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A SET OF INTEGRAL EQUATIONS FOR PION-PION SCATTERING
VALID AT ALL ENERGIES

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ABSTRACT

We derive a set of coupled integral equations for pion-pion scattering by using fixed- t dispersion relations combined with suitable fixed- s analytic continuation. They represent a possible way to extend the validity of Roy-type equations to all energies. The expected low-energy properties of the amplitude are also properly taken into account.

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1. INTRODUCTION

The unitarity and analyticity of scattering amplitudes, together with their crossing properties, are without any doubt strong restrictive conditions. It is presently not known which might be the "correct" input, which by means of the above principles, would produce a unique amplitude. Nevertheless, these three principles can be used either directly [as Atkinson did in existence proofs or in finding iterative constructive procedures within some functional regions, via the Schauder and the contraction mapping principles ¹⁾], or in providing us with integral equations, which one then tries to solve using various inputs. In this latter category are Roy's equations for m partial waves ²⁾ which also have the advantage of being valid in the purely axiomatic framework. Indeed, one has to be precise about the word "analyticity", since it has been used in various senses, ranging from the Mandelstam conjecture to the analyticity domain rigorously deduced from the axioms of local field theory ³⁾. Beside their pure theoretical interest, these partial wave equations can be (and were) employed to construct phenomenological amplitudes, both compatible with the first principles and the experimental data ⁴⁾.

When deriving a set of coupled integral equations for the partial waves, there are nevertheless some problems in combining the requirements of unitarity (which involve the amplitude at all scattering angles) and the assumed amount of analyticity. Actually, the lines on which dispersion relations can be written, have to pass through some "défilé" allowed by the analyticity properties of the amplitude and, consequently, not all the angles θ can be scanned above some critical energy $\sqrt{s_c}$ ⁵⁾. Therefore, one often feels compelled to consider the amplitude as given from other sources of information for $s > s_c$, and limit the integral equations to the low energy region. Clearly, this is not a physical limitation, but merely a technical one, which should be overcome by purely mathematical means. A possible way would be to construct a reasonably simple family of analytic manifolds covering the whole physical region and entirely contained in the (two variables) holomorphy domain. So far, the attempts to achieve such a construction have failed. An alternative method, involving an analytic continuation of the amplitude in the transfer variable, will be presented in this paper.

The problem of extending partial wave equations to all energies has also been considered by Lopez ⁶⁾ in the case where the Mandelstam representation is assumed to hold together with some conditions on the double spectral functions. In the analytic continuation method proposed here,

we shall use only the information on the shape of the analyticity domain in $\cos \theta$ at each physical s . This can be done irrespective of the assumed analyticity domain (axiomatic or Mandelstam), the only difference consisting in the actual form of the functions used in the analytic continuation procedure. Let us notice that the idea of using near-forward scattering information in this context goes back to the early sixties, when attempts were made to write integral equations by means of the first t derivatives of the dispersion relation at $t = 0$ ⁷⁾. The germ of this approach can be found in the early work of Chew, Goldberger, Low and Nambu ⁸⁾.

As will be seen at the beginning of the next section, the analytic continuation involves an expansion of the amplitude not in terms of the usual Legendre polynomials, but in terms of some functions which embody the shape of the assumed analyticity domain and provide a fast convergence there. In that section, for the sake of clarity, the method is explained in the case of neutral pion scattering. In Section 3 we discuss briefly the extension of the method to the general isospin case. Section 4 contains some final remarks. Our procedure involves various conformal mappings, the construction of which is displayed in the Appendices.

2. THE NEUTRAL MODEL

2.1 Unitarity and convergence problems

In order to find integral equations, one first looks for manifolds in the (s, t) space, on which the amplitude $F(s, t)$ can be written as a Cauchy integral (dispersion relation) in terms of its boundary values on some known cuts. In this paper we shall restrict ourselves to (twice-subtracted) fixed- t dispersion relations:

$$F(s, t) = F(s, t) + \frac{1}{\pi} \int_4^{\infty} ds' \frac{A(s', t)}{s'^2} \left[\frac{s^2}{s' - s} - \frac{s_0^2}{s' - s_0} + \frac{u^2}{s' - u} - \frac{(4 - s_0 - t)^2}{s' - 4 + s_0 + t} \right]. \quad (1)$$

To turn these Cauchy integrals into equations, one uses the unitarity condition which relates (at a given s) the absorptive part $A(s, t)$ to an integral involving the amplitude at all physical angles. At this point, one usually introduces the partial wave expansion in terms of Legendre polynomials:

$$F(s, t) = \sum_{\ell=0,2,\dots} (2\ell+1) f_{\ell}(s) P_{\ell}(\cos \theta) \quad , \quad \cos \theta = 1 + \frac{2t}{s-4} \quad (2)$$

for which unitarity takes a very simple form:

$$\text{Im } f_l(s) \geq \frac{1}{2} \sqrt{\frac{s-4}{s}} |f_l(s)|^2, \quad (s \geq 4, l = 0, 2, \dots) \quad (3)$$

The misleading fact here is the belief that approximations of the form $\sum_0^N (2l+1) f_l(s) P_l(\cos \theta)$ should always converge over physical $\cos \theta$'s, and even over the whole (small) Martin-Lehmann ellipses³⁾, namely within:

$$\frac{(\text{Re } \cos \theta)^2}{a^2(s)} + \frac{(\text{Im } \cos \theta)^2}{b^2(s)} < 1, \quad (4)$$

where

$$b^2(s) = a^2(s) - 1 = \begin{cases} \frac{16s}{(s-4)^2} & , \quad 4 < s < 16 \\ \frac{256}{s(s-4)} & , \quad 16 \leq s < 32 \\ \frac{4s}{(s-4)(s-16)} & , \quad s \geq 32 \end{cases} \quad (4')$$

[in the case of Mandelstam analyticity, the first expression (4') for $b^2(s)$ extends to $16 \leq s < \infty$]. In fact, as s increases, the amplitude from which the expansion coefficients $f_l(s)$ are to be computed is "known" only within a shrinking subinterval of the physical region $-1 \leq \cos \theta \leq 1$. This is due to the fact that the dispersion relations (1) are valid only for t ranging between t_m to 4, where $t_m = -28$ in the axiomatic framework and $t_m = -32$ if the Mandelstam analyticity is conjectured. Then, even if one tries to approximate $F(s, t)$ by sums of the form $\sum_0^N (2l+1) f_l(s) P_l(\cos \theta)$, the coefficients $f_l(s)$ can no longer be computed by integrals of $F[s, t = \frac{1}{2}(s-4)(\cos \theta - 1)]$ over $-1 \leq \cos \theta \leq 1$, for the simple reason that F is "unknown" (out of the reach of dispersion relations), when^{*)} $\cos \theta < 1 + 2 t_m / (s-4) \equiv \cos \theta_0(s)$. This happens as soon as s exceeds the critical value s_c defined by $\cos \theta_0(s_c) = -1$, namely $s_c = 32$ or 36 if $t_m = -28$ or -32 .

*) For the moment we ignore the $t \rightleftharpoons u$ symmetry of $F(s, t)$.

