reached, while for $Y \sim Y_0$ perturbation theory breaks down. Most of these results are shown to be stable against the addition of a small quartic coupling of the simplest type $[\lambda'(\Psi\Psi)^2]$ up to the "magic" value $\lambda' = \lambda^2/\mu$. The existence of a level crossing at this value is confirmed by an analytic continuation in λ' .

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1. INTRODUCTION

Recently a great deal of progress has been made in the study of Reggeon Field Theory (RFT) for $\mu = \alpha(0) - 1 > 0$ ¹⁻⁶⁾ [$\alpha(0)$ is the Pomeron intercept]. It has in fact been shown^{4,5)} that, at least in an analogue quantum spin model, a phase transition occurs when the intercept $\alpha(0)$ is above a critical value α_c . For $\alpha(0) < \alpha_c$ the total cross-section decreases asymptotically, while for $\alpha(0) > \alpha_c$ it saturates the Froissart bound^{*)}.

The peculiar features of this phase transition are due to the interplay of an unstable perturbation theory [$\alpha(0) > 1$] with the absorptive corrections due to an imaginary triple Pomeron coupling, $i\lambda$.

The fact that the RFT Hamiltonian is non-hermitian is on the one hand essential for the theory to make sense, but on the other is a source of mathematical troubles. These are the following:

- i) The space of states on which the Hamiltonian acts is ill-defined, even for $\alpha(0) < 1$. Particularly when the number of transverse dimensions $D_t \neq 0$, it seems necessary to work with an indefinite metric space⁴⁾.
- ii) The definition of the Hamiltonian and of the S-matrix for $\alpha(0) > 1$ is ambiguous, because perturbation theory does not make sense any more.

This paper is devoted to an analysis of these points in the case $D_t = 0$, where the problem reduces to a quantum-mechanical one. Here a qualitative picture of the spectrum and of the asymptotic S-matrix has already been given both in a Hamiltonian formalism^{2,3)} and in a Green's function framework⁷⁾. However, in view of the relevance of the $D_t = 0$ case for the spin model approximation of the complete theory in $D_t \geq 1$, we believe that it is important to put the whole scheme on a more firm mathematical basis. Moreover, the $D_t = 0$ theory can be useful in itself as a theoretical laboratory where one can test approximations used in a higher number of dimensions.

In particular we will concern ourselves with the questions:

- i) of defining the space of states in the Hamiltonian formalism and formulating the related eigenvalue problem in a rigorous way;
- ii) of proving, at finite rapidity, $Y = \log s$, the spectral representation of the S-matrix, and its analyticity in the parameter μ for $-\infty < \mu < +\infty$.

This latter fact is peculiar to $D_t = 0$, where no phase transition is known to occur, and will allow an unambiguous definition of the S-matrix for $\mu > 0$.

We also investigate the stability of our results against the addition of a symmetric quartic coupling. Our analysis shows that nothing changes, even for

*) See, however, Ref. 6 for a different conclusion.

$\mu \geq 0$, if the four-Pomeron coupling, λ' , is less than λ^2/μ , and moreover that $\lambda' \neq 0$ acts in a sense as a regulator for $\mu > 0$. We also confirm the existence of a level crossing at $\lambda' = \lambda^2/\mu$, in agreement with recent results⁸⁾. We find, however, that a delicate process of analytic continuation is needed in order to reach the "magic" value λ^2/μ .

We start, for $\alpha(0) < 1$, by representing the Hamiltonian in the Bargmann's space of holomorphic functions of the complex variable z . This is a genuine Hilbert space and the eigenvalue problem for the RFT Hamiltonian is well defined (Section 2.1). We find however that, with the only exception of the vacuum, all eigenstates satisfy an eigenvalue condition on the semi-axis $z = -iq$ ($q > 0$). In this way we obtain a "radial" type Schrödinger equation, thus justifying the procedure used in Ref. 2 on the basis of the Feynman path integral. The problem can then be reduced to the study of a hermitian kernel, K , defined in $L^2(0, \infty)$ which is closely related to Jengo's³⁾ Hamiltonian (Section 2.2).

The essential point is that we are able to prove that K is Hilbert-Schmidt and operator analytic in μ for $-\infty < \mu < +\infty$ (Appendix A). We can therefore obtain the spectral decomposition of the S-matrix for $\mu < 0$ in terms of eigenfunctions of K (Section 2.3). Then we define the S-matrix and the Hamiltonian for $\mu > 0$ by analytic continuation in the $L^2(0, \infty)$ space (Section 3.1). It turns out that for $\alpha(0) > 1$ the states and the Hamiltonian cannot be represented in the original Bargmann's space and it is necessary to enlarge it into an indefinite metric space. We then discuss for $\mu/\lambda \gg 1$ the main properties of the spectrum of the Hamiltonian. Beside the lowest state with zero energy -- the perturbative vacuum -- the solution of the Schrödinger equation shows (Section 3.2) that there exists a second state almost degenerate with it with an energy splitting proportional to $\exp[-\mu^2/2\lambda^2]$, that is, typical of a tunnel effect. For the higher excited states we find (Section 3.3) pairs of states which, surprisingly, in perturbation theory seem to remain exactly degenerate. This may be related to an exact cyclic symmetry which the RFT Hamiltonian possesses.

In Section 4 we discuss the semi-classical limit of the S-matrix, showing that an important feature of it is the disappearance of the tunnel effect. We give arguments to show that for this reason in the classical approximation the analyticity of the S-matrix breaks down at some Y -dependent value of μ , owing to the existence of bifurcation points in the classical solutions of Ref. 2.

The inclusion of a four-Pomeron coupling can be discussed (Section 5) in a straightforward way in this approach, since it amounts to the inclusion (in our Schrödinger equation) of a centrifugal barrier corresponding to an angular momentum $\ell = -1 + (\lambda/\lambda' - \mu/\lambda)\lambda/2\lambda'$. By letting λ' vary from zero to the "magic" value λ^2/μ , ℓ changes from large positive values to $\ell = -1$. The level crossing

between the two lowest-lying states occurs at $\ell = -1$ and can be established by analytical continuation in ℓ through the point $\ell = -\frac{1}{2}$.

The conclusions are stated in Section 6.

2. EIGENVALUE PROBLEM FOR $\alpha(0) < 1$

The RFT Hamiltonian in zero transverse dimensions, $D_t = 0$, can be written in terms of creation and annihilation operators, a^+, a , in the form

$$H = H(a^+, a) = -\mu a^+ a + i\lambda a^+ (a^+ + a) a \quad (2.1)$$

where $\mu = \alpha(0) - 1 < 0$ to start with.

In the Schrödinger picture a and a^+ are genuinely hermitian conjugate^{*)} and the vacuum state is defined by

$$a |0\rangle = 0 \quad (2.2)$$

The quantities which have physical interpretation for high-energy scattering processes are: the "Pomeron propagator"

$$G^{(1,1)}(\gamma) = \langle 0 | a e^{-H\gamma} a^+ | 0 \rangle \quad (2.3a)$$

and the Pomeron Green's functions

$$G^{(n,m)}(\gamma) = \langle 0 | a^n e^{-H\gamma} a^{+m} | 0 \rangle \quad (2.3b)$$

We shall often refer to the generating functional of $G^{(n,m)}$,

$$S(f, g; \gamma) = \langle 0 | e^{-i\phi f} e^{-H\gamma} e^{-i\phi^+ g} | 0 \rangle \quad (2.4)$$

which has the physical interpretation of the elastic scattering S-matrix only if the couplings to the external particles f_n, g_m (Fig. 1) are of eikonal type, i.e. if

$$f_n = (-if)^n \quad g_m = (-ig)^m, \quad f, g > 0 \quad (2.5)$$

We first concentrate on the definition of the eigenvalue problem for the (non-hermitian) operator H , which we write

$$H |\psi_n\rangle = E_n |\psi_n\rangle \quad (2.6a)$$

$$H^\dagger |\bar{\psi}_n\rangle = \bar{E}_n^* |\bar{\psi}_n\rangle \quad (2.6b)$$

) We denote by a "+" the hermitian conjugation and by a "" the complex conjugation.

where, owing to the imaginary coupling in (2.1), the (left) eigenvectors $\langle \bar{\psi}_n |$ differ from the (right) eigenvectors $|\psi_n\rangle$. A crucial point in the discussion turns out to be the definition of the Hilbert space in which H acts. A careful analysis of all these problems will enable us to recover in a rigorous way previous results^{2,3)} on eigenvalues and eigenfunctions. We shall then discuss the existence of a spectral representation of the type

$$e^{-HY} = 1 + \sum_{n=1}^{\infty} \nu_n e^{-E_n Y} |\psi_n\rangle \langle \bar{\psi}_n| \quad \nu_n = \pm 1 \quad (2.7)$$

that was assumed in Ref. 4 for evaluating the asymptotic behaviour of the S-matrix. It seems difficult to prove (2.7) in general. We shall show, however, that if one is only interested in (2.3) or (2.4) with f and g real and positive, it is sufficient to solve a hermitian problem for which the spectral representation can be rigorously proved.

2.1 Coherent state representation -- Bargmann's space

The perturbation expansion of e^{-HY} which one can obtain from the Hamiltonian (2.1) is *a priori* well defined only for $\mu < 0$. We shall then start from the case $\mu < 0$ and see later on that the S-matrix can be analytically continued in μ across $\mu = 0$. In any case the space of states we start from is the Hilbert space constructed by applying creation operators to the (perturbative) vacuum $|0\rangle$.

This space can be put in correspondence with the Bargmann's space of holomorphic functions of the complex variable z by projecting a state $|\psi\rangle$ on a coherent basis:

$$\langle z^* | \psi \rangle = \langle 0 | e^{a^* z} | \psi \rangle = \psi(z) \quad (2.8)$$

The coherent state completeness relation

$$\int \frac{dz dz^*}{2i\pi} e^{-zz^*} |z^*\rangle \langle z^*| = 1 \quad (2.9)$$

induces the scalar product

$$\langle \Phi | \psi \rangle = \int \frac{dz dz^*}{2i\pi} e^{-zz^*} \Phi^*(z) \psi(z) \quad (2.10)$$

In this representation the operators a and a^+ are given by

$$a \rightarrow \frac{d}{dz} \quad a^+ \rightarrow z \quad (2.11)$$

so that Eq. (2.6a) becomes a second-order differential equation

$$H\psi = z \left[-\mu + i\lambda \left(z + \frac{d}{dz} \right) \right] \frac{d}{dz} \psi(z) = E \psi(z) \quad (2.12)$$

and the S-matrix elements (2.4) can be written

$$S(f, g; Y) = \langle if | e^{-HY} | -ig \rangle \quad (2.13)$$

Since $\psi(z)$ in (2.12) has to belong to the Bargmann's space, it must satisfy the following conditions:

- i) $\psi(z)$ is an entire function of z .
- ii) $\psi(z)$ is normalizable, i.e. from (2.10),

$$\|\psi\|^2 = \langle \psi | \psi \rangle = \int \frac{dz d\bar{z}}{2i\pi} e^{-|z|^2} |\psi(z)|^2 < \infty \quad (2.14)$$

We then have to look for solutions of (2.12) which obey the conditions (i) and (ii).

The differential equation (2.11) has two singular points: a regular one at $z = 0$ and an irregular one at $z = \infty$.

