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ORDER α^2 CORRECTIONS TO THE POSITRONIUM HYPERFINE INTERVAL
ARISING FROM THREE PHOTON VIRTUAL ANNIHILATION^{*)}

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ABSTRACT

We present the details of our calculation of three photon virtual annihilation contributions to the ground state triplet-singlet (hyperfine) splitting in positronium. The result is obtained in an analytic form to order $m\alpha^6$ ($\alpha^4 R_\infty$) and constitutes a contribution of -0.91 MHz to the hyperfine interval. Potential infrared ($\ln\alpha$) contributions, which appear at intermediate stages of the calculation, are shown to cancel and the gauge invariance of the result is demonstrated.

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I. INTRODUCTION

Together with the spectrum of muonium, the calculation of the positronium spectrum and its decay widths constitute the most crucial areas remaining to be tested in low-energy quantum electrodynamics. The positronium bound states are of particular interest since they afford a sensitive test of two-body relativistic equations such as the Bethe-Salpeter¹ equation in a case where the interaction is essentially electrodynamic in origin.

Calculations of the hyperfine splitting² and the fine structure³ to the first non-leading order, $\alpha^3 R_\infty$, were performed quite some time ago. The former calculations were checked to this order by experiments of the same vintage,⁴ while the latter have only recently been confirmed by experiment.⁵ The next terms in the perturbation expansion produce corrections of order $\alpha^4 \ln \alpha^{-1} R_\infty$ and $\alpha^4 R_\infty$. These terms are interesting because their calculation requires a more detailed knowledge of the two-body wave function and a thorough understanding of the infrared problems that accompany self-mass and vertex corrections. Various contributions of order $\alpha^4 \ln \alpha^{-1} R_\infty$ have been calculated^{6,7,8,9,10} and confirmed. While the total contribution to this order appears to be small,¹⁰ indications from the known $\alpha^4 R_\infty$ contributions¹¹ are that terms of this order can occur with large numerical coefficients. This, coupled with the fact that present experiments^{12,13} on the hyperfine interval are sufficiently accurate to test $\alpha^4 R_\infty$ contributions, makes it essential that these terms be computed if any

significant comparison between theory and experiment is to be made.

With this in mind, we have begun a systematic calculation of the $\alpha^4 R_\infty$ contributions arising from virtual annihilation processes. As a first stage in this program, we present the details of our calculation¹⁴ of the three photon virtual annihilation contribution to the hyperfine interval. This process appears for the first time in order $\alpha^4 R_\infty$. As a consequence, we cannot borrow various calculational techniques from lower order calculations, but must devise appropriate methods of handling this new class of graphs. Our job is simplified to some extent by the fact that the only wave function information needed is the amplitude of the non-relativistic wave function at the origin. There are, however, infrared ($\ln\alpha$) terms to contend with at intermediate stages of the calculation.

In the next section, the relevant diagrams are displayed. It is shown that, at threshold, only two of the six diagrams are independent and that the set is gauge invariant. Section III contains the reduction of the independent diagrams to a collection of integrals, and the evaluation of these integrals is discussed in Section IV. The results are presented in Section V. A number of useful formulae are collected in the Appendix.

II. ANALYSIS OF THE VIRTUAL ANNIHILATION DIAGRAMS

The six three photon virtual annihilation diagrams are shown in Fig. 1. To order $\alpha^4 R_\infty$, the calculation of the energy shift associated with three photon annihilation is equivalent to computing the real part of the electron-positron forward scattering amplitude just below threshold when the intermediate state consists of three photons. Specifically, we take the electron and positron momenta p_μ and $\bar{p}_\mu$ to be

$$p_\mu = \bar{p}_\mu = (\vec{0}, i\alpha(1 - \frac{\alpha^2}{8})), \quad (2.1)$$

where the binding is being used to provide a natural low-momentum cutoff. The imaginary part of our result is related to the width of orthopositronium into three photons. Comparison with the well-known value for this width¹⁵ will serve as a partial check on the calculation and provide a convenient way of determining the overall sign of the result.

We begin by showing that when $p = \bar{p}$ the contributions of Figs. (1a) and (1b) are equal as are those of Figs. (1e) - (1f). Using the labellings in the figure, the integrands associated with Figs. (1a) and (1b) are, apart from common factors.

$$\bar{u}(p)\gamma_\mu S_F(p-k_1)\gamma_\nu S_F(-\bar{p}+k_3)\gamma_\lambda v(\bar{p})\bar{v}(p)\gamma_\lambda S_F(-\bar{p}+k_3)\gamma_\nu S_F(p-k_1)\gamma_\mu u(p) \quad (2.2a)$$

$$\bar{u}(p)\gamma_\lambda S_F(p-k_3)\gamma_\nu S_F(-\bar{p}+k_1)\gamma_\mu v(\bar{p})\bar{v}(p)\gamma_\lambda S_F(-\bar{p}+k_3)\gamma_\nu S_F(p-k_1)\gamma_\mu u(p), \quad (2.2b)$$

where we have used the Feynman gauge. The gauge independence of the result is established below. Note that the

underlined factors above are the same for any of the diagrams encountered. The first factors vary from diagram to diagram depending on the permutation of the photon vertices. To establish the equivalence of Eqs. (2.2a) and (2.2b), we use the following relations involving the charge conjugation matrix C