Of course, one could try to find the $f_\ell(s)$'s by minimizing

$$\sup_{\cos \theta_0 \leq \cos \theta \leq 1} \left| F - \sum_{\ell=0}^N (2\ell+1) f_\ell(s) P_\ell(\cos \theta) \right| \quad (5)$$

(other norms are equally well admitted). But then, although the above polynomial approximation "looks Legendre", the convergence of the sequence of these approximations is limited by a well-known theorem, to the largest ellipse contained in the $\cos \theta$ analyticity domain of F , and having its foci at 1 and $\cos \theta_0(s)$. This ellipse ceases to cover all the physical angles when s is larger than a new critical value s'_0 , easily found to be $s'_0 = 33.79$ (resp. 40) for $t_m = -28$ (resp. -32). Hence, the sequence of approximations (5) cannot be used in the unitarity integrals when $s > s'_0$, because it is divergent. This being a mild mathematical disease, curing it is a simple task.

2.2 Expanding around segments

It is well known that the Taylor expansion of a holomorphic function $F(z)$ around some point z_0 converges in the interior of the circle passing through the nearest singularity, say z_1 , the convergence rate at z being $|(z-z_0)/(z_1-z_0)|^n$ (up to less relevant factors). In our case ($z = \cos \theta$), one sees that the available information about $F(z)$ is spread out along the finite interval $[\cos \theta_0, 1]$ (which for $s > s_0$ ceases to cover the interval $[-1, 1]$), so that one has to "expand around" this segment rather than a point. In this case, a similar convergence theorem is appropriate ⁹⁾ telling us that the best polynomial approximations converge like the Laurent series:

$$F(z) = \sum_{n=0}^{\infty} h_n (Z^n(z) + Z^{-n}(z)) \quad , \quad (6)$$

where $Z(z)$ is the variable which maps the segment $[\cos \theta_0, 1]$ onto the unit circle $|Z| = 1$ without changing the point at infinity. The corresponding convergence rate is $|Z(z)/Z(z_1)|^n$. This suggests that in order to obtain the best convergence rate, as well as the largest convergence domain, one has to resort to that variable $W(z)$ which maps conformally the whole holomorphy domain D of $F(z)$ onto an annulus, the inner (unit) circle being the image of $[\cos \theta_0, 1]$. A proof ¹⁰⁾ of this statement is sketched in Appendix A, whereas the construction of the various mappings $W(z)$ corresponding to the actual domains D is given in Appendices B to D.

Then, $F(z)$ can be written as:

$$F(z) = \sum_{n=0}^{\infty} t_n [W^n(z) + W^{-n}(z)] \quad (7)$$

Obviously, the occurrence of negative powers in the expansion (7) is due to the fact that the analyticity domain in W has a hole, so that one has to resort to Laurent series. Furthermore, as the image of the end-points $\cos \theta_0$ and 1 through the mapping $z \rightarrow W$ are chosen to be -1 and 1, both $W = e^{i\varphi}$ and $W = e^{-i\varphi}$ correspond to the same point z of the segment $[\cos \theta_0, 1]$. Hence, $F(z)$, as a function of W , satisfies:

$$F(W) = F(W^{-1}) \quad (8)$$

for any W on the unit circle. By analytic continuation, this is true for all $W(z)$ with z in D . The equality of the coefficients of W^n and W^{-n} in Eq. (7) is the practical consequence of Eq. (8). One can alternatively work with the Cutkosky variable ¹¹⁾:

$$w(z) = \frac{1}{2} [W(z) + W^{-1}(z)] \quad (9)$$

which brings the annulus $1 < |W| < R$ into an ellipse with foci at $w = \pm 1$ (the unit circle $|W| = 1$ is mapped onto the segment $[-1, 1]$ since $W = e^{i\varphi}$ implies $w = \cos \theta$). In this case, the expansion (7) takes the form:

$$F(z) = 2 \sum_{n=0}^{\infty} t_n T_n(w(z)) \quad (7')$$

where $T_n(w)$ is the n^{th} Tschebyshev polynomial of the variable w . Here, merely for reasons of convenience, we shall use instead:

$$F(z) = \sum_{n=0}^{\infty} (2n+1) g_n P_n(w(z)) \quad (7'')$$

where $P_n(w)$ is the n^{th} Legendre polynomial. Both expansions (7') and (7'') are maximally convergent, since the corresponding polynomials are the best approximants in a L^2 norm, the difference being only in the weight factors.

2.3 The unitarity condition

The expansion (7'') will now be used, in conjunction with the unitarity condition, to express the imaginary part $A(s, t)$ for $s \geq 4$ and $t_m < t \leq 0$ in terms of the values of the full amplitude $F(s, t')$ and the inelastic term $I(s, t)$, with (s, t') also restricted to the same "défilé". Unitarity reads:

$$A(s, t) = \frac{1}{2} \sqrt{\frac{s-4}{s}} \int_{K^2 \geq 0} dt_1 dt_2 K(s; t_1, t_2) F(s, t_1) F^*(s, t_2) + I(s, t) \quad (10)$$

where

$$K(s; t, t_1, t_2) = \frac{1}{\pi(s-4)} \left[\frac{4}{s-4} t t_1 t_2 - (t^2 + t_1^2 + t_2^2) + 2(t t_1 + t_1 t_2 + t_2 t) \right]^{-1/2}. \quad (11)$$

In terms of the partial wave inelasticities $\eta_\ell(s)$, defined by:

$$f_\ell(s) = 2\sqrt{\frac{s}{s-4}} \frac{\eta_\ell(s) e^{2i\delta_\ell(s)} - 1}{2i}, \quad \begin{cases} \eta_\ell(s) = 1, & 4 < s < 16, \\ 0 \leq \eta_\ell(s) \leq 1, & s \geq 16, \end{cases} \quad (12)$$

$I(s, t)$ has the expansion:

$$I(s, t) = 2\sqrt{\frac{s}{s-4}} \sum_{\ell=0,2,\dots} (2\ell+1) \frac{1 - \eta_\ell^2(s)}{4} P_\ell\left(1 + \frac{2t}{s-4}\right) \quad (13)$$

which is convergent in the region $s \geq 16$, $t_m < t \leq 0$. For the amplitude $F(s, t)$, we shall use an expansion of the form (7''), forcibly symmetrized in z :

$$F(s, t) = \sum_{n=0}^{\infty} (2n+1) g_n(s) \bar{P}_n(w(s, z)) \quad , \quad z = 1 + \frac{2t}{s-4} \quad , \quad (14)$$

where:

$$\bar{P}_n(w(s, z)) \equiv \frac{1}{2} [P_n(w(s, z)) + P_n(w(s, -z))] .$$

Owing to the orthogonality of the $P_n(w)$'s, one gets from Eq. (7''):

$$g_n(s) = \frac{1}{2} \int_{\bar{t}}^0 dt \frac{\partial w(s, z)}{\partial t} P_n(w(s, z)) F(s, t) \quad , \quad \bar{t} = \begin{cases} -(s-4), & s \leq s_c, \\ -|t_m|, & s > s_c. \end{cases} \quad (15)$$

Again, one can work out a symmetrized expression, namely:

$$g_n(s) = \frac{1}{4} \int_{\bar{t}}^0 dt \left[\frac{\partial w(s, z)}{\partial t} P_n(w(s, z)) - \frac{\partial w(s, -z)}{\partial t} P_n(w(s, -z)) \right] F(s, t). \quad (15')$$

Inserting Eq. (14) into Eq. (10), one gets:

$$A(s, t) = \sum_{n, m=0}^{\infty} K_{nm}(s, t) g_n(s) g_m^*(s) + I(s, t) \quad (16)$$

where

$$K_{nm}(s, t) = (2n+1)(2m+1) \frac{1}{2} \sqrt{\frac{s-4}{s}} \int_{K^2 \geq 0} dt_1 dt_2 K(s; t, t_1, t_2) \cdot \bar{P}_n(w(s, z_1)) \bar{P}_m(w(s, z_2)) \quad (17)$$