At $z = 0$ one finds two linearly independent solutions which behave as

$$\psi(z) \underset{z \sim 0}{\sim} c_1 + c_2 z \log z \quad c_1 \text{ and } c_2 \text{ constants} \quad (2.15a)$$

$$\psi(z) \underset{z \sim 0}{\sim} z \quad (2.15b)$$

The behaviour (2.15a) is an acceptable one only if $E = 0$; in this case in fact $c_2 = 0$ and the logarithmic branch point is absent. This implies that the normalized perturbative vacuum in the Bargmann's space, $\psi_0(z)$, is

$$\langle z^* | 0 \rangle = \psi_0(z) = 1 \quad (2.16)$$

For $E \neq 0$ we must consider functions which vanish at the origin [Eq. (2.15b)]; let us note that they are automatically orthogonal to the vacuum, since

$$\langle 0 | \psi \rangle = \psi(0) = 0 \quad (2.17)$$

We now turn to the point $z = \infty$, which is the only possible singular point for the solution of Eq. (2.12). The two independent asymptotic behaviours of $\psi(z)$ for $z \rightarrow \infty$ are

$$\psi(z) \underset{|z| \rightarrow \infty}{\sim} \text{Constant} \quad (2.18a)$$

$$\psi(z) \underset{|z| \rightarrow \infty}{\sim} \frac{1}{z} \exp \left[-\frac{1}{2} z^2 + \frac{\mu}{i\lambda} z \right] \quad (2.18b)$$

so that for any given direction in the complex z -plane an eigenfunction will be asymptotically a linear combination of (2.18a) and (2.18b).

We now show that the normalizability requirement gives a non-trivial boundary condition in some direction of the z -plane. In fact for $\text{Im } z \rightarrow -\infty$ ($\mu < 0$) the asymptotic behaviour (2.18b) is incompatible with (2.14) because

$$e^{-|z|^2} \left| \frac{1}{z} e^{-\frac{1}{2} z^2 + \frac{\mu}{i\lambda} z} \right|^2 \underset{|z| \rightarrow \infty}{\sim} \frac{1}{|z|^2} e^{\frac{2\mu}{\lambda} \text{Im } z} e^{-2(\text{Re } z)^2} \quad (2.19)$$

This means that an eigenfunction of (2.12) must be asymptotically a constant in the direction of the negative imaginary axis.

In particular this analysis excludes a formal second solution of (2.12) with $E = 0$ *)

$$\psi_c(z) = \int_0^z dz' \exp \left[-\frac{1}{2} z'^2 - \frac{\mu}{i\lambda} z' \right] \quad (2.20)$$

This function cannot be considered an eigenfunction because of its unacceptable behaviour (2.18b), as $z \rightarrow -i\infty$.

To sum up in the Hilbert space orthogonal to the vacuum (the only state of exactly zero energy), the eigenvalue conditions are

$$\psi(z) \text{ holomorphic} \quad (2.21a)$$

$$\psi(z) \underset{z \rightarrow -i\infty}{\sim} z \quad \psi(z) \underset{z \rightarrow -i\infty}{\sim} \text{Const.} \quad (2.21b)$$

Completely analogous conditions must be imposed on the conjugate equation, obtained from (2.6b). Let us note that in the Bargmann space we have **)

$$H^+(z, \lambda) = H(z, -\lambda) = [H(z^*, \lambda)]^* \quad (2.22)$$

*) We are indebted to A. Schwimmer for focusing our attention on this paradoxical feature of the existence of two zero-energy solutions, even for $\mu < 0$, and for correspondence on this subject.

**) For clearness we indicate explicitly in Eqs. (2.22) and (2.23) the dependence on the parameter λ .

Therefore eigenfunctions of the conjugate equation (2.6b) to eigenvalues $E_n^* = E_n^*$ are automatically provided by

$$\bar{\psi}_n(z, \lambda) = \psi_n(z, -\lambda) = \psi_n^*(z^*, \lambda) \quad (2.23)$$

if $\psi_n(z, \lambda)$ are eigenfunctions of (2.6a) with eigenvalues E_n .

Note that if a spectral representation like (2.7) is valid, the states $|\psi_n\rangle$ have to be normalized according to

$$\langle \bar{\psi}_n | \psi_m \rangle = \nu_n \delta_{nm} \quad (2.24)$$

where, by (2.23),

$$\langle \bar{\psi}_n | \psi_m \rangle = \int \frac{dz dz^*}{2i\pi} e^{-|z|^2} \psi_n(z^*) \psi_m(z) \quad (2.25)$$

This is not a norm and does not lead to any eigenvalue conditions (in general $|\langle \bar{\psi} | \psi \rangle|^{\frac{1}{2}} \leq \|\psi\| = \|\bar{\psi}\|$).

2.2 Restriction to the negative imaginary axis

Since the holomorphy of the solutions of (2.12) is automatic, it is possible to restrict ourselves to the negative imaginary axis, $z = -iq$, $q > 0$, where we impose the boundary conditions (2.21b). Let us write therefore

$$H_R = H(-iq, i \frac{d}{dq}) = \lambda q \left[-\frac{d^2}{dq^2} + (q-\rho) \frac{d}{dq} \right], \quad \rho = \frac{\mu}{\lambda} \quad (2.26)$$

The term proportional to d/dq can be eliminated by the transformation

$$\Psi(q) = F(q) \varphi(q), \quad F(q) = \exp \left[\frac{1}{4} (q-\rho)^2 \right] \quad (2.27)$$

giving the eigenvalue equation

$$G^{-1} \varphi_n = \left[-\frac{d^2}{dq^2} + \frac{1}{4} (q-\rho)^2 - \frac{1}{2} \right] \varphi_n(q) = \frac{E_n}{\lambda q} \varphi_n(q) \quad (2.28a)$$

or formally

$$F^{-1} H_R F = V^{-1} G^{-1}, \quad V(q) = \frac{1}{\lambda q} \quad (2.29)$$

where G^{-1} is a hermitian operator. According to (2.21b) and (2.27), ϕ_n must satisfy the boundary conditions

$$\varphi_n(q) \underset{q \rightarrow 0}{\sim} q, \quad \varphi_n(q) \underset{q \rightarrow \infty}{\sim} \exp \left[-\frac{1}{4} q^2 + \frac{1}{2} \rho q \right] \quad (2.28b)$$

which imply that $\phi_n \in L^2(0, \infty)$, and vanishes at the origin. We denote by $L_R^2(0, \infty)$ the space of functions satisfying both constraints.

We have shown in this way that the original eigenvalue problem is equivalent to (2.28) in the space $L_R^2(0, \infty)$ ^{*}, thus justifying the method used in Refs. 2 and 3.

In Eq. (2.28a) the energy, E_n , appears as a coupling constant of a Coulomb-like $(1/\lambda q)$ potential. The problem can be easily recast in a Hamiltonian form by the procedure of Refs. 2 and 3. In fact the change of variable $q = x^2$ and the similarity transformation induced by $q^{1/4}$ give

$$\begin{aligned} \hat{H}(x^2) &= q^{-1/4} F^{-1} H_R F q^{1/4} \\ &= \frac{\lambda}{4} \left[-\frac{d^2}{dx^2} + \frac{3}{4x^2} + x^2((x^2 - \rho)^2 - 2) \right] \end{aligned} \quad (2.30a)$$

$$\varphi(q) = (x^2)^{1/4} \hat{\varphi}(x), \quad q = x^2 \quad (2.30b)$$

which is Jengo's ³⁾ Hamiltonian. The angular momentum barrier with $\ell = \frac{1}{2}$ in (2.30a) corresponds to a regular behaviour

$$\hat{\varphi} \underset{|x| \rightarrow 0}{\sim} |x|^{3/2}$$

consistent with (2.28b). We therefore agree with Jengo's results except that here the vacuum state has been taken out from the beginning and does not belong to $L_R^2(0, \infty)$.

The latter fact is important in the formal steps that follow and implies in particular that all the E_n 's in (2.28a) are (real and) positive. In fact the eigenfunctions of G^{-1} which belong to $L_R^2(0, \infty)$ are explicitly known ⁹⁾ in terms of parabolic cylinder functions and the corresponding eigenvalues are strictly positive, so that G^{-1} is positive definite in $L_R^2(0, \infty)$. Note that in $L^2(-\infty, +\infty)$ G^{-1} coincides with the harmonic oscillator Hamiltonian and has eigenvalues, $\epsilon_n = n \geq 0$. It is the fact that we have restricted G^{-1} on the space of functions which vanish at the origin that eliminates the zero energy eigenvalue [remember that $\psi_0(z) = 1$]. By Eq. (2.28a), we get

$$0 < (\varphi_n, G^{-1} \varphi_n) = E_n (\varphi_n, \varphi_n) \quad (2.31)$$

and hence $E_n > 0$.

^{*}) We stress again that the equivalence relies on the holomorphy of the solutions of the type (2.15b) that we have chosen.

The qualitative behaviour of E_n can be obtained by studying the zero energy eigenstates in the effective potential of Fig. 2

$$W(q) = \frac{1}{4} (q - \rho)^2 - \frac{1}{2} - \frac{E}{\lambda q} \quad q \geq 0 \quad (2.32)$$

When $|\rho| \gg 1$, a self-consistent approximation is to take

$$W(q) \approx \frac{1}{4} \rho^2 - \frac{E}{\lambda q} \quad (2.33)$$

since it can be checked *a posteriori* that the other terms in (2.32) are indeed negligible in the range $0 < q \lesssim 1/\rho$ of the Coulombic wave function. In this way one gets

$$\frac{1}{4} \rho^2 - \left(\frac{E_n}{\lambda} \right)^2 \frac{1}{4n^2} \approx 0, \quad E_n \approx n |\rho| \quad (2.34a)$$

$$\varphi_n^c(q) \approx |\rho| \sqrt{\frac{\lambda}{n}} q e^{-|\rho|q/2} L_{n-1}^1(|\rho|q) \quad (2.34b)$$

where $L_n^\alpha(x)$ are the generalized Laguerre polynomials, as defined, for instance, in Ref. 10. In Eqs. (2.34b) we have adopted the normalization $(\phi_n, V\phi_n) = 1$ for reasons that will be clear soon.

After this qualitative discussion we now proceed to the more formal proof of the spectral representation and μ -analyticity.

The importance of the positivity properties of G^{-1} and E_n lies in the fact that the inverse operator G -- the Green's function of the problem -- exists in $L_R^2(0, \infty)$ and therefore Eq. (2.28a) can be rewritten as an integral equation

$$\varphi_n(q) = E_n \int_0^\infty dq' G(q, q') V(q') \varphi_n(q') \quad (2.35)$$

whose symmetrical form is

$$K \bar{\varphi}_n = \frac{1}{E_n} \bar{\varphi}_n \quad (2.36a)$$

$$K = V^{1/2} G V^{1/2} \quad (2.36b)$$

$$\bar{\varphi}_n = V^{1/2} \varphi_n \in L^2(0, \infty) \quad (2.36c)$$

$$(\bar{\varphi}_n, \bar{\varphi}_m) = (\varphi_n, V \varphi_m) = \delta_{nm} \quad (2.36d)$$

The kernel K , defined above, is roughly related to the inverse of $\hat{H}$ [Eq. (2.30a)] in $L^2_R(0, \infty)$.