$$C^{-1} \gamma_\mu C = -\gamma_\mu^T \quad (2.3)$$

$$C^{-1} S_F(p) C = S_F^T(-p) \quad (2.4)$$

and

$$u(p) = C \bar{v}^T(p), \quad v(\bar{p}) = C \bar{u}^T(\bar{p}) \quad (2.5)$$

$$\bar{u}(p) = -v^T(p) C^{-1}, \quad \bar{v}(\bar{p}) = -u^T(\bar{p}) C^{-1}, \quad (2.6)$$

where T denotes transpose. When used on the first factor in Eq. (2.2b), these relations give¹⁶

$$\begin{aligned} & -v^T(p) C^{-1} \gamma_\lambda C C^{-1} S_F(p-k_3) C C^{-1} \gamma_\nu C C^{-1} S_F(-\bar{p}+k_1) C C^{-1} \gamma_\mu C \bar{u}^T(\bar{p}) = \\ & -(-)^3 \bar{u}(\bar{p}) \gamma_\mu S_F(\bar{p}-k_1) \gamma_\nu S_F(-p+k_3) \gamma_\lambda v(p). \end{aligned} \quad (2.7)$$

This is just the first factor in Eq. (2.2a) when $p=\bar{p}$, which establishes the equivalence of Figs. (1a) and (1b). As can be seen by this example, the process of introducing C and C^{-1} in an electron line is equivalent to reflecting that line about a horizontal axis passing through it, with the photon lines remaining attached. By reflecting both lines in Figs. (1d) and (1f) and then relabeling the dummy variables k_1 and k_3 , it is easily seen that these diagrams are equivalent to Figs. (1e) and (1e), respectively. A reflection of the right hand electron line in Fig. (1e)

shows that it is equivalent to Fig. (1c), and hence the total contribution is twice that of Fig. (1a) plus four times that of Fig. (1c).

The expressions to be computed are therefore of the form

$$I_I = \int d^4 k_1 d^4 k_2 d^4 k_3 \frac{1}{k_1^2} \frac{1}{k_2^2} \frac{1}{k_3^2} \delta^{(4)}(2p - k_1 - k_2 - k_3) \\ \times \bar{u}(p) \gamma_\mu S_F(p - k_1) \gamma_\nu S_F(-p + k_3) \gamma_\lambda v(p) \\ \times \bar{v}(p) \gamma_\lambda S_F(-p + k_3) \gamma_\nu S_F(p - k_1) \gamma_\mu u(p), \quad (2.8)$$

$$I_{II} = \int d^4 k_1 d^4 k_2 d^4 k_3 \frac{1}{k_1^2} \frac{1}{k_2^2} \frac{1}{k_3^2} \delta^{(4)}(2p - k_1 - k_2 - k_3) \\ \times \bar{u}(p) \gamma_\mu S_F(p - k_1) \gamma_\lambda S_F(-p + k_2) \gamma_\nu v(p) \\ \times \bar{v}(p) \gamma_\lambda S_F(-p + k_3) \gamma_\nu S_F(p - k_1) \gamma_\mu u(p). \quad (2.9)$$

Before attempting any further simplification of these integrands, it is appropriate to show that the entire expression is gauge invariant. For a general gauge with photon propagator $D_{F,\mu\nu}(k)$, the integrand has the form

$$\bar{u}(p) M_{\mu_1 \mu_2 \mu_3} v(\bar{p}) \bar{v}(\bar{p}) \gamma_{\mu_3} S_F(-\bar{p} + k_3) \gamma_{\mu_2} S_F(p - k_1) \gamma_{\mu_1} u(p) \\ \times D_{F,\mu_1 \mu_1'}(k_1) D_{F,\mu_2 \mu_2'}(k_2) D_{F,\mu_3 \mu_3'}(k_3), \quad (2.10)$$

with $k_1 + k_2 + k_3 = p + \bar{p}$ and

$$M_{\mu_1 \mu_2 \mu_3} = \int \gamma_{\mu_1} S_F(p - k_1) \gamma_{\mu_2} S_F(-\bar{p} + k_3) \gamma_{\mu_3}, \quad (2.11)$$

where the summation is over all permutations of (1,2,3).
To prove that Eq. (2.10) is gauge invariant, it is sufficient to show that this expression vanishes when

$\bar{u}(p)M_{\mu_1\mu_2\mu_3}v(\bar{p})$ is contracted with $(k_1)_{\mu_1}$, $(k_2)_{\mu_2}$ or $(k_3)_{\mu_3}$. Consider, for example, the contraction with $(k_1)_{\mu_1}$, which yields

$$\begin{aligned} (k_1)_{\mu_1} \bar{u}(p)M_{\mu_1\mu_2\mu_3}v(\bar{p}) &= \bar{u}(p) [\gamma \cdot k_1 S_F(p-k_1) \gamma_{\mu_2} \\ &\times S_F(-\bar{p}+k_3) \gamma_{\mu_3} + \gamma_{\mu_2} S_F(p-k_2) \gamma_{\mu_3} S_F(-\bar{p}+k_1) \gamma \cdot k_1 \\ &+ \gamma_{\mu_3} S_F(p-k_3) \gamma \cdot k_1 S_F(-\bar{p}+k_2) \gamma_{\mu_2} + \gamma_{\mu_2} S_F(p-k_2) \gamma \cdot k_1 \\ &\times S_F(-\bar{p}+k_3) \gamma_{\mu_3} + \gamma_{\mu_3} S_F(p-k_3) \gamma_{\mu_2} S_F(-\bar{p}+k_1) \gamma \cdot k_1 \\ &+ \gamma \cdot k_1 S_F(p-k_1) \gamma_{\mu_3} S_F(-\bar{p}+k_2) \gamma_{\mu_2}] v(\bar{p}). \end{aligned} \quad (2.12)$$