2.4 Unitarity at low energies

Let us notice here a minor technical point. When $s < s_c = 4 + |t_m|$, $A(s, t)$ is needed for unphysical angles also ($t_m < t < 4 - s$). Then the formula (16) still holds, but the expressions (17) for the kernels $K_{nm}(s, t)$ become meaningless (the ellipse $K^2 \geq 0$ disappears when t becomes less than $4 - s$). Clearly, an analytic continuation is required, which of course is allowed, due to the analytic properties in t of the functions $P_n[w(s, z)]$. This can be done explicitly (see Appendix E), and leads to the alternative form of Eq. (17), valid for $t_m < t < 4 - s$:

$$K_{nm}(s, t) = (2n+1)(2m+1) \frac{1}{2} \sqrt{\frac{s-4}{s}} \int_{\tilde{K}^2 \geq 0} dt_r dt_i \tilde{K}(s; t, t_r, t_i) \cdot \bar{P}_n(w(s, \tilde{z})) \bar{P}_m^*(w(s, \tilde{z})) \quad (17')$$

where:

$$\tilde{K}(s; t, t_r, t_i) = \frac{2}{\pi(s-4)} \left[t^2 - 4t t_r - 4t_i^2 - \frac{4t}{s-4} (t_r^2 + t_i^2) \right]^{-1/2} \quad (11')$$

and

$$\tilde{z} = 1 + 2(t_r + i t_i) / (s-4).$$

Notice that the modified unitarity equation (16) and (17'), necessary when $s < s_c$, implicitly involves amplitudes at complex t obtained by analytic continuation from their values at real t (the integration range $\tilde{K}^2 \geq 0$ is the interior of an ellipse in the $\tilde{z}$ complex plane enclosing the physical region). At this stage, the fact that one is allowed to write dispersion integrals also for (some) complex t should command attention. One could in this way riddle the surface of all the complex unitarity ellipses. Moreover,

the enlargement of the "basis" of the w expansions would certainly have a damping effect on the final computational errors. But we do not want to dig further into this question here.

2.5 Integral equations

We are now ready to turn Eqs. (1) into a set of integral equations for the functions $g_n(s)$. To this end, we shall apply Roy's trick ²⁾ in order to eliminate the unwanted subtraction function $F(s_0, t)$ in favour of a subtraction constant $F(s_0, t_0)$. One first writes:

$$F(s, t) = F(t, s_0) \quad (18)$$

on account of the $s \rightleftharpoons t$ crossing and then, assuming that $-28 < s_0 < 4$, one expresses $F(t, s_0)$ as a fixed s_0 dispersion integral of the form (1), subtracted at $t = t_0$. Inserting this expression into Eq. (1), one gets:

$$F(s, t) = F(s_0, t_0) + \frac{1}{\pi} \int_4^\infty ds' \left[N(s, t, s'; s_0) A(s', t) + N(t, s_0, s'; t_0) A(s', s_0) \right] \quad (19)$$

where

$$N(s, t, s'; s_0) = \frac{1}{s'^2} \left[\frac{s^2}{s' - s} - \frac{s_0^2}{s' - s_0} + \frac{(4 - s - t)^2}{s' - 4 + s + t} - \frac{(4 - s_0 - t)^2}{s' - 4 + s_0 + t} \right] \quad (19')$$

By using the expression (16) for the absorptive parts and projecting out onto $g_k(s)$ according to Eq. (15'), we finally obtain the desired equations:

$$g_k(s) = \delta_{k0} F(s_0, t_0) + \frac{1}{\pi} \int_4^\infty ds' \sum_{n,m=0}^\infty M_{k,nm}(s, s'; s_0, t_0) g_n(s') g_m^*(s') + I_k(s; s_0, t_0) \quad (k = 0, 1, \dots) \quad (20)$$

where:

$$M_{k,nm}(s, s'; s_0, t_0) = \frac{1}{4} \int_{\frac{1}{t}}^0 dt \left[\frac{\partial W(s, z)}{\partial t} P_k(w(s, z)) - \frac{\partial W(s, -z)}{\partial t} P_k(w(s, -z)) \right] \cdot \left[N(s, t, s'; s_0) K_{nm}(s', t) + N(t, s_0, s'; t_0) K_{nm}(s', s_0) \right] \quad (21)$$

and

$$I_k(s; s_0, t_0) = \frac{1}{4\pi} \int_4^\infty ds' \int_{\frac{1}{t}}^0 dt \left[\frac{\partial W(s, z)}{\partial t} P_k(w(s, z)) - \frac{\partial W(s, -z)}{\partial t} P_k(w(s, -z)) \right] \cdot \left[N(s, t, s'; s_0) I(s', t) + N(t, s_0, s'; t_0) I(s', s_0) \right] \quad (22)$$

If we assume that the inelastic terms $I_k(s; s_0, t_0)$ are completely known (which of course is not the case in practice), Eqs. (20) constitute a set of non-linear integral equations valid for all $s \geq 4$, which embody the full content of unitarity and analyticity, both at fixed s and fixed t . Moreover, as for the usual partial wave equations, these equations are singular inasmuch as integrals in Eqs. (20) have to be understood as Cauchy integrals (with the $\pm i\epsilon$ prescription). Actually, the kernels (21) are easily seen to be of the form:

$$M_{k,nm}(s, s'; s_0, t_0) = \frac{Q_{k,nm}(s; s_0, t_0)}{s' - s} + R_{k,nm}(s, s'; s_0, t_0) \quad (23)$$

where $R_{k,nm}(s, s'; s_0, t_0)$ is regular except for "integrable" (logarithmic) singularities at the points $s' = 4 - s$ and $s' = 4 - s - \bar{t}$, due to the t integration of the term $u^2/(s' - u)$ contained in $N(s, t, s'; s_0)$.

2.6 Choice of the bases and threshold behaviour of the partial waves

Until now, the basis of the conformal mapping $w(z)$ (i.e. that part of the real z -axis which is sent onto $-1 \leq w \leq 1$) has been chosen as:

$$\begin{cases} -1 \leq z \leq 1 & , \quad 4 < s \leq s_c \\ 1 + \frac{2t_m}{s-4} \leq z \leq 1 & , \quad s > s_c \end{cases} \quad (24)$$

It must be realized, however, that such a choice is by no means compulsory. In principle, any basis which is entirely contained in the assumed analyticity domain of $F(z)$ and does not enter the region $z < 1 + 2t_m/(s-4)$ (out of the reach of dispersion relations) is acceptable. In particular, choices are possible for which the basis does not shrink up to zero (in the t variable) when $s \rightarrow 4$ (an example is given in Fig. 1a). We shall not attempt here to determine the optimal basis, a question which actually remains to be solved^{*)}. Let us simply notice that the behaviour of the basis near $s=4$ has no direct effect on the threshold behaviour of the partial waves $f_\ell(s)$, as long as the solutions of our equations are not "pathological" [i.e. produce amplitudes $F(s, t)$ which remain finite when $s \rightarrow 4$ at fixed t , $|t| < 4$]

*) See also the remark following Eq. (11').