There are two advantages in introducing K : on the one hand the condition that ϕ_n vanishes at $q = 0$ is embodied in the requirement that $\bar{\phi}_n = \sqrt{V}\phi_n \in L^2(0, \infty)$; in fact if $\phi_n(0) \neq 0$, $\sqrt{V} = 1/(\lambda q)^{1/2}$ would introduce a singularity in $\bar{\phi}_n$ at $q = 0$. So Eqs. (2.36) define a usual $L^2(0, \infty)$ eigenvalue problem. On the other hand, we show in Appendix A, by giving an explicit representation of G , that K is a Hilbert-Schmidt kernel for $-\infty < \mu < \infty$. As a consequence¹¹⁾ K is completely continuous and operator analytic in a neighbourhood of the real μ axis. In this way we will be able to prove both the spectral representation and the μ -analyticity of the S -matrix (2.4).

2.3 Spectral representation of the S -matrix

We want to relate the matrix elements of e^{-HY} between coherent states, of the type (2.4), to the kernel K defined in Eq. (2.36b). We start by recalling that for any operator $O = O(a^+, a)$, and in particular for the RFT Hamiltonian of Eq. (2.1), one has in the Bargmann's space the formula

$$\langle if | O | -ig \rangle = O(-if, -ig) e^{-fg} \quad (2.37)$$

On the semi-axis $z = -iq$, $q > 0$, let us denote by $H(q, q')$ the kernel which represents the action of H on a function $\psi(q)$:

$$H(-iq, i \frac{d}{dq}) \psi(q) = \int_0^\infty dq' H(q, q') \psi(q') \quad (2.38)$$

It is easy to see^{*)} that

$$\langle if | H | -ig \rangle = \int_0^\infty dq' H(q, q') e^{-gq'} \quad (2.39)$$

A similar relation holds for any power of H and therefore for e^{-HY} .

Denoting by H_R the restriction of H to the states orthogonal to the vacuum [i.e. to functions $\psi(q)$ which vanish at the origin], we can express the S -matrix (2.4) in terms of H_R by separating out the vacuum contribution. From (2.39) we obtain

*) It is sufficient to verify (2.39) on monomials of the type $N(a^+, a) = a^{+n} a^m$. Since for the corresponding kernel, as defined in (2.38), one has $N(q, q') = (-i)^{n+m} q^n \delta^{(m)}(q-q')$, Eq. (2.39) gives $\langle if | a^{+n} a^m | -ig \rangle = (-i)^{n+m} f^n g^m e^{-fg}$, in agreement with the general formula (2.37).

$$\begin{aligned}
 S(f, g; Y) - 1 &= \langle f | e^{-HY} (1 - |g\rangle\langle g| - |0\rangle\langle 0|) \\
 &= \int_0^\infty dq e^{-H_R Y} (f, q) (e^{-gq} - 1) = [e^{-H_R Y} M](f, g)
 \end{aligned}
 \quad (2.40)$$

where we have introduced the kernel

$$M(q, q') = e^{-qq'} - 1 \quad (2.41)$$

The relation between H_R and K is now easily established through (2.29) and (2.36b) and it is given by the similarity transformation

$$e^{-H_R Y} = F V^{-1/2} e^{-Y/K} V^{1/2} F^{-1} \quad (2.42)$$

Therefore

$$S(f, g; Y) - 1 = [F V^{-1/2} e^{-Y/K} V^{1/2} F^{-1} M](f, g) \quad (2.43)$$

Since the function

$$[V^{1/2} F^{-1} M](q) = \exp\left[\frac{1}{4}(q-p)^2\right] \frac{e^{-gq} - 1}{\sqrt{\lambda q}} \quad (2.44)$$

belongs to $L^2(0, \infty)$ for any real positive g , and $e^{-EnY} \rightarrow 0$ as $n \rightarrow \infty$, we can use the spectral representation [see Eqs. (2.36)]

$$e^{-Y/K} = \sum_{n=1}^{\infty} e^{-E_n Y} \bar{\varphi}_n \otimes \bar{\varphi}_n \quad (2.45)$$

to get

$$S(f, g; Y) = 1 + \sum_{n=1}^{\infty} e^{-E_n Y} [F \varphi_n](f) [M F^{-1} V \varphi_n](g) \quad (2.46)$$

The form of (2.46) is rather peculiar, because it does not look symmetrical in f and g , as it should as a consequence of the $a \leftrightarrow -a^\dagger$ symmetry of H . However, introducing the metric operator

$$\mathcal{M} = (-1)^{a^\dagger a} = \mathcal{M}^\dagger \quad (2.47)$$

with the properties

$$\begin{aligned}
 \mathcal{M} a^\dagger \mathcal{M} &= -a^\dagger & \mathcal{M} a \mathcal{M} &= -a \\
 \mathcal{M} H^\dagger \mathcal{M} &= H
 \end{aligned}
 \quad (2.48)$$

one gets

$$(\langle f | e^{-HY} | -ig \rangle)^* = \langle -ig | e^{-H^\dagger Y} | f \rangle = \langle ig | e^{-HY} | -if \rangle \quad (2.49)$$

This implies that the S-matrix, being real, is actually symmetric in f and g , so that necessarily one must have

$$F^{-1} M F^{-1} V \varphi_n = \nu_n N_n^2 \varphi_n, \quad \nu_n = \pm 1 \quad (2.50)$$

where $N_n^2 > 0$ is to be determined by the eigenvalue problem (2.50) itself*).

We can rewrite (2.46) in a form explicitly symmetric in f and g . In fact putting [see Eq. (2.27)]

$$\psi_n(-iq) = N_n F(q) \varphi_n(q) \quad (2.51)$$

by (2.50) we have

$$[M F^{-1} V \varphi_n](q) = \nu_n N_n \psi_n(-iq) \quad (2.52)$$

and (2.46) becomes

$$S(f, g; Y) = 1 + \sum_{n=1}^{\infty} \nu_n e^{-\epsilon_n Y} \psi_n(-if) \psi_n(-ig) \quad (2.53)$$

This is the spectral representation of (2.4) we were looking for. To see that it is precisely of the kind (2.7), it is enough to remember [Eq. (2.23)] that

$$\psi_n(-ig) = [\bar{\psi}_n(ig)]^* \quad (2.54)$$

so that (2.53) can be written in the form

$$S(f, g; Y) = 1 + \sum_{n=1}^{\infty} \nu_n e^{-\epsilon_n Y} \langle if | \psi_n \rangle \langle \bar{\psi}_n | -ig \rangle \quad (2.55)$$

It remains to be checked that the ψ_n 's are correctly normalized according to (2.24). This is done in Appendix B.

3. CONTINUATION TO $\alpha(0) > 1$. EXCITED STATES

In this section we want to discuss the definition of the S-matrix for $\mu > 0$, starting from the Eq. (2.43) valid for $\mu < 0$, by exploiting the analyticity property of the kernel K [Eq. (2.36b)] as a function of μ . We then study the spectrum

*) Eqs. (2.36) and (2.50) imply that the operators $V^{\frac{1}{2}} F^{-1} M F^{-1} V^{\frac{1}{2}}$ and K have the same eigenfunctions, therefore they commute. This fact can be checked directly.

of excited states of the Hamiltonian and we find that, beside a state almost degenerate with the vacuum with an energy gap $\sim e^{-\rho^2/2}$, there are pairs of levels whose energy difference turns out to be zero, when computed as a perturbative series in $1/\rho^2$.

3.1 Continuation to $\alpha(0) > 1$

For $\mu > 0$ perturbation theory, being a power series in $e^{\mu Y}$, makes no sense. Therefore neither H nor e^{-HY} can be defined as operators in the Bargmann's space to start with. However, we have proved in the previous section that the S-matrix can be given through Eqs. (2.43) and (2.46) in terms of the kernel K . Since the parameter μ enters analytically only in the Green's function G , we shall define the S-matrix for $\mu > 0$ by the analytic continuation of the expressions (2.43) and (2.46).

We prove in Appendix A that K is operator analytic in μ for $-\infty < \mu < +\infty$ in the $L^2(0, \infty)$ space. So is, therefore, the operator $e^{-Y/K}$. Since in Eq. (2.43) it acts on functions which belong to $L^2(0, \infty)$, this proves the analyticity in μ of the S-matrix (2.4) and of the Pomeron Green's functions (2.3).

Note that our procedure is meaningful only because we are working with fixed boundary conditions [vanishing of $\phi(q)$ at 0 and ∞], which in fact define the Hilbert space $[\sqrt{V} \phi = \bar{\phi} \in L^2(0, \infty)]$ on which our operators are acting. Had we tried to work directly in the Bargmann's space, at $\mu = 0$ for example, there would have been no boundary condition at all, while the boundary condition for $\mu > 0$ would have been on the positive imaginary axis^{*)}. If we want to recover perturbation theory for $\mu < 0$ and λ small, then our definition of the S-matrix is the only physically acceptable one.

3.2 Discussion of the eigenvalue problem for $\mu > 0$

The eigenvalue problem for K is equivalent to the eigenvalue problem (2.28), both for μ positive and negative^{**)}. Therefore we again have to look for zero energy states in the effective potential $W(q)$ of Eq. (2.32), which for $\mu > 0$ looks as in Fig. 3.

The novel feature of this potential is the two-well structure around $q = 0$ and $q = \rho$ which has suggested the existence of the phase transition (vacuum degeneracy) found⁴⁾ in a higher number of dimensions.

*) Indeed the eigenvectors $\psi_n(z)$ of H , defined by analytic continuation from $\psi_n(-iq)$ do not have finite Bargmann's norm, $\langle \psi | \psi \rangle$, for $\mu > 0$. They are, however, properly normalized according to (2.24).

**) This is the main advantage of the present approach with respect to that of Ref. 7.

In the $D_t = 0$ problem we do not have ground-state degeneracy, but there is a state orthogonal to the vacuum, of nearly zero energy for λ/μ small²⁾.

We can understand intuitively the existence of this state as follows: if $\phi_n(q)$ in (2.28a) did not have to vanish at $q = 0$, a zero energy solution would have simply been the ground state of the harmonic well (see Fig. 3). However, we must modify this harmonic ground state wave function close to the origin to match the boundary condition, so that its energy becomes slightly positive. Since the value of the wave function at the origin is $\sim e^{-\rho^2/4}$, this energy will be of the order of $e^{-\rho^2/2}$. This must be compensated by the attractive Coulomb potential $E/\lambda q \approx E/\mu$, giving $E \propto e^{-\rho^2/2}$.

This behaviour is typical of a tunnelling effect and was indeed interpreted as such in Ref. 2. However, in the present formalism it appears in a slightly different way. Let us now give a more precise derivation of it.