Now, the equations of motion

$$\bar{u}(p)(i\gamma \cdot p + m) = 0, \quad (i\gamma \cdot \bar{p} - m)v(\bar{p}) = 0, \quad (2.13)$$

together with the formula

$$-iS_F(p-k_2) \gamma \cdot k_1 S_F(-\bar{p}+k_3) = S_F(-\bar{p}+k_3) - S_F(p-k_2), \quad (2.14)$$

can be used to reduce Eq. (2.12) to

$$\begin{aligned} i(k_1)_{\mu_1} \bar{u}(p)M_{\mu_1\mu_2\mu_3}v(\bar{p}) &= \bar{u}(p) [-\gamma_{\mu_2} S_F(-\bar{p}+k_3) \gamma_{\mu_3} \\ &+ \gamma_{\mu_2} S_F(p-k_2) \gamma_{\mu_3} + \gamma_{\mu_3} S_F(-\bar{p}+k_2) \gamma_{\mu_2} - \gamma_{\mu_3} S_F(p-k_3) \gamma_{\mu_2} \\ &+ \gamma_{\mu_2} S_F(-\bar{p}+k_3) \gamma_{\mu_3} - \gamma_{\mu_2} S_F(p-k_2) \gamma_{\mu_3} + \gamma_{\mu_3} S_F(p-k_3) \gamma_{\mu_2} \\ &- \gamma_{\mu_3} S_F(-\bar{p}+k_2) \gamma_{\mu_2}] v(\bar{p}) = 0. \end{aligned} \quad (2.15)$$

The same procedure can be used on each index μ_i , and hence the expression Eq. (2.10) is gauge invariant.

III. REDUCTION OF THE CONTRIBUTING DIAGRAMS

To eliminate the Dirac matrices from the integrands of Eqs. (2.8) and (2.9), we first use the equations of motion Eqs. (2.13) and the representation

$$S_F(p) = \frac{m - i\gamma \cdot p}{p^2 + m^2 - i\epsilon} \quad (3.1)$$

to obtain, for example,

$$\begin{aligned} \bar{v}(p) \gamma_\lambda S_F(-p+k_3) \gamma_\nu S_F(p-k_1) \gamma_\mu u(p) &= \frac{1}{(p-k_1)^2 + m^2} \\ &\times \frac{1}{(p-k_3)^2 + m^2} [4p_\lambda p_\mu \gamma_\nu - 2\gamma_\lambda \gamma \cdot k_3 \gamma_\nu p_\mu - 2p_\lambda \gamma_\nu \gamma \cdot k_1 \gamma_\mu \\ &+ \gamma_\lambda \gamma \cdot k_3 \gamma_\nu \gamma \cdot k_1 \gamma_\mu]_{\beta\alpha} u_\alpha(p) \bar{v}_\beta(p). \end{aligned} \quad (3.2)$$

This expression can be reduced to the form of a trace by noting that, at threshold, the last factors can be written as

$$u_\alpha(p) \bar{v}_\beta(p) = \frac{-i}{2\sqrt{2}} [(1+\gamma_4) \vec{\gamma} \cdot \vec{\epsilon}]_{\alpha\beta}. \quad (3.3)$$

Here, $\vec{\epsilon}$ denotes the three independent normalized polarization vectors associated with the spin-1 triplet state.

Since the factor in square brackets in Eq. (3.2) contains only odd numbers of gamma matrices, we find

$$\begin{aligned}
\bar{v}(p)\gamma_\lambda S_F(-p+k_3)\gamma_\nu S_F(p-k_1)\gamma_\mu u(p) &= \frac{1}{(p-k_1)^2+m^2} \\
&\times \frac{1}{(p-k_3)^2+m^2} \frac{-i}{2\sqrt{2}} [4p_\lambda p_\mu \langle \gamma_\nu \gamma \cdot \epsilon \rangle - 2\langle \gamma_\lambda \gamma \cdot k_3 \gamma_\nu \gamma \cdot \epsilon \rangle p_\mu \\
&- 2p_\lambda \langle \gamma_\nu \gamma \cdot k_1 \gamma_\mu \gamma \cdot \epsilon \rangle + \langle \gamma_\lambda \gamma \cdot k_3 \gamma_\nu \gamma \cdot k_1 \gamma_\mu \gamma \cdot \epsilon \rangle], \quad (3.4)
\end{aligned}$$

where the angular brackets denote a trace, and $\epsilon_\mu = (\vec{\epsilon}, i0)$, so that $p \cdot \epsilon = 0$. The remaining factors in Eqs. (2.8) and (2.9) can be analyzed in a similar fashion using

$$v_\alpha(p)\bar{u}_\beta(p) = \frac{-i}{2\sqrt{2}} [(1-\gamma_4)\vec{\gamma} \cdot \vec{\epsilon}]_{\alpha\beta}, \quad (3.5)$$

where we have assumed, without loss of generality, that the polarization vectors are real. In this way, we obtain

$$\begin{aligned}
\bar{u}(p)\gamma_\mu S_F(p-k_1)\gamma_\nu S_F(-p+k_3)\gamma_\lambda v(p) &= \frac{1}{(p-k_1)^2+m^2} \\
&\times \frac{1}{(p-k_3)^2+m^2} \frac{-i}{2\sqrt{2}} [4p_\mu p_\lambda \langle \gamma_\nu \gamma \cdot \epsilon \rangle - 2\langle \gamma_\mu \gamma \cdot k_1 \gamma_\nu \gamma \cdot \epsilon \rangle p_\lambda \\
&- 2p_\mu \langle \gamma_\nu \gamma \cdot k_3 \gamma_\lambda \gamma \cdot \epsilon \rangle + \langle \gamma_\mu \gamma \cdot k_1 \gamma_\nu \gamma \cdot k_3 \gamma_\lambda \gamma \cdot \epsilon \rangle], \quad (3.6)
\end{aligned}$$