Indeed, this last condition [together with the proper analyticity in t of $F(s, t)$] is known to be sufficient to guarantee a normal threshold behaviour of the partial waves, $f_\ell(s) \sim \text{const.} (s-4)^\ell$. It is only the way by which the normal behaviour is achieved, which depends on the choice of the bases. For instance, with the set of bases (24), the functions $g_n(s)$ are found to behave like $(s-4)^n$ for $s \rightarrow 4$, which is sufficient to ensure the normal behaviour of the $f_\ell(s)$ [notice that, in particular, the $g_n(s)$ coincide with the $f_n(s)$ for $s \leq s_c$ if, in addition to (24), one uses only the small Lehmann-Martin ellipses in the construction of the mapping $w(z)$, and not the full axiomatic domain].

If, on the other hand, one chooses the bases as in Fig. 1a, the $g_n(s)$ themselves do not vanish for $s \rightarrow 4$, but still the $f_\ell(s)$ have the proper threshold behaviour. To show this, clearly it is enough to consider the partial wave projection of the functions $w^n(s, z) = 2^{-n} [W(\bar{s}, z) + W^{-1}(s, z)]^n$:

$$(w^n)_\ell(s) \equiv \frac{1}{2} \int_{-1}^1 dz P_\ell(z) w^n(s, z) \quad (25)$$

The functions $w^n(s, z)$ being, by construction, holomorphic in the (s dependent) analyticity domain D of the amplitude, one can rewrite Eq. (25) as:

$$(w^n)_\ell(s) = \frac{1}{2i\pi} \oint_C dz' Q_\ell(z') w^n(s, z') \quad (25')$$

where C is the boundary of D , transformed by the W mapping into the outer circle $|W| = R(s)$ of the annulus $1 < |W| < R(s)$. For $z' \in C$ and $s \rightarrow 4$, we have $\text{const.}/(s-4) < |z'| < \text{const.}/(s-4)$, so that, by using the asymptotic behaviour of the Legendre functions of the second kind, $Q_\ell(z') \sim \text{const.} z'^{-\ell-1}$, we obtain:

$$\begin{aligned} |(w^n)_\ell(s)| &\leq \text{const.} \int_C \left| \frac{dz'}{z'^{\ell+1}} \right| [R(s) + R^{-1}(s)]^n \\ &< \text{const.} [R(s) + R^{-1}(s)]^n (s-4)^\ell \end{aligned} \quad (26)$$

Now, for the choice of bases under consideration, it is readily seen that $R(s)$ remains finite as $s \rightarrow 4$, which entails the desired behaviour of $(w^n)_\ell(s)$, and hence of the partial waves.

3. EXTENSION TO CHARGED PIONS

The derivation of integral equations for the scattering of neutral pions given in the previous section can serve as a paradigm for a similar treatment of other scattering processes. In the following, we shall briefly describe how the isospin complications of the $\pi\pi$ scattering case are taken into account. The amplitudes with definite isospin $I(=0,1,2)$ in the s channel are denoted by $F^{(I)}(s,t)$, and written in the matrix form:

$$\underline{F}(s,t) = \begin{pmatrix} F^{(0)}(s,t) \\ F^{(1)}(s,t) \\ F^{(2)}(s,t) \end{pmatrix}. \quad (27)$$

Then, the dispersion relations in the "Roy form" ²⁾ [the analogue of Eq. (19) with the particular choice $s_0=0$, $t_0=4$] can be written as

$$\underline{F}(s,t) = \frac{1}{2} (s+t C_{st} + u C_{su}) \underline{a}_0 + \frac{1}{\pi} \int_4^\infty ds' \left[N_1(s,t,s') \underline{A}(s',t) + N_2(s,t,s') \underline{A}(s',0) \right] \quad (28)$$

where

$$\begin{cases} N_1(s,t,s') = \frac{-su}{s'(s'-4+t)} \left(\frac{1}{s'-s} + \frac{1}{s'-u} C_{su} \right), \\ N_2(s,t,s') = \frac{-t}{s'(s'-4)} (u+s C_{su}) C_{st} \left(\frac{1}{s'-t} + \frac{1}{s'-4+t} C_{su} \right) \end{cases} \quad (29)$$

Here, C_{st} and C_{su} are the crossing matrices:

$$C_{st} = \begin{pmatrix} 1/3 & 1 & 5/3 \\ 1/3 & 1/2 & -5/6 \\ 1/3 & -1/2 & 1/6 \end{pmatrix}, \quad C_{su} = \begin{pmatrix} 1/3 & -1 & 5/3 \\ -1/3 & 1/2 & 5/6 \\ 1/3 & 1/2 & 1/6 \end{pmatrix}, \quad (30)$$

and

$$\tilde{a}_0 = \begin{pmatrix} a_0^{(0)} \\ 0 \\ a_0^{(2)} \end{pmatrix} \equiv \frac{1}{2} \tilde{F}(4,0) \quad (31)$$

represents the two S wave scattering lengths $a_0^{(I)} = \frac{1}{2} f_0^{(I)}$ (4). Similarly to Eq. (14), the functions $g_n^{(I)}(s)$ are now defined by the expansions:

$$F^{(I)}(s,t) = \left\{ \begin{matrix} 1 \\ (t-u) \end{matrix} \right\} \sum_{n=0}^{\infty} (2n+1) g_n^{(I)}(s) \bar{P}_n(w(s,z)) \quad , \quad I = \begin{matrix} 0, 2 \\ 1 \end{matrix} \quad (32)$$

The extra factor $(t-u)$ in the case $I=1$ has been inserted to enforce the antisymmetry in z of $F^{(1)}(s,t)$. The unitarity relations read:

$$\begin{aligned} A^{(I)}(s,t) &= \frac{1}{2} \sqrt{\frac{s-4}{s}} \int_{k^2 \geq 0} dt_1 dt_2 K(s; t, t_1, t_2) F^{(I)}(s, t_1) F^{(I)*}(s, t_2) + I^{(I)}(s, t) \\ &= \sum_{n,m=0}^{\infty} K_{nm}^I(s, t) g_n^{(I)}(s) g_m^{(I)*}(s) + I^{(I)}(s, t) \end{aligned} \quad (33)$$

where:

$$\begin{aligned} K_{nm}^I(s, t) &= (2n+1)(2m+1) \frac{1}{2} \sqrt{\frac{s-4}{s}} \int_{k^2 \geq 0} dt_1 dt_2 K(s; t, t_1, t_2) \cdot \\ &\cdot \bar{P}_n(w(s, z_1)) \bar{P}_m(w(s, z_2)) \left\{ \begin{matrix} 1 \\ (2t_1+s-4)(2t_2+s-4) \end{matrix} \right\} \quad , \quad I = \begin{matrix} 0, 2 \\ 1 \end{matrix} \end{aligned} \quad (34)$$

and $I^{(I)}(s, t)$ is defined similarly to $I(s, t)$ in Eq. (13). Then, proceeding exactly as in the previous section, one derives the following set of integral equations for the $g_n^{(I)}(s)$'s:

$$\begin{aligned} g_k^{(I)}(s) &= \int_{k_0} L^{(I)}(s) + \frac{1}{\pi} \int_4^{\infty} ds' \sum_{\substack{n,m=0,1,\dots \\ I=0,1,2}} M_{k,nm}^{II'}(s, s') g_n^{(I')}(s') g_m^{(I')}(s') \\ &+ I_k^{(I)}(s) \quad (k=0,1,\dots) \end{aligned} \quad (35)$$

where

$$M_{k,nm}^{II'}(s,s') = \frac{1}{4} \int_t^0 dt \left[\frac{\partial W(s,z)}{\partial t} P_k(W(s,z)) - \frac{\partial W(s,-z)}{\partial t} P_k(W(s,-z)) \right] \cdot \left[N_1^{II'}(s,t,s') K_{nm}^{I'}(s',t) + N_2^{II'}(s,t,s') K_{nm}^{I'}(s',0) \right] \quad (36)$$