As we said, the eigenfunctions of G^{-1} in $L_R^2(0, \infty)$ are parabolic cylinder functions, $D_\nu(q - \rho)$, where⁹⁾

$$G^{-1} D_\nu(q - \rho) = \nu D_\nu(q - \rho) \quad (3.1)$$

The energy ν of the wave function is determined by the boundary condition [Eq. (2.28b)]

$$D_\nu(-\rho) = 0 \quad (3.2)$$

Since $\rho \gg 1$, the lowest value of ν can be computed by using the asymptotic expansion of the D_ν function for large arguments⁹⁾

$$D_\nu(x) \underset{x \rightarrow -\infty}{\sim} x^\nu \exp\left(-\frac{1}{4}x^2\right) \left(1 - \frac{\nu(\nu-1)}{2x^2} + \dots\right) - \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i\pi\nu} e^{\frac{1}{4}x^2} x^{-\nu-1} \left(1 + \frac{(\nu+1)(\nu+2)}{2x^2} + \dots\right) \quad (3.3)$$

and one easily gets

$$\nu = \frac{\rho}{\sqrt{2\pi}} e^{-\frac{1}{2}\rho^2} [R(\rho)]^{-1} \approx \frac{\rho}{\sqrt{2\pi}} e^{-\frac{1}{2}\rho^2} \left(1 - \frac{1}{\rho^2} + O\left(\frac{1}{\rho^4}\right)\right) \quad (3.4)$$

where

$$R(\rho) = 1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{\rho^{2n}} \quad (3.5)$$

We now compute the average value of the Coulomb-like potential, $E/\lambda q$, in Eq. (2.28a), by noticing that in the region where $D_v(q - \rho)$ is large (i.e. $q \approx \rho$), one can write

$$D_v(q - \rho) \underset{q \sim \rho}{\simeq} e^{-\frac{1}{4}(q - \rho)^2} (1 + O(e^{-\frac{1}{2}\rho^2})) \quad (3.6)$$

so we get

$$E(D_v, V D_v) = \frac{E}{\lambda \rho} R(\rho) \simeq \frac{E}{\mu} \left(1 + \frac{1}{\rho^2} + O\left(\frac{1}{\rho^4}\right)\right) \quad (3.7)$$

Equating (3.4) and (3.7), we obtain for the energy, E_1 , of the lowest non-zero energy state

$$E_1 \simeq \frac{\mu \rho}{\sqrt{2\pi}} e^{-\frac{1}{2}\rho^2} [R(\rho)]^{-2} \simeq \frac{\mu \rho}{\sqrt{2\pi}} e^{-\frac{1}{2}\rho^2} \left(1 - \frac{2}{\rho^2} + O\left(\frac{1}{\rho^4}\right)\right) \quad (3.8)$$

in agreement with previous results^{4,7)}. Here we have also been able to evaluate all the corrections of order $(1/\rho^2)^n$. Indeed we see *a posteriori* that, consistently, all the results (wave function, energy) are valid up to the order of the tunnelling energy, $e^{-\rho^2/2}$.

The wave function is especially interesting. Going back to the ψ -like wave function [Eq. (2.27)], we find for this state which we will indicate by $|\psi_1\rangle$:

$$\psi_1(z) = \langle z^x | \psi_1 \rangle \simeq e^{\frac{1}{4}(iz - \rho)^2} D_v(iz - \rho) \quad (3.9)$$

with v given by (3.4). Around $z = 0$ we can write

$$\psi_1(z) \underset{z \sim 0}{\simeq} 1 + \rho \frac{e^{-i\rho z - \frac{1}{2}z^2}}{iz - \rho} \simeq 1 - e^{-i\rho z} \quad (3.10)$$

This last equation shows that $|\psi_1\rangle$ is approximately the sum of the vacuum state and a coherent state

$$|\psi_1\rangle = (1 - e^{-i\rho a^\dagger}) |0\rangle \quad (3.11)$$

The normalization of (3.11) is correct [$\langle \bar{\psi}_1 | \psi_1 \rangle = -1 + O(e^{-\rho^2})$], as for the more accurate expression (3.9)], but the evaluation of small matrix elements through it must be handled with some care. For example, one would find $\langle \bar{\psi}_1 | H | \psi_1 \rangle \simeq e^{-\rho^2}$, while the correct result is $\langle \bar{\psi}_1 | H | \psi_1 \rangle \simeq e^{-\rho^2/2}$. However, one can safely compute matrix elements of field operators like $\langle \bar{\psi}_1 | (a^\dagger)^n | \psi_0 \rangle$. In any case it is always possible to use the more accurate representation (3.9).

3.3 Higher excited states

Together with the two almost degenerate lowest states, $|\psi_0\rangle$ and $|\psi_1\rangle$, we confirm the existence of two sets of nearly degenerate (for $\rho \gg 1$) excited

states with eigenfunctions which are either mainly Coulombic or mainly harmonic. In fact from the potential (2.32) one can argue that close to the Coulomb well ($0 < q \lesssim \lambda/\mu$) the harmonic potential is a constant $\sim \frac{1}{4}\rho^2$, so that

$$\frac{1}{4}\left(\frac{\mu}{\lambda}\right)^2 - \left(\frac{\bar{E}_n^c}{\lambda}\right)^2 \frac{1}{4n^2} \approx 0, \quad \bar{E}_n^c \approx n\mu, \quad n \geq 1 \quad (3.12)$$

Similarly for $|q - \rho| \lesssim 1/\rho$, $V = 1/\lambda q \approx 1/\mu$ and the harmonic well gives

$$n - \frac{E_n^h}{\mu} \approx 0, \quad E_n^h \approx n\mu, \quad n \geq 1 \quad (3.13)$$

One interesting question is whether this degeneracy is broken by terms of order $\mu(1/\rho)^k$, with $k \geq 1$, or instead only by terms of order $e^{-\rho^2/2}$ (tunnel effect). The latter possibility is trivially realized in $\lambda\phi^4$ theory, where because of the $\phi \leftrightarrow -\phi$ symmetry, the energy shifts coming from the anharmonic part of the potential are the same in the two wells, to each order of perturbation theory in $1/\rho$. Therefore only the off-diagonal (tunnelling) matrix elements contribute to the shift. The situation is different in our case, because there is no apparent symmetry operation connecting the two wells, neither for our kernel K , nor for Jengo's Hamiltonian $\hat{H}$ of Eq. (2.30a). The $a \leftrightarrow -a^+$ symmetry in (2.49) induced by (2.48) [or the classical $p \leftrightarrow q$ exchange symmetry^{*)}] has to do with the relation between left and right states and not with the connection between the two vacua. Classically there is a cyclic symmetry^{**)}

$$p \rightarrow q \rightarrow \frac{\mu}{\lambda} - p - q \quad (3.14)$$

which implies the degeneracy of the potential close to $(0,0)$ and $(0, \mu/\lambda)$ [or $(\mu/\lambda, 0)$]. However, one expects this classical symmetry, which involves both p and q , to be broken in the quantum problem by relative order $1/\rho^2$ due to the commutation relation $[q, p] = 1$.

We have been unable to rigorously establish (or disprove) the $e^{-\rho^2/2}$ -shift for the pairs of higher excited states. However, we have ascertained two facts which point in the direction of the tunnelling result:

- i) Taking as zero-order approximation the two sets of Coulomb-like and harmonic-like wave functions which lead to (3.12) and (3.13), we have found that in perturbation theory the energy shifts due respectively to the harmonic

*) We define, following the notations of Ref. 4, $p = ia$ and $q = ia^+$. Then (2.47) becomes $\mathcal{M} p^r \mathcal{M} = q$.

***) The Lagrangian itself can be expressed in a completely symmetric way in the variables q, p and r , with the constraint $r = \mu/\lambda - p - q$.

correction and to the Coulombic correction are equal for all levels up to second order, i.e. up to relative order $(1/\rho^2)^2$. The explicit perturbative result is^{*)}:

$$E_n^c = E_n^h = n\mu \left[1 - \frac{1}{\rho^2} (1+3n) - \frac{1}{\rho^4} (2+6n+12n^2) + O\left(\frac{1}{\rho^6}\right) \right] \quad (3.15)$$

ii) The semi-classical Bohr's quantization rule, applied to Jengo's Hamiltonian gives exact degeneracy of the two wells' energy levels.

The latter result can be proved by a simple analytical argument. Let us in fact consider the action variables ($\varepsilon = 4E/\lambda$)

$$J_c(\varepsilon) = \int_{a_1}^{a_2} dx [\varepsilon - U_\alpha(x)]^{1/2} \quad (3.16a)$$

$$J_h(\varepsilon) = \int_{b_1}^{b_2} dx [\varepsilon - U_\alpha(x)]^{1/2} \quad (3.16b)$$

where

$$U_\alpha(x) = \frac{\alpha^2}{x^2} + x^2[(x^2 - \rho)^2 - 2] \quad (3.17)$$

and the integration limits are defined in Fig. 4. J_c and J_h are easily related to the complex integration contours C_c and C_h in Fig. 5 of the function $[\varepsilon - U_\alpha(x)]^{1/2}$, whose determinations are explicitly shown. It follows from Cauchy's theorem

$$4(J_c - J_h) + \int_{C_\infty} dx [\varepsilon - U_\alpha(x)]^{1/2} = \int_{C_0} dx [\varepsilon - U_\alpha(x)]^{1/2} \quad (3.18)$$

where we have used the fact that $[\varepsilon - U_\alpha(x)]^{1/2}$ is odd in x and that its discontinuities along C_h and C_c have opposite signs. The pole contribution, C_0 , comes from the centrifugal potential. The C_∞ contribution can be evaluated by isolating the $1/x$ term in the asymptotic expansion of $[\varepsilon - U_\alpha(x)]^{1/2}$. After simple algebra we get

$$4(J_h - J_c) = 2i\pi(-i|\alpha|) - 2i\pi\left(-\frac{i}{2}\right)(\rho^2 - 2 - \frac{(2\rho)^2}{4}) = 2\pi(1+|\alpha|) \quad (3.19)$$

and from Bohr's quantization rule

*) In fact the degeneracy [see Eq. (3.21)] is between the fundamental Coulomb level and the first excited harmonic level and so on. The fundamental harmonic level does not enter in Eq. (3.15) and goes together with the perturbative vacuum.

$$J_h - J_c = \frac{\pi}{2} (1 + |\alpha|) = \pi (n_h(\varepsilon) - n_c(\varepsilon)) \quad (3.20)$$

Since in our case $|\alpha| = 1$, corresponding to $\ell = \frac{1}{2}$ ^{*}), we have degeneracy for

$$n_h = n_c + 1 \quad (3.21)$$

This result is consistent with the fact that no "Coulomb state" corresponds to the harmonic ground state $n_h = 0$ ^{**}).

To sum up, we have analysed in detail the state $|\psi_1\rangle$ (3.9-3.10) connected to the vacuum $|\psi_0\rangle$ by the approximate matrix elements:

$$\langle \bar{\psi}_1 | a^\dagger | \psi_0 \rangle = \langle \bar{\psi}_0 | a | \psi_1 \rangle \approx \frac{\mu}{\lambda} \quad (3.22)$$

which are fundamental in deriving the spin model of Ref. 4, and we have confirmed the existence of a small energy gap contributing to the asymptotic S-matrix. We have also derived some properties of the higher excited states. However, we have been unable to perform a complete quantitative analysis and we believe that the tunnelling picture needs further clarification.

4. SEMI-CLASSICAL LIMIT

In this section we wish to examine how one can get the classical limit of the quantum theory for small Y and determine the parameters which control this limit.