$$\begin{aligned}
\bar{u}(p)\gamma_\mu S_F(p-k_1)\gamma_\lambda S_F(-p+k_2)\gamma_\nu v(p) &= \frac{1}{(p-k_1)^2+m^2} \\
&\times \frac{1}{(p-k_2)^2+m^2} \frac{-i}{2\sqrt{2}} [4p_\mu p_\nu \langle \gamma_\lambda \gamma \cdot \epsilon \rangle - 2p_\nu \langle \gamma_\mu \gamma \cdot k_1 \gamma_\lambda \gamma \cdot \epsilon \rangle \\
&- 2p_\mu \langle \gamma_\lambda \gamma \cdot k_2 \gamma_\nu \gamma \cdot \epsilon \rangle + \langle \gamma_\mu \gamma \cdot k_1 \gamma_\lambda \gamma \cdot k_2 \gamma_\nu \gamma \cdot \epsilon \rangle]. \quad (3.7)
\end{aligned}$$

To obtain the desired integrands, Eqs. (3.6) and (3.7) must be multiplied by Eq. (3.4) and the products contracted over the indices μ, ν, λ . We refer to the

respective integrands (and the corresponding diagrams) as type I and type II. Examination of these equations reveals that the process of contracting μ , ν and λ can involve the contraction of products of third rank tensors obtained from evaluating a trace of up to six γ -matrices. At first sight, it might seem that the number of terms encountered in this procedure would quickly become unmanageable. It turns out, however, that invariance arguments can be used to reduce the number of terms considerably.

To illustrate the use of such arguments, consider the following product, which occurs in the type I diagrams

$$\begin{aligned} \langle \gamma_\nu \gamma \cdot k_3 \gamma_\lambda \gamma \cdot \epsilon \rangle \langle \gamma \cdot p \gamma \cdot k_1 \gamma_\nu \gamma \cdot k_3 \gamma_\lambda \gamma \cdot \epsilon \rangle &= A k_3^2 k_1 \cdot p \\ &+ B k_1 \cdot k_3 k_3 \cdot p + C p \cdot k_3 k_1 \cdot \epsilon k_3 \cdot \epsilon + D p \cdot k_1 k_3 \cdot \epsilon k_3 \cdot \epsilon. \end{aligned} \quad (3.8)$$

The right side of this equation is constructed by observing that the left side is homogeneous of degree two in k_3 and ϵ and of degree one in p and k_1 . Using the constraint $p \cdot \epsilon = 0$ reduces the number of possible terms to those shown. To determine the coefficients, we look at two special cases, $k_1 = k_3$ and $k_1 = p$. With no difficulty, the former gives

$$32 k_3^2 k_3 \cdot p = (A+B) k_3^2 k_3 \cdot p + (C+D) p \cdot k_3 k_3 \cdot \epsilon k_3 \cdot \epsilon, \quad (3.9)$$

or

$$(A+B) = 32, \quad (C+D) = 0. \quad (3.10)$$

For the latter, we find

$$\begin{aligned}
 -m^2 \langle \gamma_\nu \gamma \cdot k_3 \gamma_\lambda \gamma \cdot \epsilon \rangle \langle \gamma_\nu \gamma \cdot k_3 \gamma_\lambda \gamma \cdot \epsilon \rangle &= -32m^2 (k_3^2 + k_3 \cdot \epsilon \ k_3 \cdot \epsilon) \\
 &= -m^2 A k_3^2 + B p \cdot k_3 p \cdot k_3 - m^2 D k_3 \cdot \epsilon \ k_3 \cdot \epsilon. \quad (3.11)
 \end{aligned}$$

Hence, the coefficients are

$$A = -C = D = 32, \quad B = 0. \quad (3.12)$$

This procedure can be applied in a similar fashion to all the traces encountered, and the results are summarized in Table 1.

With the aid of this table, the integrands of the type I and type II diagrams are found to be

$$\begin{aligned}
 \frac{-32}{D_I} \{ m^2 [-\frac{1}{2} (2p - k_1 - k_3)^2 - k_1^2 + k_1 \cdot \epsilon \ k_3 \cdot \epsilon - 2(k_1 \cdot \epsilon)^2] \\
 + 2p \cdot k_1 \ k_1 \cdot \epsilon \ k_3 \cdot \epsilon + p \cdot k_1 p \cdot k_3 - 2k_1^2 p \cdot k_3 \\
 - 2p \cdot k_3 (k_1 \cdot \epsilon)^2 + \frac{3}{4} k_1^2 k_3^2 + k_1^2 (k_3 \cdot \epsilon)^2 - k_1 \cdot k_3 \ k_1 \cdot \epsilon \ k_3 \cdot \epsilon \\
 + \frac{1}{2} (k_1 \cdot k_3)^2 \}, \quad (3.13)
 \end{aligned}$$