Here, $N_{1,2}^{II'}$ are the I, I' elements of the matrices N_1 and N_2 defined in Eq. (29), and

$$\underline{L}(s) = \begin{pmatrix} 2a_0^{(0)} \\ \frac{4}{6}a_0^{(0)} - \frac{5}{12}a_0^{(2)} \\ 2a_0^{(2)} \end{pmatrix} + (s-4) \begin{pmatrix} \frac{4}{3}a_0^{(0)} - \frac{5}{6}a_0^{(2)} \\ 0 \\ -\frac{4}{6}a_0^{(0)} + \frac{5}{12}a_0^{(2)} \end{pmatrix}, \quad (37)$$

while the $I_k^{(I)}(s)$ have obvious expressions analogous to Eq. (22).

Of course, the remarks following Eq. (22), including those pertaining to threshold behaviours, are, mutatis mutandis, still in order here.

4. SUMMARY AND CONCLUDING REMARKS

In this paper we have written integral equations [Eqs. (20) and (35)] not for the usual partial waves, but for the coefficient functions $g_n(s)$ of the series (14). As compared to the Roy-type equations, they have some advantages and some disadvantages. The main computational drawback is that, in contradistinction to the partial waves, unitarity couples the various g_n 's. On the other hand, the series (14) have, by construction (see Section 2.2 and Appendix A) the fastest convergence rate, so that it is expected that a few terms will be sufficient for practical applications. Further, the range of validity of the integral equation is no longer limited to some finite energy region (see Section 2.2), and the "correct" threshold behaviour of the partial waves is granted (see Section 2.6). After displaying in a detailed form the main features of these equations in the neutral case, we have written them (Section 3) for physical pions. In principle, there is no difficulty to rewrite them for any two-particle scattering process: only the actual form of the functions w (and, of course, the isotopics) will change. We think that our equations should deserve further study, both at the pure mathematical level and also, because of their improved convergence property, as a tool for phenomenological analysis. Furthermore, it might be desirable to formulate them in terms of the Wanders symmetric variables ¹²⁾ in order to take a better account of the crossing properties of the amplitude, an ingredient which is not fully contained

in Eqs. (20) and (35). An intermediate step would be to use consistently the variables s and $v = (t-u)^2$ together with new mappings $W(s,v)$ transforming the vertical segments of the shaded region in Fig. 1b (the bases) into unit circles. In this way, the final symmetrization (14) becomes superfluous, so that one has, in place of Eq. (32):

$$F^{(I)}(s,t) = \left\{ \begin{matrix} 1 \\ (t-u) \end{matrix} \right\} \sum_{n=0}^{\infty} (2n+1) g_n^{(I)}(s) P_n(w(s,v)) \quad , \quad I = \begin{matrix} 0, 2 \\ 1 \end{matrix} .$$

In other words, the symmetrized $\bar{P}_n$'s entering the definition of K_{nm}^I [Eq. (34)] have to be replaced by the Legendre polynomials P_n of the new w , while the bracket $\frac{1}{2}\{\partial w(s,z)/\partial t \cdot P_k[w(s,z)] - \partial w(s,-z)/\partial t \cdot P_k[w(s,-z)]\}$ entering the definition (36) of $M_{k,nm}^{II'}$ have to be replaced by $\partial w(s,v)/\partial t \cdot P_k[w(s,v)]$.

As a final remark, let us remind the reader that an efficient way of reducing the effects of the errors and (or) lack of knowledge of the (high energy) inelasticity terms is the use of suitable damping outer functions ¹³⁾.

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APPENDIX A

Conformal mappings and convergence rates

In this appendix, we shall discuss the analytic continuation abilities of the expansions of the scattering amplitude (at fixed energy) in terms of different mapping variables. The actual construction of the optimal expansion variables $w(z)$ will be exhibited in the subsequent three appendices: for the Martin axiomatic domain in Appendix B (in that case, because of the complicated shape of the domain, we shall give only a general algorithm which, nevertheless, is ready for computer processing), for the Mandelstam domain in Appendix C, and finally for the Lehmann-Martin ellipse (which is a subset of the full axiomatic domain) in Appendix D.

Let $W_a(z)$ be the function that conformally maps some sub-domain D_a of the analyticity domain D of $F(z)$ (artificially cut along the basis, C_1) onto the annulus:

$$1 < |W_a(z)| < R_a \quad . \quad (A.1)$$

In this mapping, C_1 becomes the unit circle $|W_a| = 1$, while the outer boundary C_a of D_a becomes the circle $|W_a| = R_a$. We shall suppose that this boundary C_a already touches some of the singularities of $F(z)$. Then, an expansion of the form (7),

$$F(z) = \sum_{n=0}^{\infty} t_n [W_a^n(z) + W_a^{-n}(z)] \quad (A.2)$$

clearly converges for z in D_a and diverges for z not in (the closure of) D_a . The simple connectedness of the image set is restored by means of the auxiliary function:

$$w_a(z) = \frac{1}{2} [W_a(z) + W_a^{-1}(z)] \quad (A.3)$$

which maps D_a onto an ellipse with foci at $w_a = \pm 1$ (the basis C_1 becomes the segment $[-1, 1]$), and it appears that the convergence domain in z of polynomial expansions of $F(z)$ in terms of the variable $w_a(z)$ will be precisely D_a . Now the point is - and this fact is of great practical importance - that not only the convergence domain increases with D_a , but also the convergence rate of the expansion (A.2) at a given point z .

To show that ¹⁰⁾, let us consider two domains D_a and D_b with $D_a \subset D_b \subset D$, and the corresponding functions $W_a(z)$ and $W_b(z)$ which map D_a and D_b onto the respective annuli:

$$1 < |W_a| < R_a \quad \text{and} \quad 1 < |W_b| < R_b . \quad (\text{A.4})$$

In both cases, C_1 is sent onto the unit circle. We shall prove in a moment that, for every z in D_a :

$$\frac{|W_a(z)|}{R_a} \geq \frac{|W_b(z)|}{R_b} . \quad (\text{A.5})$$

Then, since the boundaries C_a and C_b of D_a and D_b are assumed to touch some singularities of $F(z)$, the convergence rates of the expansions of the form (A.2) are respectively given by $(|W_a(z)|/R_a)^n$ and $(|W_b(z)|/R_b)^n$. Hence, the property (A.5) tells us that the best convergence rate is indeed obtained by means of that conformal mapping, $W(z)$, which send the whole holomorphy domain D of $F(z)$ onto an annulus:

$$|W(z \in C_1)| = 1 , \quad |W(z \in C)| = R . \quad (\text{A.6})$$

The property (A.5) basically rests upon the principle of Groetzsch, which states that if $D_a \subset D_b$:

$$R_a \leq R_b \quad (\text{A.7})$$

[for a proof, see Appendix A of Ref. ¹⁰⁾]. Introducing then the functions:

$$\begin{cases} \sigma_a(z) = \text{Log} (|W_a(z)|/R_a) , \\ \sigma_b(z) = \text{Log} (|W_b(z)|/R_b) , \end{cases} \quad (\text{A.8})$$