We start from the expression of the S-matrix obtained in Eq. (2.46), which we write explicitly

$$S(p, q; Y) = 1 + e^{\frac{1}{4}(p-p)^2} \int_0^\infty dq \sum_{n=1}^\infty e^{-\bar{\epsilon}_n Y} \frac{\varphi_n(p) \varphi_n(q)}{\lambda q} e^{-\frac{1}{4}(q-p)^2} \frac{e^{-gq} - 1}{\lambda q} \quad (4.1)$$

and we approximate for $p \gg 1$ the evolution operator in Eq. (2.45) by its semi-classical expression

$$\begin{aligned} V^{-1/2} e^{-Y/K} V^{1/2} &= \sum_{n=1}^\infty e^{-\bar{\epsilon}_n Y} \frac{\varphi_n(p) \varphi_n(q)}{\lambda q} \approx \\ &\approx \int_0^\infty d\bar{\epsilon} \gamma(\bar{\epsilon}) \exp \left[- \int_p^q dq' \sqrt{\frac{1}{4}(q'-p)^2 - \frac{1}{2} - \frac{\bar{\epsilon}}{\lambda q'}} - \bar{\epsilon} Y \right] \end{aligned} \quad (4.2)$$

where $\gamma(E)$ is the level density.

^{*}) The correct phase in Bohr's quantization is obtained by putting $\alpha^2 = (\ell + \frac{1}{2})^2$ and not $\alpha^2 = \ell(\ell + 1)$.

^{**}) See note on page 17.

The integrals over q in (4.1) and E in (4.2) will now be evaluated by the saddle point method. The exponent in (4.1) is stationary in q when

$$q \equiv q_i = \frac{1}{2} (p - g) - \sqrt{\frac{1}{4} (p - g)^2 - \frac{E}{\lambda g}} \quad (4.3)$$

where we have neglected the quantum zero point energy $\frac{1}{2}$ in comparison with $\frac{1}{4}(q - p)^2$. This is legitimate provided we do not arrive too close to a turning point of the classical trajectory. As we shall see, this is always the case when Y is small enough.

Notice that Eq. (4.3) means that the point ($p_i = g$, $q = q_i$) is located on the classical trajectory of energy

$$E = q_i \left(\mu - \lambda (q_i + p_i) \right) p_i \quad (4.4)$$

The stationary condition in E gives

$$Y(E) = \frac{1}{2} \int_p^{q_i} \frac{dq'}{\lambda q'} \left[\frac{1}{4} (q' - p)^2 - \frac{E}{\lambda q'} \right]^{-1/2} \quad (4.5)$$

Equation (4.5) determines the energy $E(f, g; Y)$ of the classical trajectory corresponding to the boundary conditions¹⁾

$$p(0) = p_i = g \quad q(Y) = f \quad (4.6)$$

The exponential factor in the integral (4.1) after the approximation (4.2) is evaluated at $q = q_i(f, g; Y)$ [Eq. (4.3)] and $E = E(f, g; Y)$ [Eq. (4.5)]. By putting all terms together we get the semi-classical result for the S-matrix

$$\begin{aligned} S &\sim e^{-A} \\ A(f, g; Y) &= - \int_p^{q_i} dq \left[\frac{1}{4} (p - q)^2 - \sqrt{\frac{1}{4} (p - q)^2 - \frac{E}{\lambda q}} \right] + EY + g q_i \\ &= - \int_p^{q_i} p dq + EY + q_i g \end{aligned} \quad (4.7)$$

Let us now see how q_i , E and A depend on f, g and Y for small λ/μ . The trajectory which corresponds to the boundary conditions (4.6) for moderate μY is the (a_1, b_1) trajectory of Fig. 6, in which we have taken $g < f$. In the limit

$$\frac{f}{p} e^{\mu Y} \ll 1 \quad \frac{g}{p} e^{\mu Y} \ll 1 \quad (4.8)$$

explicit formulae can be obtained, using the expansions

$$\left(\frac{1}{4} (q - p)^2 - \frac{E}{\lambda q} \right)^{1/2} \approx \frac{p}{2} \left(1 - \frac{1}{p} \left(q + \frac{2E}{\mu q} \right) + O\left(\frac{1}{p^2}\right) \right) \quad (4.9a)$$

$$p \approx \frac{E}{q\mu} \left(1 + \frac{1}{\rho} \left(q + \frac{E}{q\mu} \right) + O\left(\frac{1}{\rho^2}\right) \right) \quad (4.9b)$$

which allow to compute

$$\int_f^{q_i} p \, dq \approx \frac{E}{\mu} \left[\log \frac{q_i}{f} + \frac{1}{\rho} (q_i - f) \left(1 + \frac{g}{f} \right) \right] \quad (4.10)$$

and

$$Y \approx \frac{1}{\mu} \left[\log \frac{q_i}{f} + \frac{1}{\rho} (q_i - f) \left(1 + \frac{2g}{f} \right) \right] \quad (4.11)$$

It then follows

$$q_i = f e^{\mu Y} \left[1 - \frac{f+2g}{\rho} (e^{\mu Y} - 1) + O\left(\frac{1}{\rho^2}\right) \right] \quad (4.12a)$$

$$E = \mu f g e^{\mu Y} \left[1 - \frac{f+g}{\rho} (2e^{\mu Y} - 1) + O\left(\frac{1}{\rho^2}\right) \right] \quad (4.12b)$$

$$A = f g e^{\mu Y} \left[1 - \frac{f+g}{\rho} (e^{\mu Y} - 1) + O\left(\frac{1}{\rho^2}\right) \right] \quad (4.12c)$$

Equations (4.12) explicitly show that, if the conditions (4.8) hold, q_i , E and A are given by the perturbative result.

When one leaves the perturbative regime (4.8) the integrals cannot be computed analytically any more; however, an intuitive picture of what happens as Y increases, can be obtained with the help of Fig. 6, by looking at the trajectories compatible with the boundary conditions (4.6).

For $Y = 0$ the trajectory is reduced to a point, P , which is located on the orbit of energy $E_0 \approx \mu f g$ at the intersection of the lines $p = g$ and $q = f$. Let us increase Y : the trajectory is now the portion (a_1, b_1) of the orbit of energy E_1 , with $E_1 > E_0$. If we continue to increase Y , at some rapidity Y_g we will reach an orbit which is tangent to the line $p = g$ and whose energy, $E_g (> E_1)$, is the maximum allowed value of E . In fact for $Y > Y_g$ the only possibility is to go back to one of the previous orbits (say the orbit of energy E_1), but taking now the portion (a'_1, b_1) of it. This corresponds to choosing the opposite determination of the square root in the expression of q_i :

$$q_i = \frac{1}{2}(\rho - g) + \sqrt{\frac{1}{4}(\rho - g)^2 - \frac{E}{\lambda g}} \quad , \quad Y > Y_g \quad (4.13)$$

It is easy to compute E_g and Y_g and one finds

$$E_g = \frac{\lambda}{4} g(\rho - g)^2 \quad (4.14a)$$

$$Y_g = \frac{1}{2} \int_f^{\frac{1}{2}(\rho - g)} \frac{dq}{\lambda q} \left(\frac{1}{4} (\rho - q)^2 - \frac{E_g}{\lambda q} \right)^{-1/2} \quad (4.14b)$$

From Eqs. (4.5) and (4.14) one concludes that the function $Y = Y(E)$, shown in Fig. 7, has a square root branch point at $E = E_g$ (B_1 , in Fig. 7). For $Y > Y_g$, E decreases and finally tends to zero as e^{-Y} . A typical trajectory for very large Y is the portion (a'_2, a_2) of the orbit of energy E_2 .

Up to now we have followed the trajectory for increasing values of Y and we have found that for the same value of E , say E_1 , there are two values of Y corresponding respectively to the portions (a_1, b_1) -- the perturbative trajectory -- and (a'_1, b_1) of the orbit of energy E_1 . But from Fig. 6 it is clear that for the same energy there are other possible (larger) values of Y , that is, other trajectories compatible with the boundary conditions (4.6). They are, in order of increasing Y , the portions (a_1, b'_1) and (a'_1, b'_1) of the orbit of energy E_1 *).

The general structure of the function $Y = Y(E)$ in this region can be obtained by a detailed study of the properties of these trajectories. A trajectory of the type (a_1, b'_1) can only appear at a sufficiently large value of Y , which we call Y_c , corresponding to an orbit of energy $E_c < E_g$. For $Y > Y_c$ two different orbits are allowed (Fig. 6): one with $E > E_c$ [in which $q(0) > a_c$ and $p(Y) < b'_c$] and one with $E < E_c$ [in which $q(0) < a_c$ and $p(Y) > b'_c$]. This means that $Y = Y(E)$ has a minimum around $E = E_c$, and $Y'(E_c) = 0$.

This behaviour of $Y(E)$ goes on until Y reaches a value, Y'_g , for which the largest of the two possible values of the energy is again E_g . Here there is a new branch point (B_2 , in Fig. 7) of $Y(E)$, and the trajectory in which (a_g, b'_g) is turned for $Y > Y'_g$, is of the type (a'_1, b'_1) . The value of Y'_g can be computed by noticing that, by symmetry, the line $q = g$ is tangent to the orbit of energy E_g . So that we can write

$$\begin{aligned} Y'_g &= \frac{1}{2} \left(\int_{q_i}^f + \int_f^g - \int_g^f \right) \frac{dq}{\lambda q} \left(\frac{1}{4} (\rho - q)^2 - \frac{E_g}{\lambda q} \right)^{-1/2} = \\ &= Y_g + \int_f^g \frac{dq}{\lambda q} \left(\frac{1}{4} (\rho - q)^2 - \frac{E_g}{\lambda q} \right)^{-1/2} \end{aligned} \quad (4.15)$$

*) Of course, at any given value of E , there are infinitely many other trajectories, obtained by adding any number of closed orbit loops (see Fig. 7).

To sum up we have seen that starting from $Y \approx 0$ with the unique trajectory of energy $E \sim \mu f g e^{\mu Y} = E_0 e^{\mu Y}$, we meet a square root branch point in $Y = Y(E)$ at $Y = Y_g$, $E = E_g$ [Eqs. (4.14)] which leads to a reflection of the energy. The perturbative trajectory (a_1, b_1) joins smoothly with the asymptotic trajectory of minimal action (a'_1, b'_1) . The trajectories (a_1, b'_1) and (a'_1, b_1) appear for $Y > Y_c$. All three contribute to the asymptotic behaviour of the S-matrix with the exponential factors¹⁾

$$(a_1, b_1) \rightarrow e^{-\rho Y} \quad (a_1, b'_1) \rightarrow e^{-\rho f Y} \quad (a'_1, b'_1) \rightarrow e^{-\rho(f+g)Y} \quad (4.16)$$

This exercise shows that the S-matrix must have singularities at (f, g, μ) -dependent values of Y , when $e^{-(\mu/\lambda)^2 Y} \rightarrow 0$ (classical limit). Since $E(Y)$ has a singularity at Y_c , this is presumably also a singularity of the action and of the classical S-matrix. This means that the analyticity in μ of the quantum-mechanical action does not go through in the classical limit.

For fixed f, g and μ/λ , if Y stays below Y_c , the classical limit is smooth and, at small enough μY and λ/μ , such that the conditions (4.8) are satisfied, our quantum-mechanical treatment joins with perturbation theory. To actually see how the perturbative result is re-obtained in the quantum theory, one may use the semi-classical argument developed in this section to conclude that only the values of q not too far from $q_1 \approx f e^{\mu Y} \ll \rho$ [Eq. (4.8)] contribute appreciably in Eq. (4.1). Therefore, only the states of the Coulomb well [Eq. (2.34b)] will be important in the sum (4.1). Following Jengo's^{*)} method, this sum can be explicitly evaluated, giving

$$S(f, g; Y) \approx 1 + \sum_{n=1}^{\infty} e^{\frac{1}{4}(\rho-f)^2 Y} \varphi_n(f) e^{-n\mu Y} \cdot \int_0^{\infty} dq \varphi_n(q) e^{-\frac{1}{4}(q-\rho)^2 Y} \frac{e^{-gY}}{\lambda g} \approx \exp[-fg e^{\mu Y}] \quad (4.17)$$

in accordance with the semi-classical result (4.7) and (4.12c).