and

$$\begin{aligned}
 \frac{-32}{D_{II}} \{ m^2 [2(k_2 + k_3) \cdot p - k_1 \cdot \epsilon (k_2 + k_3) \cdot \epsilon + k_1 \cdot (k_2 + k_3) \\
 - 2k_2 \cdot k_3 - (k_1 \cdot \epsilon)^2 - 2k_2 \cdot \epsilon k_3 \cdot \epsilon] + 3k_1 \cdot p (k_2 + k_3) \cdot p \\
 - k_1^2 (k_2 + k_3) \cdot p + \frac{1}{2} (k_1^2 - 2p \cdot k_1) k_2 \cdot k_3 - p \cdot k_1 \ k_2 \cdot k_3 \\
 + (k_1 \cdot p)^2 + k_1 \cdot \epsilon k_2 \cdot p k_3 \cdot \epsilon + k_1 \cdot \epsilon k_2 \cdot \epsilon k_3 \cdot p \\
 - 2k_1 \cdot p k_2 \cdot \epsilon k_3 \cdot \epsilon + (2k_2 \cdot k_3 - k_2 \cdot p - k_3 \cdot p) (k_1 \cdot \epsilon)^2 \\
 - k_1 \cdot \epsilon (k_2 + k_3) \cdot \epsilon \ k_1 \cdot p \}, \quad (3.14)
 \end{aligned}$$

respectively. The denominators D_I and D_{II} are

$$D_I = k_1^2 k_2^2 k_3^2 (k_1^2 - 2p \cdot k_1 + \gamma^2)^2 (k_3^2 - 2p \cdot k_3 + \gamma^2)^2 \quad (3.15)$$

$$D_{II} = k_1^2 k_2^2 k_3^2 (k_1^2 - 2p \cdot k_1 + \gamma^2)^2 (k_2^2 - 2p \cdot k_2 + \gamma^2)^2 \times (k_3^2 - 2p \cdot k_3 + \gamma^2)^2, \quad (3.16)$$

with $\gamma = (m\alpha)/2$.

IV. EVALUATION OF THE INTEGRALS

All of the integrals encountered can be evaluated analytically to order $\alpha^4 R_\infty$ by standard Feynman parametrization techniques. Given the number of terms in Eqs. (3.13) and (3.14), this evaluation is, needless to say, quite tedious. Also, special care must be exercised, since the majority of the integrals have either infrared contributions or an imaginary part associated with the logarithm of a negative number. It is here that the use of the binding to provide a natural low-momentum cutoff is particularly useful.

The binding term, represented by γ^2 in Eqs. (3.15) and (3.16), cuts off potentially divergent terms in regions of the momentum integration where the denominator is otherwise too singular. For example, in D_I as $k_1, k_3 \rightarrow 0$, we have

$$D_I \rightarrow k_1^2 k_3^2 (2p)^2 (-2p \cdot k_1)^2 (-2p \cdot k_3)^2, \quad (4.1)$$

if the binding is neglected. Thus, D_I behaves as $(k_1)^4 (k_3)^4$ in this limit. The numerator has the form

$d^4k_1 d^4k_3 f(k_1, k_3)$, so that if $f(k_1, k_3)$ does not tend to zero sufficiently rapidly, we can expect a logarithmic divergence.

To illustrate the technique of identifying infrared contributions, consider the integrand k_1^2/D_I . In the limit discussed above, we have

$$\frac{k_1^2}{D_I} \xrightarrow{k_1, k_3 \rightarrow 0} \frac{1}{(2p)^2 (2p \cdot k_1)^2 k_3^2 (2p \cdot k_3)^2}, \quad (4.2)$$

which has a logarithmic singularity in k_3 . However, when $k_1 \rightarrow 2p$, $k_3 \rightarrow 0$, we find

$$\frac{k_1^2}{D_I} \xrightarrow[k_3 \rightarrow 0]{k_1 \rightarrow 2p} \frac{1}{k_2^2 (2p \cdot k_2)^2 k_3^2 (2p \cdot k_3)^2}, \quad (4.3)$$

which has logarithmic singularities in both k_2 and k_3 . As will be shown, retaining the binding converts these singularities to $\ln \gamma^2$ or $\ln^2 \gamma^2$ contributions.

Integrals which have an imaginary part can be distinguished from those which do not by remembering that the discontinuity in the photon variables is obtained by the replacement¹⁷ $D_F(k_i^2) \rightarrow \delta(k_i^2) \theta(k_i, 0)$ for $i = 1, 2, 3$. Any integral containing a factor k_i^2 in the numerator is therefore real, and all other integrals are complex. Due to the $-i\eta$ prescription used to displace the poles of the fermion propagator, γ^2 can be treated as having a small negative imaginary part. This device uniquely defines the phase of the complex logarithms encountered.

The parameter integrals which occur can always be reduced to one-dimensional integrals involving the product of a rational function and at most two logarithms with arguments that are linear in the integration variable. A table of useful integrals is given in the Appendix.

The integration of Eqs. (3.13) and (3.14) is summarized in Tables 2 and 3, respectively. These tables are constructed to reflect what we found to be a convenient separation of the integrands into sub-parts. In these tables, $\ln(-1)$ is interpreted as $\ln(-1-i\eta)$. The result of the integration is converted into an expression for the energy shift in the next section.