one notices that they are harmonic in the (artificially cut) domain $D_a - C_1$, since both $W_a(z)$ and $W_b(z)$ are holomorphic and have no zeros there (by construction, all points of $D_a - C_1$ are mapped outside the unit circle!). Furthermore:

$$\sigma_a(z \in C_1) = -\text{Log} R_a , \quad \sigma_b(z \in C_1) = -\text{Log} R_b , \quad (\text{A.9})$$

so that, owing to Eq. (A.7):

$$\sigma_a(z) \geq \sigma_b(z) \quad , \quad z \in C_1 \quad . \quad (A.10)$$

The same inequality also holds on the outer boundary C_a . Actually:

$$\sigma_a(z \in C_a) = \log(R_a/R_a) = 0 \quad (A.11)$$

and

$$\sigma_b(z) \leq 0 \quad \text{for } z \in D_b \quad , \quad (A.12)$$

because the function $\sigma_b(z)$ is harmonic in $D_b - C_1$ and non-positive on the boundary of $D_b - C_1$ ($\sigma_b(z \in C_1) = -\log R_b < 0$, $\sigma_b(z \in C_b) = -\log(R_b/R_b) = 0$).

Since C_a is contained in the closure of D_b , Eqs. (A.11) and (A.12) imply

$$\sigma_a(z) \geq \sigma_b(z) \quad , \quad z \in C_a \quad (A.13)$$

and, together with (A.10) and the harmonicity of $\sigma_a(z) - \sigma_b(z)$ in $D_a - C_1$:

$$\sigma_a(z) \geq \sigma_b(z) \quad , \quad z \in D_a - C_1 \quad (A.14)$$

This proves the inequality (A.5), and establishes the optimality of the conformal mapping $W(z)$ defined above. The effective construction of this mapping is the purpose of the forthcoming appendices.

APPENDIX B

Mapping the axiomatic domain

For fixed $s > 4$, the analyticity domain obtained by Martin³⁾ is essentially the union of two pieces^{*)}:

a) the "small" ellipse:

$$\frac{(\operatorname{Re} z)^2}{a^2(s)} + \frac{(\operatorname{Im} z)^2}{b^2(s)} < 1 \quad (z = \cos \theta) \quad (\text{B.1})$$

where $a(s)$ and $b(s)$ have been defined in Eq. (4');

b) the symmetrized image, through $z = \pm[1 + 2t/(s-4)]$, of the (s independent) "potato" made in the t -plane by the intersection

$$\bigcap_{s>4} E[0, 4-s | r(s)] \quad (\text{B.2})$$

Here, $E[0, 4-s | r(s)]$ stands for the large Lehmann-Martin ellipse, i.e., the ellipse with foci at $t=0$ and $4-s$ and with right extremity at $t=r(s)$:

$$r(s) = \begin{cases} \frac{16s}{s-4} & , \quad 4 < s < 16 \\ \frac{256}{3} & , \quad 16 \leq s < 32 \\ \frac{4s}{s-16} & , \quad s \geq 32 \end{cases} \quad (\text{B.3})$$

For $s < 28$, the t - and u -cuts $[z \geq (s+4)/(s-4)]$ and $[z \leq -(s+4)/(s-4)]$ enter the region b) and have to be taken out (Fig. 2). The problem is to find the function $w_A(z)$ which maps conformally this domain D_A onto a unifocal ellipse, the image of the basis $C_1 = [z_0, z_1]$ being the segment $[-1, 1]$ {if C_1 is chosen according to Eq. (24), $z_0 = \operatorname{Max}[-1, 1 + 2 t_m / (s-4)]$ and $z_1 = 1$ }. This will be done in two steps: i) construct the function $V(z)$ which maps D_A onto the unit disk $|V| < 1$ [a further prescription is needed to define uniquely this mapping, e.g., $V(z_0) = 0$, $V'(z_0) \geq 0$], ii) map this unit disk onto a unifocal ellipse in such a way that $V(C_1)$ be sent onto $[-1, 1]$.

*) We ignore some further enlargements of minor importance here.

Step i)

Since the boundary C_A of D_A cannot be expressed in a simple analytical form, we shall appeal to a constructive method ¹⁴⁾, suitable for computational purposes. The idea is to relate the function $V(z)$ to some integral kernel (the Bergman reproducing kernel) which can be written in terms of its eigenfunctions. Then these functions can easily be computed by means of a Schmidt orthogonalization process. It turns out, however, that the procedure applies only if D_A is the complement of the closure of a domain ¹⁴⁾. Clearly, this is not the case for the slit domain D_A of Fig. 2a). Thus, for $s < 28$, one has first to "open" the two cuts, e.g., by means of the auxiliary mapping $z \rightarrow \tilde{z} = \text{Arc cos} [z(s-4)/(s+4)]$. In the following, we shall assume that this has been done when necessary, so that z may represent the variable $\tilde{z}$ as well.

Now, the Bergman kernel $K(z, z')$ for a (bounded, simply connected) domain D is defined by its reproducing property:

$$f(z) = \iint_D dx' dy' K^*(z, z') f(z') \quad (z' = x' + iy') \quad , \quad (B.4)$$

where $f(z)$ is any square integrable, holomorphic function in D [$f \in L^2(D)$]. In order to relate the mapping $V(z)$ to this kernel, one needs the Green formula for the product $f(z)g^*(z)$, where both f and g are holomorphic in D :

$$\iint_D dx dy f(z) g'^*(z) = \frac{1}{2i} \int_{C=\partial D} dz f(z) g^*(z) \quad . \quad (B.5)$$

[Proof: using the definitions $d/dz = \frac{1}{2}(\partial/\partial x - i\partial/\partial y)$, $d/dz^* = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$, one has:

$$\frac{df}{dz^*} = 0 \quad , \quad g'^* \equiv \left(\frac{dg}{dz} \right)^* = \frac{dg^*}{dz^*} \quad .$$

Hence:

$$\begin{aligned} 2i \iint_D dx dy f g'^* &= 2i \iint_D dx dy \frac{d}{dz^*} (f g^*) = \iint_D dx dy \left[\frac{\partial}{\partial x} (i f g^*) - \frac{\partial}{\partial y} (f g^*) \right] \\ &= \int_C (f g^* dx + i f g^* dy) = \int_C dz f g^* \quad] \end{aligned}$$

Let us take $g(z) = V(z)$ in formula (B.5):

$$\begin{aligned} \iint_D dx dy f(z) V'^*(z) &= \frac{1}{2i} \int_C dz f(z) V'^*(z) = \frac{1}{2i} \int_C dz \frac{f(z)}{V(z)} \\ &= \pi \frac{f(z_0)}{V'(z_0)}. \end{aligned} \quad (B.6)$$

In the second equality (B.6), we have used the property $|V(z)| = 1$; the third one follows from the residue theorem and the fact that $V(z)$ has only one zero in D (arbitrarily chosen at $z = z_0$), since this function is a conformal mapping. Thus, z_0 being real, we can write:

$$f(z_0) = \frac{1}{\pi} \iint_D dx dy V'^*(z_0) V'^*(z) f(z), \quad (B.7)$$

and by comparing with Eq. (B.4):

$$K(z_0, z) = \frac{1}{\pi} V'(z_0) V'(z). \quad (B.8)$$

Taking $z = z_0$ in this equation, one obtains $V'(z_0) = \sqrt{\pi K(z_0, z_0)}$ and, finally, the desired relation^{*)}:

$$V(z) = \sqrt{\frac{\pi}{K(z_0, z_0)}} \int_{z_0}^z dz' K(z_0, z'). \quad (B.9)$$

The practical interest of formula (B.9) is that the Bergman kernel $K(z, z')$ can be expressed in the form:

$$K(z, z') = \sum_{n=1}^{\infty} \varphi_n^*(z) \varphi_n(z') \quad (B.10)$$

where $\{\varphi_n(z)\}$ is any complete orthonormal set of holomorphic functions in D :

$$\iint_D dx dy \varphi_n(z) \varphi_m^*(z) = \delta_{nm}. \quad (B.11)$$

*) In this proof, we have skipped the technicalities required to treat properly the convergence problems in the $L(D)$ norm. They can be found in Ref. 14).