We conclude this section by stressing the importance of the parameters $(f/\rho)e^{\mu Y}$ and $(g/\rho)e^{\mu Y}$, which in fact control the approach to asymptopia when $\mu Y \rightarrow \infty$. To see this in the simplest way, let us consider the "single field"

$$-\frac{\partial S}{\partial g} \Big|_{g=0} = \frac{1}{\lambda} e^{\frac{1}{4}(\rho-f)^2 Y} \sum_{n=1}^{\infty} e^{-n\mu Y} \varphi_n(f) \int_0^{\infty} dq \varphi_n(q) e^{-\frac{1}{4}(q-\rho)^2 Y} \quad (4.18)$$

*) For $\mu < 0$ the sum has been first evaluated in Refs. 2 and 3. An analogous computation can be done for $\mu < 0$ ¹²⁾.

in the "tunnelling region"

$$\frac{\lambda}{\mu} \ll f \ll \frac{\mu}{\lambda} \quad (4.19)$$

between the Coulomb well and the harmonic well^{*)}. We want to argue that it is precisely the parameter $(f/\rho)e^{\mu Y}$ that controls the relative magnitude of the non-leading terms in (4.18), when $\mu Y \rightarrow \infty$. We start by noticing that for $q \ll \mu/\lambda$, the eigenvalue equation (2.12) becomes simply

$$q \left(\mu + \lambda \frac{d}{dq} \right) \frac{d}{dq} \psi_n(q) \simeq -n\mu \psi_n(q) \quad (4.20)$$

and the two possible behaviours of $\psi_n(q)$ are

$$\psi_n(q) \simeq q^n e^{-\rho q} \left(1 + O\left(\frac{1}{\rho q}\right) \right) \quad (4.21a)$$

$$\psi_n(q) \simeq q^{-n} \left(1 + O\left(\frac{1}{\rho q}\right) \right) \quad (4.21b)$$

The first type of behaviour appears in the Coulomb-like wave functions [Eq. (2.34b)], while the second one matches the harmonic-like wave functions

$$\psi_n^h(q) \simeq \frac{1}{q^n} \left(\frac{\mu}{\lambda} - f \right)^n \simeq \left(\frac{\mu}{\lambda q} \right)^n \quad (4.22)$$

and dominates in the tunnelling region. By introducing (4.22) in (4.18), we get a power series in $[(f/\rho)e^{\mu Y}]^{-1}$ which therefore appears as the parameter which controls the asymptotic behaviour of the "single field":

$$-\frac{\partial S}{\partial g} \Big|_{g=0} \xrightarrow{\mu Y \rightarrow \infty} \frac{\mu}{\lambda} + O\left(\frac{\rho}{f} e^{-\mu Y}\right) \quad (4.23)$$

The leading term in (4.23) comes from the state almost degenerate with the vacuum.

Of course, both the series expansion of the "single field" in powers of $(\rho/f)e^{-\mu Y}$ and the behaviour (4.23), are features of the classical expression, obtained by Schwimmer¹³⁾

$$-\frac{\partial S}{\partial g} \Big|_{g=0}^{\text{class}} = \frac{\rho e^{\mu Y}}{1 + \frac{\lambda \rho}{\mu} (e^{\mu Y} - 1)} \underset{\mu Y \gg 1}{\simeq} \frac{\rho}{1 + \frac{\rho}{f} e^{-\mu Y}} \quad (4.24)$$

*) For $f \ll \lambda/\mu$, $-(\partial S/\partial g)_{g=0}$ can be directly computed from (4.7) and (4.12c).

which we have been unable to recover exactly, owing to our lack of detailed knowledge of the higher excited states.

5. INTRODUCTION OF A FOUR-POMERON COUPLING

In this section we wish to discuss the introduction of a four-Pomeron coupling in the theory for two reasons: first we want to check that the results of Sections 2 and 3 are stable with respect to it; secondly we want to understand whether the very simple results obtained by Bronzan and Sugar⁸⁾ for a special ("magic") value of the four-Pomeron coupling can be continuously related to the case in which this coupling is zero.

There are two possible forms of a four-Pomeron coupling:

$$i) \quad \lambda' \bar{\psi}^2 \psi^2 \quad (5.1a)$$

$$ii) \quad \lambda'' (\bar{\psi}^3 \psi + \bar{\psi} \psi^3) \quad (5.1b)$$

The second type of interaction is more complicated to analyse, since it leads to a third-order differential equation. We shall, in the following, restrict ourselves to the case (i).

The Hamiltonian we have to study is:

$$H = -\mu a^\dagger a + i\lambda a^\dagger (a^\dagger + a) a + \lambda' a^{\dagger 2} a^2 \quad (5.2)$$

Let us begin with a brief discussion of the classical problems, generalizing the work of Ref. 1.

5.1 Classical analysis

As in Ref. 1 we set

$$p = i a \quad q = i a^\dagger \quad (5.3)$$

p and q are classical variables which depend on the rapidity y . Their equations of motion are

$$\frac{dp}{dy} = \mu p - \lambda p^2 - 2\lambda q p + 2\lambda' q p^2 \quad (5.4a)$$

$$\frac{dq}{dy} = -\mu q + \lambda q^2 + 2\lambda q p - 2\lambda' q^2 p \quad (5.4b)$$

The fixed points (obtained by solving $dp/dy = dq/dy = 0$) are given by the following pairs of (q, p) values

$$\begin{array}{lll} 1) (0, 0) & 2) (p, 0) & 3) (0, p) \\ 4) (f_-, f_-) & 5) (f_+, f_+) & \end{array} \quad (5.5)$$

where

$$f_{\pm} = \frac{3}{4} \sigma \left(1 \pm \sqrt{1 - \frac{8}{9} \frac{p}{\sigma}} \right), \quad \sigma = \frac{\lambda}{\lambda'}, \quad (5.6)$$

Note that when $\sigma \rightarrow \infty$ ($\lambda' \rightarrow 0$) the last point in (5.5) moves to infinity, since

$$f_+ \xrightarrow{\sigma \rightarrow \infty} \frac{3}{2} \sigma \quad f_- \xrightarrow{\sigma \rightarrow \infty} \frac{1}{3} p \quad (5.7)$$

while the others stay finite.

As the energy E is conserved, the lines $E = 0$ in the (q, p) plane cannot be crossed by any trajectory. These lines are given by

$$1) \quad p = 0 \quad (5.8a)$$

$$2) \quad q = 0 \quad (5.8b)$$

$$3) \quad p = \sigma \frac{p - q}{\sigma - q} \quad (5.8c)$$

It is then easy to draw the classical trajectories. They are shown for $\sigma > p$ in Fig. 8a. In Fig. 8b we recall for completeness the well-known situation for $\lambda' = 0$ ¹⁾.

For small λ' there is no qualitative modification of the results of Ref. 1. However, when λ' reaches the ("magic") value

$$\lambda' = \lambda'_0 = \frac{\lambda^2}{p}, \quad \text{i.e.} \quad \sigma = p \quad (5.9)$$

there appear drastic modifications. Indeed the hyperbola (5.8c) becomes degenerate with its asymptotes and the triangle of Fig. 8b becomes a square. The fixed point f_+ is at a vertex of this square and there are no trajectories going from the point $(p, 0)$ to $(0, p)$.

Explicitly for $\lambda' < \lambda'_0$ the trajectory of zero energy is given by

$$q(y) = \frac{1}{2} p \left(1 + \tanh \frac{p}{2} (y + y_1) \right) \quad (5.10)$$

$$p(y) = \frac{1}{2} p \left(1 - \tanh \frac{p}{2} (y - y_1) \right)$$

where

$$\tanh \mu y_+ = \frac{\rho/2\sigma}{1 - \rho/2\sigma} \quad (5.11)$$

When $\sigma = \rho$ the trajectories, by varying y , go either from $(\rho, 0)$ to (f_+, f_+) or from (f_+, f_+) to $(0, \rho)$.

Going to smaller values of σ , the fixed points (f_+, f_+) finally disappear for $8\rho/9\sigma = 1$ and the situation bears no resemblance to the case $\lambda' = 0$.

Let us remark that the existence of a critical value $(1/\mu)$ in the parameter λ'/λ^2 is not surprising, because some approaches to RFT, as the topological expansion¹⁴⁾ predict a Pomeron interaction potential of the form $1/\lambda^2 V(\lambda\psi, \lambda\bar{\psi})$. Then λ' is measured in the natural unit λ^2 , and the value λ'/μ is to be considered large, because it corresponds to comparable magnitudes of non-enhanced and enhanced graphs.

5.2 Quantum problem

Using the coherent state representation, we get from the Hamiltonian (5.2) the differential equation

$$\left[-\frac{d^2}{dz^2} - i\sigma \frac{iz - \rho}{iz - \sigma} \frac{d}{dz} + i\sigma \frac{E/\lambda}{iz - \sigma} \right] \psi(z) = 0 \quad (5.12)$$

Remember that we must look for solutions which are analytic in the whole complex z -plane and which have a finite norm in the Bargmann's space [Eq. (2.14)]. We also demand that $\psi(z)$ vanishes at the origin, in order to ensure orthogonality to the vacuum.

The possible asymptotic behaviours of the solutions of (5.12) are

$$\psi(z) \sim \begin{matrix} 1 \\ |z| \rightarrow \infty \end{matrix} \quad (5.13a)$$

$$\psi(z) \sim \begin{matrix} z^{\sigma(\rho-\sigma)} e^{-i\sigma z} \\ |z| \rightarrow \infty \end{matrix} \quad (5.13b)$$

which means that the integral (2.14) is always convergent. Unlike the case $\lambda' = 0$, there is no boundary condition from the behaviour at infinity.

However, the point $z = -i\sigma$ is a new singular point of the equation. Since this point is located on the negative imaginary axis, we shall again put $z = -iq$, $q \geq 0$. Around $q = \sigma$ the possible behaviours are

$$\psi(q) \sim \begin{matrix} 1 \\ q \approx \sigma \end{matrix} \quad (5.14a)$$

$$\psi(q) \underset{q \approx \sigma}{\sim} (q - \sigma)^{1 - \sigma(\sigma - \rho)} = (q - \sigma)^{-(2\ell + 1)} \quad (5.14b)$$

where we have defined

$$\ell = -1 + \frac{1}{2} \sigma(\sigma - \rho) \quad (5.15)$$

for reasons which will be clear in a moment.

The eigenvalue condition is now obtained by requiring that: (i) $\psi(q)$ vanishes at the origin

$$\psi(0) = 0 \quad (5.16a)$$

and (ii) behaves as a constant at the point $q = \sigma$:

$$\psi(q) \approx \text{const} \quad (5.16b)$$

in order to enforce analyticity at that point.

Both requirements are in general incompatible, except for some (discrete) values of E , which are precisely the eigenvalues.