V. SUMMARY

To obtain an expression for the energy $\Delta E_{3\gamma}^A$ from Tables 2 and 3, we need the factor 32 from Eqs. (3.13) and (3.14), the wave function factor $|\phi(0)|^2 = (m\alpha)^3/8\pi$ and the factor $(4\pi\alpha)^3/(2\pi)^8$ from the Feynman rules associated with the diagrams of Fig. 1. This results in the expression

$$\begin{aligned} \Delta E_{3\gamma}^A = \alpha^2 R_\infty \frac{\alpha^2}{\pi^2} \left[-4\ln 2 - \frac{19\pi^2}{36} - \frac{\pi^2}{18} \ln 2 + \frac{3}{4} \zeta(3) \right. \\ \left. + \frac{13}{2} - \frac{2\pi i}{9} (\pi^2 - 9) \right]. \end{aligned} \quad (5.1)$$

The real part of Eq. (5.1) corresponds to the desired energy shift, while the imaginary part is related to the lifetime of orthopositronium by

$$\text{Im}\Delta E_{3\gamma}^A = -\frac{1}{2} \Gamma_{3\gamma}. \quad (5.2)$$

The agreement of Eq. (5.2) with the standard result¹⁵ serves as a partial check on our calculation.

Evaluation of the real part of Eq. (5.1) leads to the frequency shift

$$\Delta v_{3\gamma}^A = \frac{\alpha^4}{\pi^2} c R_{\infty}(-0.96) = -.91 \text{ MHz}. \quad (5.3)$$

This result is too small by an order of magnitude to significantly alter the comparison between theory and experiment. This is due, in part, to the fact that the "natural" expansion parameter (the one with coefficient of order unity) is α/π and not α .

We are currently applying techniques analogous to those outlined here to single loop corrections to the two photon virtual annihilation diagrams, and double loop corrections to the single virtual photon annihilation diagram. There is no reliable way to estimate these contributions and hence only a detailed calculation will ultimately reveal the total effect of virtual annihilation on the hyperfine interval.

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APPENDIX

Here, we summarize formulae which are useful in evaluating parameter integrals encountered in the course of compiling Tables 2 and 3. The entries in these tables involving $\ln \alpha$ or $\ln^2 \alpha$ usually arise from integrals of the form

$$\int_0^1 \frac{dx}{(c+ex)} \ln(a+bx) = \frac{1}{e} \left[\ln\left(\frac{ae-bc}{e}\right) \ln(c+ex) - \text{Li}_2\left(-b \frac{c+ex}{ae-bc}\right) \right] \Big|_0^1, \quad (\text{A.1})$$

where

$$\text{Li}_2(y) \equiv - \int_0^1 \frac{dx}{x} \ln(1-yx) \quad (\text{A.2})$$

is the Euler dilogarithm and c and/or e are functions of α . Techniques for simplifying integrals of this type are given in Appendix A of Ref. 18.

As a typical example, consider the integral

$$\int_0^1 \frac{dx}{\epsilon^2 - i\eta - 4x} \ln x = - \frac{1}{4} \left\{ \ln \frac{\epsilon^2}{4} [\ln(-4-i\eta) - \ln \epsilon^2] - \text{Li}_2\left(\frac{-4-i\eta}{\epsilon^2}\right) + \text{Li}_2(1) \right\}, \quad (\text{A.3})$$

where $\epsilon^2 = \gamma^2/m^2$. We seek to take the limit $\epsilon^2 \rightarrow 0$, and this is facilitated by using the identity¹⁸

$$\text{Li}_2\left(-\frac{1}{y}\right) = -\text{Li}_2(-y) - \frac{1}{2} \ln^2 y - \zeta(2), \quad (\text{A.4})$$

together with

$$\text{Li}_2(0) = 0, \quad \text{Li}_2(1) = \zeta(2), \quad (\text{A.5})$$

to obtain

$$\int_0^1 \frac{dx}{\epsilon^2 - i\eta - 4x} \ln x = \frac{1}{4} \left[\frac{1}{2} \ln^2 \epsilon^2 - \ln \epsilon^2 \ln(-4 - i\eta) - 2 \ln^2 2 \right. \\ \left. + 2 \ln 2 \ln(-4 - i\eta) - \frac{\pi^2}{3} \right]. \quad (\text{A.6})$$

We also encounter a variety of integrals which are not singular as $\alpha \rightarrow 0$ but require some effort to evaluate. The more useful of these are given in Tables 4 and 5. Some of these results can also be found in Ref. 18.

Table 1. Products of traces.

$\langle \gamma_\mu \gamma \cdot a \gamma_\nu \gamma \cdot \epsilon \rangle \langle \gamma_\mu \gamma \cdot b \gamma_\nu \gamma \cdot \epsilon \rangle$	$32(a \cdot b + a \cdot \epsilon \ b \cdot \epsilon)$
$\langle \gamma_\mu \gamma \cdot a \gamma \cdot p \gamma \cdot \epsilon \rangle \langle \gamma_\mu \gamma \cdot b \gamma \cdot p \gamma \cdot \epsilon \rangle$	$16(a \cdot p \ b \cdot p + p^2 a \cdot \epsilon \ b \cdot \epsilon)$
$\langle \gamma_\mu \gamma \cdot a \gamma_\nu \gamma \cdot \epsilon \rangle \langle \gamma \cdot p \gamma \cdot b \gamma_\mu \gamma \cdot c \gamma_\nu \gamma \cdot \epsilon \rangle$	$32(a \cdot c \ b \cdot p - a \cdot p \ b \cdot \epsilon \ c \cdot \epsilon + b \cdot p \ a \cdot \epsilon \ c \cdot \epsilon)$
$\langle \gamma_\mu \gamma \cdot a \gamma_\nu \gamma \cdot \epsilon \rangle \langle \gamma_\mu \gamma \cdot a \gamma \cdot p \gamma \cdot b \gamma_\nu \gamma \cdot \epsilon \rangle$	$32(a^2 b \cdot p + b \cdot p \ a \cdot \epsilon \ a \cdot \epsilon + a \cdot p \ a \cdot \epsilon \ b \cdot \epsilon)$
$\langle \gamma_\mu \gamma \cdot a \gamma_\nu \gamma \cdot b \gamma_\lambda \gamma \cdot \epsilon \rangle \langle \gamma_\mu \gamma \cdot a \gamma_\nu \gamma \cdot b \gamma_\lambda \gamma \cdot \epsilon \rangle$	$32[3a^2 b^2 + 2(a^2 b \cdot \epsilon \ b \cdot \epsilon + b^2 a \cdot \epsilon \ a \cdot \epsilon + (a \cdot b)^2) - 4a \cdot b \ a \cdot \epsilon \ b \cdot \epsilon]$
$\langle \gamma_\mu \gamma \cdot a \gamma_\nu \gamma \cdot b \gamma_\lambda \gamma \cdot \epsilon \rangle \langle \gamma_\mu \gamma \cdot a \gamma_\lambda \gamma \cdot c \gamma_\nu \gamma \cdot \epsilon \rangle$	$32[a^2 + 4(a \cdot \epsilon)^2] \ b \cdot c$