Actually, all holomorphic functions in $L^2(D)$ can be expanded as:

$$f(z) = \sum_{n=0}^{\infty} f_n \varphi_n(z) \quad \text{with} \quad f_n = \iint_D dx dy f(z) \varphi_n^*(z) \quad (\text{B.12})$$

and the reproducing property of the expression (B.10) is obvious.

In order to compute the $\varphi_n(z)$, one can use the Schmidt orthogonalization process. Namely, one starts with an arbitrary, complete set of functions in $L^2(D)$ (e.g., the powers z^n , $n=0, 1, \dots$) and produce successively orthogonal combinations. Let us mention that formula (B.5) is also here of computational help, because all integrals are reduced to contour integrals. For instance, if one decides to work with the powers of z , one has:

$$I_{nm} \equiv \iint_D dx dy z^n z^{m*} = \frac{1}{2i(m+1)} \int_C dz z^n (z^*)^{m+1}. \quad (\text{B.13})$$

Step ii)

At this stage, $V(z)$ maps the axiomatic domain D_A onto the unit disk $|V| < 1$ and transforms the basis $[z_0, z_1]$ into the segment:

$$0 \leq V \leq k_A, \quad k_A = V(z_1). \quad (\text{B.14})$$

To complete the conformal mapping of D_A onto the appropriate ellipse, it is convenient to introduce the intermediate mapping:

$$\zeta_A(V) = - \frac{1 - \frac{2V}{k_A} + V^2}{1 - 2k_A V + V^2} \quad (\text{B.15})$$

which transforms the unit disk into the complex ζ_A -plane cut along $[-\infty, -1/k_A]$ and $[1/k_A, \infty]$, the basis (B.14) becoming $-1 \leq \zeta_A \leq 1$. In this way we recover a cut plane analyticity of the Mandelstam type, and we can rely on the calculations of Appendix C: the variable ζ_A here plays the same role as the variable ζ_M there, and the Eqs. (C.5) - (C.9) provide us with the proper mapping of the cut ζ_A plane onto the unifocal ellipse. Thus:

$$w_A(z) = \sin \left[\frac{\pi}{2K(k_A)} F(\zeta_A(V(z)), k_A) \right] \quad (\text{B.16})$$

where $K(k_A)$ and $F(\zeta_A, k_A)$ are the elliptic integrals defined in Eqs. (C.6) and (C.5).

APPENDIX C

Mapping the Mandelstam domain

If one assumes the full Mandelstam analyticity, the somewhat lengthy computations required to deal with the axiomatic domain can be avoided, for the corresponding optimal mapping $w_M(z)$ can be written entirely in terms of known (and easily computable!) functions.

The Mandelstam analyticity domain D_M is the z plane cut along $z \geq c$ and $z \leq -c$, with:

$$c = \frac{s+4}{s-4} \quad (C.1)$$

The basis C_1 , if chosen according to Eq. (24), is the segment $[z_0, 1]$, where

$$z_0 = \begin{cases} -1 & , \quad 4 < s \leq 36 \\ \frac{s-68}{s-4} & , \quad s > 36 \end{cases} \quad (C.2)$$

We first want to change variable in such a way that C_1 become $[-1, 1]$ with the two cuts remaining in symmetric positions. This is done by the homographic transformation:

$$\xi_M(z) = \frac{1}{k_M} \frac{(c-k_M)z + c(k_M c - 1)}{(c-k_M)c + z(k_M c - 1)} \quad (C.3)$$

where

$$\frac{1}{k_M} = \frac{c^2 - z_0 + \sqrt{(c^2 - z_0^2)(c^2 - 1)}}{c(1 - z_0)} \quad (C.4)$$

Then D_M becomes the ζ_M plane cut along $\zeta_M \geq 1/k_M$ and $\zeta_M \leq -1/k_M$ (notice that $1/k_M = c$ and $\zeta_M \equiv z$ for $s \leq 36$).

Consider now the incomplete elliptic integral of the first kind with modulus k :

$$F(\xi, k) \equiv \int_0^\xi \frac{dx}{\sqrt{(1-x^2)(1-k^2 x^2)}} \quad (0 < k < 1) \quad (C.5)$$

Just from this definition, it is easy to check that the mapping $\zeta \rightarrow F$

sends the upper half ζ plane onto the rectangle with vertices $(-K, K, K+iK', -K+iK')$, where K is the complete elliptic integral of the first kind:

$$K(k) \equiv F(1, k) \quad (C.6)$$

and

$$K'(k) \equiv \int_1^{1/k} \frac{dx}{\sqrt{(x^2-1)(1-k^2x^2)}} \quad (C.7)$$

Thus, the function $F(\zeta_M, k_M)$ maps the upper half ζ_M -plane onto such a rectangle, whose lower side $[-K(k_M), K(k_M)]$ is the image of the basis $-1 \leq \zeta_M \leq 1$ and whose upper side $[K(k_M)+iK'(k_M), -K(k_M)+iK'(k_M)]$ is the image of the upper lip of the cuts $\zeta_M \geq 1/k_M$ and $\zeta_M \leq -1/k_M$.

Finally, the transformation

$$w(F) = \sin \frac{\pi F}{2 K(k_M)} \quad (C.8)$$

brings this rectangle onto the upper half of the ellipse with foci $w = \pm 1$ and right extremity $w = \cosh[\pi K'(k_M)/2 K(k_M)]$. The upper side of the rectangle (the "cut") is sent onto the upper boundary of this ellipse and the lower side (the "basis") onto $-1 \leq w \leq 1$, as required. Moreover, by using the reflection principle, one immediately verifies that the mapping $\zeta_M \rightarrow w(F(\zeta_M))$ extends to the lower half ζ_M plane, which is sent onto the lower half of the ellipse.

Therefore, the conformal mapping $w_M(z)$ which transforms D_M into a unifocal ellipse and C_1 into $[-1, 1]$ is found to be:

$$w_M(z) = \sin \left[\frac{\pi}{2 K(k_M)} F(\zeta_M(z), k_M) \right] \quad (C.9)$$

APPENDIX D

Mapping the Lehmann-Martin ellipses

If one wants to stay within the axiomatic framework, a way to escape the extensive computation of the optimal mapping described in Appendix B is to restrict oneself to a part of the axiomatic domain: the small Lehmann-Martin ellipse (B.1). Then the corresponding mapping $w_L(z)$ has again a closed analytical form, but of course the full optimality is lost [at high energy at least, this should be harmless, as the enlargement due to the adjunction of the subdomain b) of Appendix B is rather poor (see Fig. 2)].