To discuss the eigenvalue problem, just defined, it is useful to recast Eq. (5.12), restricted on the negative imaginary axis, into a Schrödinger equation by making the transformation

$$\psi(q) = e^{-\frac{1}{2}\sigma q} (\sigma - q)^{-(\ell+1)} \phi(q) \quad (5.17)$$

The differential equation for $\phi(q)$ is then

$$H_q \phi(q) = \left(-\frac{d^2}{dq^2} + V(q) \right) \phi(q) = 0 \quad (5.18a)$$

$$V(q) = \frac{\ell(\ell+1)}{(\sigma - q)^2} - \frac{1}{\sigma - q} \left(\sigma(\ell+1) + \frac{E}{\lambda} \right) + \frac{1}{4} \sigma^2 - \frac{E}{\lambda} q \quad (5.18b)$$

The potential $V(q)$ has been written in such a way as to show its structure around the point $q = \sigma$: there is a centrifugal term, a Coulomb-like potential and a slowly varying function of q .

The eigenvalues are obtained by looking for solutions of (5.18) which behave as

$$\phi(q) \underset{q \approx \sigma}{\sim} (\sigma - q)^{\ell+1}$$

at $q \approx \sigma$ [regular solutions (5.14a)] and vanish at the origin. All this amounts to solving a Schrödinger equation with the potential shown in Fig. 9, with boundary conditions as in the case in which the potential is restricted to the interval $(0, \sigma)$. The potential for $\sigma = \rho$ is shown in Fig. 10.

To discuss qualitatively the solutions of Eqs. (5.18), we distinguish two cases.

- i) $\sigma \gg \rho \gg 1$. This is the situation in which the four-Pomeron coupling is very small. The potential (5.18b) has a Coulomb well at $q = 0$ and an approximately harmonic well at $q = \rho$. This situation is similar to the one discussed in Sections 2 and 3, except that the boundary condition (vanishing of the wave function) is now at $q = \sigma$ and not at infinity. But, if $\sigma \gg \rho$, it is clear that the solutions of (5.18) will display the same qualitative behaviour as the solutions of (2.28). We conclude that the quantum results are stable with respect to the addition of a four-Pomeron coupling.
- ii) $\sigma \geq \rho \gg 1$ and $\ell \geq -1$. This occurs when $\sigma - \rho \ll 1/\sigma$ and λ' approaches λ'_0 from above. The limit $\lambda' = \lambda'_0$ corresponds to $\ell = -1$ and has some practical importance since, in this situation one can write, even for $D_t \neq 0$, an exact expression for a state which is degenerate with the perturbative vacuum: what happens is that the coherent state representation (3.11) of $|\psi_1\rangle$ becomes exact and can be generalized to a higher number of dimensions⁸⁾.

The whole point in the following discussion is that looking at the solutions of the Schrödinger equation (5.18), there is a problem at $\ell = -\frac{1}{2}$, since at that point the regular solutions around $q = \sigma$ [$\phi(q) \sim (\sigma - q)^{\ell+1}$] and the irregular ones [$\phi(q) \sim (\sigma - q)^{-\ell}$] coincide. Therefore, it is not possible to write directly the solution for $\ell = -1$, but one has to determine it by continuity from the values $\ell > -\frac{1}{2}$.

Since the relevant behaviour is for $q \approx \sigma$, let us make the change of variables

$$x = \sigma - q \quad (5.19)$$

and write the potential (5.18b) in the form

$$V(x) = V_0(x) + V'(x) \quad (5.20)$$

$$V_0(x) = \frac{\ell(\ell+1)}{x^2} - \frac{1}{x} \left[\sigma(\ell+1) + \frac{E}{\lambda} \right] + \frac{1}{4} \sigma^2 - \frac{E}{\lambda \sigma} \quad (5.21a)$$

$$V'(x) = - \frac{E x}{\lambda \sigma (\sigma - x)} \quad (5.21b)$$

We obviously know the exact eigenfunctions of the Hamiltonian

$$H_0 = - \frac{d^2}{dx^2} + V_0(x) \quad (5.22)$$

As we are only interested in the lowest non-zero energy eigenstate of (5.18), let us consider the ground state energy of H_0 :

$$\varepsilon_0 = - \frac{E}{\lambda \sigma} - \frac{E \sigma}{2 \lambda (\ell+1)} - \frac{E^2}{4 \lambda^2 (\ell+1)^2} \quad (5.23)$$

We shall determine an approximate ground-state wave function, $\phi_1(x)$, for our eigenvalue problem by using a variational method. We look for a trial wave function which satisfies

$$(i) \quad \varphi_1(\sigma) = 0 \quad (5.24a)$$

and which minimizes the matrix element $(\phi_1, H_q \phi_1)$. Then we choose E in such a way that $\phi_1(x)$ is (approximately) a zero-energy solution of H_q :

$$(ii) \quad (\varphi_1, H_q \varphi_1) = 0 \quad (5.24b)$$

The trial wave-function $\phi_1(x)$ is built by multiplying the wave function corresponding to the lowest energy level of H_0 by a factor $[1 - \exp \beta(\sigma - x)]$ which ensures that (5.24a) is satisfied; β is the variational parameter. We write

$$\varphi_1(x) = N x^{\ell+1} e^{-\alpha x} (1 - e^{-\beta(\sigma-x)}) \quad (5.25)$$

where N is a normalization factor regular at $\ell = -1$, and

$$\alpha = \frac{1}{2} \left(\sigma + \frac{E}{\lambda(\ell+1)} \right) \quad (5.26a)$$

Since it will turn out that $E \propto (\ell+1) e^{-\sigma^2}$, we take for simplicity from the beginning

$$\alpha = \frac{1}{2} \sigma \quad (5.26b)$$

At fixed E we obtain the approximate value of the ground-state energy of H_q by minimizing the average value

$$\bar{E}(E, \beta) = (\varphi_1, H_q \varphi_1) \quad (5.27)$$

with respect to β . A straightforward calculation shows that $\bar{E}$ has a minimum at

$$\beta = \sigma \quad (5.28)$$

and the condition (5.24b) becomes

$$-\frac{\bar{E}}{\lambda\sigma} - \frac{\bar{E}\sigma}{2\lambda(\ell+1)} - \frac{E^2}{4\lambda^2(\ell+1)^2} + \sigma e^{-\sigma^2} \sigma^{2(\ell+1)} - \bar{E}(\varphi_1, V'\varphi_1) = 0 \quad (5.29)$$

As the matrix element $(\phi_1, V'\phi_1)$ is regular at $\ell = -1$, one finally gets for $\ell \geq -1$

$$E \approx 2\lambda(\ell+1) e^{-\sigma^2} \quad (5.30)$$

This equation shows that E vanishes at $\ell = -1$ and that there exists a level crossing at this point, since the ground state of H_q in (5.18) turns out to have the same energy as the perturbative vacuum.

It is interesting to remark that the tunnelling energy is now proportional to $e^{-\sigma^2}$, where σ^2 is the area of the square enclosed by the zero-energy trajectories (Fig. 8a). This is in agreement with what one can expect from an application of Polyakov's method¹⁵⁾ to this problem. Indeed in the case $\lambda' = 0$, the tunnelling energy is proportional to $e^{-\rho^2/2}$, where again $\rho^2/2$ is the area of the triangle determined by the zero-energy trajectories (Fig. 8b).

The wave function corresponding to the energy (5.30) is

$$\psi_1(q) \simeq (\sigma - q)^{\ell+1} e^{-\frac{1}{2}\sigma(\sigma-q)} (1 - e^{-\sigma q}) \quad (5.31)$$

or

$$\psi_1(q) \simeq 1 - e^{-\sigma q} \quad (5.32)$$

It can be directly checked that when $\ell = -1$, i.e. at the "magic" value of λ' , $\psi_1(q)$ in (5.32) becomes an exact eigenfunction of H_q with zero energy. The corresponding vector in the coherent state representation is

$$|1\rangle = |0\rangle - \frac{1}{\ell+1} \left(\frac{q}{\sigma} \right)^{\ell+1} |0\rangle = (1 - e^{\frac{q}{\sigma} a^\dagger}) |0\rangle \quad (5.33)$$

as was first discovered in Ref. 8. However, it is important to note that, in order to obtain (5.33), we had to choose the irregular behaviour of $\phi_1(q)$ at $q = \sigma$ [irregular in the sense that for $\ell \geq -1$, $(\sigma - q)^{\ell+1}$ is more singular than $(\sigma - q)^{-\ell}$]. Hence a careful study of the point $\ell = -\frac{1}{2}$ was necessary in order to justify the results of Ref. 8.

6. CONCLUSIONS

The object of this paper has been to prove in a rigorous way how the non-hermitian problem of RFT for $D_t = 0$ can be reduced (exception made for vacuum) to the hermitian formulation of Refs. 2 and 3. We have moreover proved the μ -analyticity of the S-matrix at Y fixed, and we have confirmed, for $\mu > 0$, all qualitative results^{*)} of Refs. 2-4 and 7. These results are also stable with respect to the addition of a small four-Pomeron coupling. We then conclude that the spin model of Ref. 4 relies on a rather firm mathematical basis.

We have also made some progress in understanding, for $\mu > 0$, the "perturbative" limit of λ/μ small.

*) It was understood in Ref. 4 that the $(a + a^\dagger)$ term in the Hamiltonian of Ref. 2 must be dropped in order to reach agreement with Refs. 3 and 7.

We have given strong arguments to show that, at fixed Y and $\mu > 0$, the classical limit is smooth for sufficiently small λ/μ , and that therefore our analytically continued S-matrix joins continuously with the perturbative result. We have also shown that there is a singularity of the classical action, as a function of Y , at some $Y_c(f, g, \lambda, \mu)$. This means that, at fixed Y , the analyticity in μ of the classical action breaks down at some $\bar{\mu}(f, g, \lambda, Y)$. This is to be contrasted with the complete analyticity in μ that we have proved for the quantum action at fixed Y . Since the classical limit ($\hbar \rightarrow 0$) is obtained in this theory for $\lambda/\mu \rightarrow 0$, we can make two remarks.

- i) On the one hand, one cannot expect to reach continuously the perturbative result for $\mu > 0$ at fixed E , in agreement with the result of Ref. 7. This is because E space and Y space are related by a Laplace transform, and therefore the E space is sensitive to the singularity at $Y = Y_c$, in the limit $\lambda/\mu \rightarrow 0$.
- ii) On the other hand, the existence of tunnelling ($\sim e^{-\frac{1}{2}\mu^2/\lambda^2}$) is essential in order to have analyticity in μ at fixed Y and fixed boundary conditions, specified by the initial and final sources g and f . Since tunnelling disappears when there are transverse dimensions [$\sim e^{-(\frac{1}{2}\mu^2/\lambda^2)V}$], we expect analyticity in μ at fixed Y , f , g only if the volume V is kept finite. This was the point of view adopted in the S-matrix definition of the spin model⁴⁾ which lead to the discovery of the phase transition.