Table 2. Evaluation of the integrals encountered in the type I diagrams.

$f(k_1, k_3, \epsilon)$	$\ln^2(\frac{\alpha^2}{4})$	$\ln(\frac{\alpha^2}{4})$	$\ln(\frac{\alpha^2}{4}) \ln(-4)$	$\ln^2 2$	$\ln 2$	1	$\pi^2 \ln 2$	π^2	$\zeta(3)$	$\pi^2 \ln(-1)$	$\ln(-1)$
$-\frac{1}{2} m^2 (2p \cdot k_1 - k_3)^2$	$\frac{1}{2}$										
$-m^2 k_1^2$	$\frac{1}{2}$	1						$\frac{1}{3}$			
$m^2 k_1 \cdot \epsilon k_3 \cdot \epsilon$					$-\frac{1}{3}$	$-\frac{1}{12}$	$-\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{8}$	$\frac{1}{72}$	$-\frac{1}{6}$
$-2m^2 (k_1 \cdot \epsilon)^2$		$-\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{3}$		$\frac{5}{72}$			$\frac{1}{6} - \frac{1}{6} \ln 2$
$2p \cdot k_1 k_1 \cdot \epsilon k_3 \cdot \epsilon$					-1	$\frac{1}{2}$	$-\frac{2}{9}$	$\frac{1}{9}$	$\frac{1}{2}$	$\frac{1}{18}$	$-\frac{1}{2}$
$p \cdot k_1 p \cdot k_3$						$-\frac{1}{4}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{3}{8}$	$\frac{1}{24}$	
$-2k_1^2 p \cdot k_3$						1		$-\frac{1}{3}$			
$-2p \cdot k_3 (k_1 \cdot \epsilon)^2$					$\frac{1}{3}$			$-\frac{1}{18}$			$\frac{1}{6}$
$\frac{3}{4} k_1^2 k_3^2$						$-\frac{3}{4}$					
$k_1^2 (k_3 \cdot \epsilon)^2$						$-\frac{1}{12}$		$-\frac{1}{36}$			
$-k_1 \cdot k_3 k_1 \cdot \epsilon k_3 \cdot \epsilon$					$\frac{8}{3}$	$-\frac{5}{12}$	$\frac{5}{9}$	$-\frac{13}{36}$	$-\frac{5}{4}$	$-\frac{5}{36}$	$\frac{4}{3}$
$\frac{1}{2} (k_1 \cdot k_3)^2$					-3	$\frac{1}{2}$	$-\frac{2}{3}$	$\frac{5}{12}$	$\frac{3}{2}$	$\frac{1}{6}$	$-\frac{3}{2}$
SUM	1	$\frac{2}{3}$	$\frac{1}{6}$	$-\frac{1}{6}$	-1	$\frac{1}{12}$	$-\frac{5}{9}$	$\frac{3}{8}$	$\frac{5}{4}$	$\frac{5}{36}$	$-\frac{1}{2} - \frac{1}{6} \ln 2$

The numbers in the rows are the coefficients in the expansion $\int d^4 k_1 d^4 k_3 \frac{f(k_1, k_3, \epsilon)}{D_1} = \frac{\pi^4}{m^2} (a_1 \ln^2(\frac{\alpha^2}{4}) + a_2 \ln(\frac{\alpha^2}{4}) + \dots)$.

Table 3. Evaluation of the integrals encountered in the type II diagrams.