In this case, $w_L(z)$ has to map the ellipse (B.1) onto another unifocal ellipse in such a way that the (asymmetric) basis $C_1 = [z_0, 1]$ be sent onto $-1 \leq w_L \leq 1$. According to Eq. (24):

$$z_0 = \begin{cases} -1 & , \quad 4 < s \leq 32 \\ \frac{s-60}{s-4} & , \quad s > 32 \end{cases} \quad (D.1)$$

For $s \leq 32$, C_1 is already in the right position and the mapping is trivial: $w_L(z) \equiv z$ [this simply means that the functions $g_n(s)$ coincide with the partial waves $f_n(s)$ in this energy range]. For $s > 32$, the Lehmann-Martin ellipse D_L is the unifocal ellipse with right extremity [see Eq. (4')]:

$$a = \frac{s-8}{\sqrt{(s-4)(s-16)}} \quad (D.2)$$

We shall construct $w_L(z)$ by the following chain of simple mappings:

$$i) \quad \xi = \text{Arc sin } z \quad (D.3)$$

The upper half of D_M is brought onto the rectangle Γ with vertices $(-\pi/2, \pi/2, \pi/2+i\alpha, -\pi/2+i\alpha)$, with $\alpha = \log(a + \sqrt{a^2 - 1})$. The image of C_1 is the segment $[\text{Arc sin } z_0, 1]$.

$$ii) \quad \eta = \text{sn} \left(\frac{2}{\pi} K(k) \xi, k \right) \quad (D.4)$$

where

$$\begin{cases} k = 4\sqrt{q} \prod_{n=1}^{\infty} \left(\frac{1+q^{2n}}{1+q^{2n-1}} \right)^4 \\ q = (a - \sqrt{a^2 - 1})^2 \end{cases} \quad (D.5)$$

and $\text{sn}(\cdot, k)$, the Jacobian elliptic sine, is the inverse of the function $F(\cdot, k)$ introduced in Eq. (C.5).

Then, according to the properties of $F(\cdot, k)$ discussed in the preceding Appendix, the mapping (D.4) clearly transforms the rectangle Γ into the upper half η -plane (the image of the upper side being $\eta \geq 1/k$, $\eta \leq -1/k$), provided that k is defined by $K'(k)/K(k) = 2\phi/\pi$. Eq. (D.5) is nothing but the solution of this last equation, when one introduces the nome $q \equiv \exp [-\pi K'(k)/K(k)]$, which is known to be related to the modulus k through the first equation (D.5) ¹⁵⁾.

Using the reflection principle, we see that $\eta \circ \zeta$ maps the whole of D_M onto the η plane cut along $[1/k, \infty]$ and $[-\infty, -1/k]$. The image of C_1 is $[\eta_0, 1]$ with:

$$\eta_0 = \text{sn} \left(\frac{2}{\pi} K(k) \text{Arc sin } z_0, k \right). \quad (\text{D.6})$$

$$\text{iii)} \quad \xi_L = \frac{1}{k_L} \frac{(1 - k k_L) k \eta + (k_L - k)}{(1 - k k_L) + k \eta (k_L - k)} \quad (\text{D.7})$$

where

$$\frac{1}{k_L} = \frac{1 - k^2 \eta_0 + \sqrt{(1 - k^2)(1 - k^2 \eta_0^2)}}{k (1 - \eta_0)}. \quad (\text{D.8})$$

This symmetrization mapping, analogous to the one given in Eq. (C.3), maps the cut η -plane onto the ζ_L -plane cut along $(\zeta_L \geq 1/k_L, \zeta_L \leq -1/k_L)$ and the basis onto $-1 \leq \zeta_L \leq 1$.

$$\text{iv)} \quad w_L = \sin \left[\frac{\pi}{2 K(k_L)} F(\zeta_L, k_L) \right]. \quad (\text{D.9})$$

It is the final, by now familiar, mapping already used in the two previous appendices.

The Eqs. (D.3) to (D.9) completely determine the function $w_L(z)$.

APPENDIX E

On the analytic continuation of unitarity

The well-defined prescription to be used in the unitarity Equation (10), namely to integrate over the ellipse $K^2 \geq 0$, loses its meaning when one has to compute the absorptive part of the amplitude for unphysical scattering angles θ . Indeed, when $\cos \theta$ goes to ± 1 , the ellipse shrinks up to a segment and disappears. Of course, the unitarity equation can be properly extended by analytic continuation. We shall follow the elegant method of Martin¹⁶⁾.

Omitting the energy dependence and the inelastic contribution which plays no role here (the analytic continuation of the expression (13) is trivial), we start from physical unitarity written in the form:

$$A(\vec{n}_1, \vec{n}_2) = \frac{1}{4\pi} \int d^2\vec{n}_3 F(\vec{n}_1, \vec{n}_3) F^*(\vec{n}_3, \vec{n}_2) \quad (\text{E.1})$$

where the $\vec{n}_i$'s are unit vectors and the $\vec{n}_i \cdot \vec{n}_j$'s cosines of scattering angles. Let (φ, χ) be the polar coordinates of $\vec{n}_3$ in a frame where the (x, z) plane contains the vectors $\vec{n}_1$ and $\vec{n}_2$ and the z axis is along the bisector of the angle $(\vec{n}_1, \vec{n}_2) = 2\alpha$ (Fig. 3). Then, Eq. (E.1) can be rewritten as

$$A(2c^2 - 1) = \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_{-1}^1 d\cos\chi F(c \cos\chi + \sqrt{1-c^2} \cos\varphi \sin\chi) \cdot F^*(c^* \cos\chi - \sqrt{1-c^{*2}} \cos\varphi \sin\chi) \quad (\text{E.2})$$

where $c = \cos \alpha$ (so far, $-1 \leq c \leq 1$ and $c^* = c$). For fixed (φ, χ) , the integrand in the r.h.s. of Eq. (E.2) considered as an analytic function of c , has square root singularities at $c = \pm 1$. However, by making the change of variable $\varphi \rightarrow \varphi + \pi$, it is readily seen that these singularities cancel out in the integration. As a result, Eq. (E.2) can be continued to unphysical values of c , provided that the argument of the amplitude F remains inside its analyticity domain. Thus, this equation allows us to recover the well-known connection between the analyticity domains of A and F : given $2c^2 - 1$ on the boundary of some "large" ellipse, the argument of F for varying (φ, χ) sweeps the corresponding "small" ellipse. In particular, for real c , $|c| > 1$, Eq. (E.2) becomes:

$$A(z) = \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_{-1}^1 d\cos\chi \left| F\left(\sqrt{\frac{z+1}{2}} \cos\chi + i\sqrt{\frac{z-1}{2}} \cos\varphi \sin\chi\right) \right|^2 \quad (\text{E.3})$$

with $z = 2c^2 - 1 > 1$.

By coming back to the variable t and inserting the correct kinematical factors, it is a straightforward matter to derive from Eq. (E.3) the modified Eqs. (16), (17') and (11'), valid for (real) unphysical values of t .

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FIGURE CAPTIONS

Fig. 1: A possible choice of bases (hatched lines) for the mappings w with:
a) the variable $z = \cos \theta$ (or t), b) the variable $v = (t - u)^2$.

Fig. 2: Shape of the axiomatic analyticity domains D_A in the z -plane
(shaded areas) for: a) "low" energies ($s < 28$), b) high energies.
The dotted lines are the small Lehmann-Martin ellipses.

Fig. 3: Definition of the variables (φ, χ) used in the unitarity equation.

a)

b)

Fig. 2

Fig. 3