However, we must say that we do not have yet a complete understanding of the various singularities in the limit $\lambda/\mu \rightarrow 0$, especially if the sources f and g are allowed to change. Technically, this is because we do not know how the excited states in the two wells are mixed and this forbids a direct study of the $\lambda/\mu \rightarrow 0$ limit by quantum methods. In particular, we have not clarified the point of whether some symmetry operation connects the two wells. Classically, there is a cyclic symmetry $p \rightarrow q \rightarrow \mu/\lambda - p - q = r$ which implies exact degeneracy, and is confirmed at the level of Bohr's quantization rule. However, since this symmetry involves both p and q , one expects a breaking, due to the commutation rules, already at relative order λ^2/μ^2 . Surprisingly enough, the degeneracy is kept by the quantum mechanical calculation, up to relative order λ^4/μ^4 , included. We have not gone further than that.

We believe that a complete satisfactory answer to this problem (limit $\lambda/\mu \rightarrow 0$) and related ones would be extremely helpful, in order to understand better the differences between the approach of Ref. 4 and that of White⁶⁾ in a case ($D_t = 0$) where no ambiguity can possibly arise.

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APPENDIX A

We want to prove that the kernel K [Eq. (2.36b)], defined by

$$K = V^{1/2} G V^{1/2}, \quad V = \frac{1}{\lambda q} \quad (A.1)$$

$$G^{-1} \varphi = \left[-\frac{d^2}{dq^2} + \frac{1}{4} (q-\rho)^2 - \frac{1}{2} \right] \varphi, \quad \varphi \in L^2_R(0, \infty) \quad (A.2)$$

is completely continuous and operator analytic in μ in the neighbourhood of the real axis $-\infty < \mu < +\infty$.

Let us first give an explicit representation of the Green's function G of the differential operator (A.2). By definition, G must satisfy the equation

$$\left[-\frac{d^2}{dq^2} + \frac{1}{4} (q-\rho)^2 - \frac{1}{2} \right] G(q, q') = \delta(q - q') \quad (A.3)$$

and must have its range and its domain contained in $L^2_R(0, \infty)$. We seek therefore a solution of (A.3) of the form

$$G(q, q') = f(q) g(q') \theta(q - q') + f(q') g(q) \theta(q' - q) \quad (A.4)$$

where

$$\begin{aligned} a) \quad G^{-1} f(q) &= G^{-1} g(q) = 0 \\ b) \quad f(q) &\approx e^{-\frac{1}{4} q^2} & q \rightarrow \infty \\ c) \quad g(q) &\approx q & q \rightarrow 0 \\ d) \quad w(f, g) &= f \frac{dg}{dq} - \frac{df}{dq} g = 1 \end{aligned} \quad (A.5)$$

The function $f(q)$ is clearly the regular solution, $\exp[-\frac{1}{4}(q - \rho)^2]$, while $g(q)$ has to involve the irregular solution, which is of the form

$$e^{-\frac{1}{4}(q-\rho)^2} T(q-\rho), \quad \text{with} \quad \frac{dT}{dx} + \lambda T(x) = 1 \quad (A.6)$$

A representation of the function $T(x)$ is

$$T(x) = e^{-\frac{1}{2} x^2} \cdot \int_0^x e^{\frac{1}{2} y^2} dy \quad (A.7)$$

One can derive an asymptotic expression of $T(x)$ of the form

$$T(x) = \frac{1}{x} \left[1 + \sum_{k=1}^{\infty} \frac{(2k-1)!!}{x^{2k}} \right] = \frac{R(x)}{x} \quad (A.8)$$

where $R(x)$ has been defined in (3.5). The conditions (A.5) determine $f(q)$ and $g(q)$ to be

$$f(q) = e^{-\frac{1}{4}(q-\rho)^2} \quad (A.9a)$$

$$\begin{aligned} g(q) &= e^{\frac{1}{4}(q-\rho)^2} \tau(q-\rho) - e^{-\frac{1}{4}(q-\rho)^2} e^{\frac{1}{2}\rho^2} \tau(-\rho) = \\ &= f(q) \int_{\rho-q}^{\rho} e^{\frac{1}{2}y^2} dy \end{aligned} \quad (A.9b)$$

With the explicit expressions (A.4) and (A.9) it is a simple matter to show that the Hilbert-Schmidt norm of K is finite for any real μ :

$$\text{Tr}(K^\dagger K) = 2 \int_0^\infty dq V(q) [f(q)]^2 \int_0^q dq' V(q') [g(q')]^2 < \infty \quad (A.10)$$

In fact at $q \sim 0$ the $1/q$ divergence of V is cancelled by a quadratic term coming from the q' -integration. At $q \sim \infty$ the q' -integration diverges at most as $q^{-4} \exp[\frac{1}{2}(q-\rho)^2]$, thus leading to an over-all q^{-5} convergence factor for the q -integration.

It is interesting to note that an even stronger property holds for K , namely that it belongs to the trace class, i.e. that

$$\begin{aligned} \text{Tr} K &= \int_0^\infty \frac{dq}{\lambda q} e^{-\frac{1}{2}(q-\rho)^2} \int_{\rho-q}^{\rho} e^{\frac{1}{2}y^2} dy = \\ &= \int_{-\rho}^\infty dq \frac{e^{-q^2/2}}{q+\rho} \int_{-q}^{\rho} dy e^{\frac{1}{2}y^2} < \infty \end{aligned} \quad (A.11)$$

This follows from the fact that for $q \rightarrow +\infty$ the y -integral diverges at most as $q^{-1} e^{q^2/2}$, while the pole at $q = -\rho$ is clearly cancelled.

Eq. (A.11) implies

$$\sum_{n=1}^{\infty} \frac{1}{E_n(\mu)} < \infty \quad (A.12)$$

where the E_n 's are the eigenvalues of H_R [see Eq. (2.42)].

$\text{Tr} K$ can be approximately computed in the limit $\rho \gg 1$, by noticing that for the values of q which are important in the integral in (A.11) one can use the approximation

$$\begin{aligned} g(q) \underset{\substack{\rho \gg 1 \\ q > \epsilon > 0}}{\sim} \int_{\rho-q}^{\rho} e^{\frac{1}{2}y^2} dy &= e^{\frac{1}{2}\rho^2} \int_0^q dy' e^{-\rho y'} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{y'}{2}\right)^n \approx \\ &\approx \exp\left(\frac{1}{2}\rho^2\right) \frac{R(\rho)}{\rho} [1 + O(e^{-\frac{1}{2}\rho^2})] \end{aligned} \quad (A.13)$$

Inserting (A.13) in (A.11) we get

$$\begin{aligned} \text{Tr } K &\underset{\rho \gg 1}{\approx} \int_{\varepsilon-\rho}^{\infty} \frac{dq}{\lambda(q+\rho)} e^{-\frac{1}{2}q^2} \frac{R(\rho)}{\rho} [1 + O(e^{-\frac{1}{2}\rho^2})] \approx \\ &\approx \frac{\sqrt{2\pi}}{\lambda\rho^2} [R(\rho)]^2 e^{\frac{1}{2}\rho^2} [1 + O(e^{-\frac{1}{2}\rho^2})]. \end{aligned} \quad (\text{A.14})$$

This shows that the state of Section 3.2 almost degenerate with the vacuum practically saturates $\text{Tr} K$. In fact we have

$$\left(\sum_{n=1}^{\infty} \frac{1}{E_n} \right)^{-1} \approx E_1 \approx \frac{\kappa\rho}{\sqrt{2\pi}} e^{-\frac{1}{2}\rho^2} [R(\rho)]^{-2} (1 + O(e^{-\frac{1}{2}\rho^2})) \quad (\text{A.15})$$

in agreement with (3.8).

APPENDIX B

In this appendix we want to prove the orthonormality condition

$$\langle \bar{\Psi}_n | \Psi_m \rangle = \nu_n \delta_{nm} \quad , \quad \nu_n = \pm 1 \quad (\text{B.1})$$

for the eigenfunctions of the eigenvalue problem (2.12) and (2.21), where

$$\langle \bar{\Psi}_n | \Psi_m \rangle = \int \frac{dz dz^*}{2i\pi} \Psi_n(z^*) \Psi_m(z) e^{-zz^*} \quad (\text{B.2})$$

By a simple change of variables Eq. (B.2) can be rewritten in the form

$$\langle \bar{\Psi}_n | \Psi_m \rangle = \int_{-\infty}^{\infty} dq \Psi_n(-iq) \tilde{\Psi}_m(-iq) \quad (\text{B.3})$$

with the definition

$$\tilde{\Psi}_m(-iq) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{ixq} \Psi_m(x) \quad (\text{B.4})$$

The important point to observe here is that $\tilde{\Psi}(-iq)$ is only different from zero for $q \geq 0$ because for $y < 0$ the x -integration contour in (B.4) can be closed for $\text{Im } x < 0$ [remember that $\Psi_m(x)$ is well behaved for $x \rightarrow -i\infty$] to give a vanishing result. We then have

$$\langle \bar{\Psi}_n | \Psi_m \rangle = \int_0^{\infty} dq \Psi_n(-iq) \tilde{\Psi}_m(-iq) \quad (\text{B.5})$$

Since

$$0 = \Psi_m(0) = \int_0^{\infty} \tilde{\Psi}_m(-iq) dq \quad (\text{B.6})$$

Eq. (B.4) is inverted by the formula [see Eq. (2.41)]

$$\Psi_m(-iq) = \int_0^{\infty} dq' (e^{-iqq'} - 1) \tilde{\Psi}(-iq') = \int_0^{\infty} dq' M(q, q') \tilde{\Psi}(-iq') \quad (\text{B.7})$$

Remembering Eq. (2.50), (B.5) can be formally written

$$\langle \bar{\Psi}_n | \Psi_m \rangle = (N_n F \Psi_n, N_m M^{-1} F \Psi_m) \quad (\text{B.8})$$

From Eq. (2.52) we finally get

$$\begin{aligned} \langle \bar{\Psi}_n | \Psi_m \rangle &= (N_n \bar{\Psi}_n, \frac{V_m}{N_m} \bar{\Psi}^{-1} V \Psi_m) = \\ &= V_m \frac{N_n}{N_m} (\bar{\Psi}_n, \bar{\Psi}_m) = V_n \delta_{nm} \end{aligned} \quad (\text{B.9})$$

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Figure captions

- Fig. 1 : $n + m$ Pomeron contribution to the elastic scattering.
- Fig. 2 : The effective potential $W(q) + \frac{1}{2}$ of Eq. (2.32) for $\mu < 0$.
- Fig. 3 : The effective potential $W(q) + \frac{1}{2}$ of Eq. (2.32) for $\mu > 0$.
- Fig. 4 : The potential in the Jengo's Hamiltonian (2.30a). The points (a_1, a_2) (b_1, b_2) are the integration limits in Eqs. (3.16).
- Fig. 5 : Integration contour in the complex x -plane. The determinations of the function $\mathcal{U}_\alpha(x)$ along the real axis are explicitly indicated.
- Fig. 6 : Classical trajectories in the (q, p) -plane.
- Fig. 7 : The function $Y = Y(E)$ associated with the classical trajectories of Fig. 6. The lowest branches of $Y(E)$ are labelled by the end-points of the corresponding trajectories of Fig. 6.
- Fig. 8 : a) The classical trajectories in the (q, p) -plane for $\sigma > \rho$, σ finite.
b) Same as before, for $\lambda' = 0$ ($\sigma = \infty$).
- Fig. 9 : The effective potential in Eq. (5.18b) for $\sigma > \rho$.
- Fig. 10 : The effective potential in Eq. (5.18b) for $\sigma = \rho$.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9

Fig. 10