$g(k_1, k_2, k_3, \epsilon)$	$\ln^2(\frac{\alpha^2}{4})$	$\ln(\frac{\alpha^2}{4})$	$\ln^2 2$	$\ln 2$	1	$\pi^2 \ln 2$	π^2	$\zeta(3)$	$\pi^2 \ln(-1)$	$\ln(-1)$
$2m^2(k_2+k_3) \cdot p$	-1		1	-1	2					$-\ln 2$
$m^2 k_1 \cdot (k_2+k_3)$						$-\frac{1}{4}$			$-\frac{1}{8}$	
$-2m^2 k_2 \cdot k_3$			-1			$-\frac{1}{4}$	$-\frac{5}{12}$		$-\frac{1}{8}$	$\ln 2$
$-2m^2 k_2 \cdot \epsilon \cdot k_3 \cdot \epsilon$		$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{12}$	$\frac{1}{72}$		$-\frac{1}{24}$	$\frac{1}{6} + \frac{1}{6} \ln 2$
$3k_1 \cdot p(k_2+k_3) \cdot p$						$\frac{1}{2}$	$\frac{1}{4}$		$\frac{1}{4}$	
$-k_1^2(k_2+k_3) \cdot p$							$-\frac{1}{6}$			
$\frac{1}{2}(k_1^2 - 2p \cdot k_1)k_2 \cdot k_3$						$\frac{1}{6}$		$-\frac{3}{8}$	$-\frac{1}{24}$	
$-p \cdot k_1 \cdot k_2 \cdot k_3$						$\frac{5}{12}$	$-\frac{1}{6}$	$-\frac{3}{4}$	$-\frac{1}{24}$	
$(k_1 \cdot p)^2$		-1			2	$\frac{1}{12}$	$-\frac{3}{8}$		$\frac{1}{24}$	
$k_1 \cdot \epsilon \cdot k_2 \cdot p \cdot k_3 \cdot \epsilon + k_1 \cdot \epsilon \cdot k_2 \cdot \epsilon \cdot k_3 \cdot p$				$-\frac{2}{3}$		$\frac{2}{9}$	$-\frac{1}{18}$	$-\frac{1}{4}$	$\frac{1}{36}$	$-\frac{1}{3}$
$-2k_1 \cdot p \cdot k_2 \cdot \epsilon \cdot k_3 \cdot \epsilon$										
$(2k_2 \cdot k_3 - k_2 \cdot p - k_3 \cdot p)(k_1 \cdot \epsilon)^2$				$-\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{5}{18}$	$\frac{5}{18}$	$\frac{1}{2}$	$\frac{1}{36}$	$-\frac{1}{3}$
$-k_1 \cdot \epsilon(k_2+k_3) \cdot \epsilon \cdot k_1 \cdot p$										
SUM	-1	$-\frac{2}{3}$	$-\frac{1}{6}$	-1	$\frac{19}{6}$	$\frac{19}{36}$	$-\frac{23}{36}$	$-\frac{7}{8}$	$-\frac{1}{36}$	$-\frac{1}{2} + \frac{1}{6} \ln 2$

The numbers in the rows are the coefficients in the expansion

$$\int d^4 k_1 d^4 k_2 d^4 k_3 \delta^{(4)}(2p+k_1-k_2-k_3) \frac{g(k_1, k_2, k_3, \epsilon)}{2^{11}} = \frac{\pi^4}{m^2} (b_1 \ln^2(\frac{\alpha^2}{4}) + b_2 \ln(\frac{\alpha^2}{4}) + \dots).$$

Table 4. $\int_0^1 dx \frac{\ln^p x \ln^q(1-x)}{(a+bx)^r}$

a	b	p	q	r	1	π^2	$\ln 2$	$\ln^2 2$
1	1	1	0	1		$-\frac{1}{12}$		
1	-1	1	0	1		$-\frac{1}{6}$		
1	1	1	0	2			-1	
1	1	1	0	3	$-\frac{1}{4}$		$-\frac{1}{2}$	
1	1	1	0	4	$-\frac{7}{24}$		$-\frac{1}{3}$	
1	1	0	1	1		$-\frac{1}{12}$		$\frac{1}{2}$
1	1	0	1	2			$-\frac{1}{2}$	
1	1	0	1	3	$-\frac{1}{8}$		$-\frac{1}{8}$	
1	1	0	1	4	$-\frac{5}{48}$		$-\frac{1}{24}$	
1	-1	2	0	2		$\frac{1}{3}$		
1	1	2	0	2		$\frac{1}{6}$		
1	1	2	0	3		$\frac{1}{12}$	1	
1	1	2	0	4	$\frac{1}{6}$	$\frac{1}{18}$	1	
1	1	1	1	2		$\frac{1}{24}$		$-\frac{1}{2}$
1	1	1	1	3		$-\frac{1}{96}$	$\frac{1}{2}$	$-\frac{1}{4}$
1	1	1	1	4	$\frac{1}{12}$	$-\frac{5}{288}$	$\frac{3}{8}$	$-\frac{1}{6}$
1	1	0	2	2		$\frac{1}{12}$		$-\frac{1}{2}$
1	1	0	2	3		$\frac{1}{48}$	$\frac{1}{4}$	$-\frac{1}{8}$
1	1	0	2	4	$\frac{1}{24}$	$\frac{1}{144}$	$\frac{1}{8}$	$-\frac{1}{24}$

Table 5. $\int_0^1 dx \frac{\ln(a+bx)\ln(c+dx)}{(e+fx)}$

a	b	c	d	e	f	$\pi^2 \ln 2$	$\ln^3 2$	$\zeta(3)$
1	-1	1	-1	0	1			2
1	-1	1	1	0	1			$-\frac{5}{8}$
0	1	1	1	1	-1	$-\frac{1}{4}$		1
1	1	1	1	0	1			$\frac{1}{4}$
0	1	1	1	1	1			$-\frac{1}{8}$
0	1	1	-1	1	1	$-\frac{1}{4}$		$\frac{13}{8}$
0	1	1	1	0	1			$-\frac{3}{4}$
1	-1	1	-1	1	1	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{7}{4}$
1	-1	1	1	1	1	$-\frac{1}{12}$	$\frac{1}{3}$	$\frac{1}{8}$
0	1	1	-1	1	-1			1

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FIGURE CAPTION

These six diagrams represent the kernel for three photon virtual annihilation. The letters i and f denote initial and final states of the electron-positron system. The labeling of the fermion momenta in the initial state and of the photon momenta emerging from this state is the same for all diagrams.