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ANOMALOUS CURRENTS IN CURVED SPACE

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A B S T R A C T

We present a detailed discussion concerning the anomalies of the Euler and Hirzenbruch currents, as well as the spinor axial-vector current in a given background metric.

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1. INTRODUCTION

Axial-current anomalies have been known for a long time in gauge theories with fermions¹⁻³⁾. According to naive canonical arguments, the axial-vector current is expected to be conserved when the fermion mass vanishes. However, the current does not comply with the naive picture, and it is impossible to redefine a gauge-invariant axial-vector current for which the anomalous term is absorbed somewhere else. In the presence of a background gravitational field, a gravitational contribution to the anomaly has also been found⁴⁾.

It has, however, been realized only very recently that these anomalies are deeply related to the asymmetry between the number of zero modes of the affiliated Euclidean Dirac equation with positive and negative chirality in the presence of a gauge field and/or a background metric with non-vanishing curvature⁵⁻⁷⁾. The Atiyah-Singer theorem⁸⁾, in turn, relates the "skewness" of the zero mode structure to the topological properties of the underlying classical Euclidean problem by expressing it in terms of characteristic classes, at least for Euclidean space time compact without boundary [the case of more general space time is dealt with elsewhere⁹⁾, solving a problem raised by Hawking¹⁰⁾]. The current anomaly is then twice the index density.

In this work we shall, in addition to a treatment of the Dirac equation on an arbitrary compact orientable Riemannian manifold M also use the Hodge calculus¹¹⁾ to construct a new Lagrangian, which is a function of all differential forms on M , i.e. of all antisymmetric tensor fields on M . The corresponding equations of motion are Hodge's equations, a linear system of first-order differential equations for the components of the tensor fields. This Lagrangian leads to two new anomalous currents, which we call the Euler and Hirzebruch currents, because their anomalies are related to the Euler characteristic and the Hirzebruch L genus, respectively. Currents whose anomalies correspond to other topological invariants as, for example, the generalized $\hat{A}$ genus can also be constructed. As an example the axial-current anomaly of supergravity will be treated in a forthcoming publication.

In Section 2 the construction of the Lagrangian density and of the Euler and Hirzebruch currents, as well as their relationship with the Euler characteristic and the L genus, is explained. In Section 3 we present the covariant point-separation procedure¹²⁾ for the definition of all three currents and use it for the evaluation of the anomalies. The technical details are relegated to three Appendices. Appendix A deals with the short-distance behaviour of Green's functions; in Appendix B the bivectors and bispinors are treated; and in Appendix C some important algebraic identities are listed.

Finally, in Appendix D we show how the Hodge operator can in analogy to the Dirac operator be expressed in terms of covariant derivatives and Duffin-Kemmer-like Γ matrices.

The principal results of this work have already been outlined in a shorter communication⁷⁾.

2. EULER AND HIRZEBRUCH CURRENTS

The general concepts necessary for the construction of the Euler current and the Hirzebruch current will be given in this section. For a general review of the subject see Ref. 11.

Let M be a manifold of dimension n . The space of complex-valued differential forms of degree p is denoted by $\Omega^p(M)$. For the space of all differential forms we write

$$\Omega(M) = \bigoplus_{p=0}^n \Omega^p(M)$$

On the differential forms we have an exterior multiplication

$$\wedge: \Omega^p(M) \times \Omega^q(M) \rightarrow \Omega^{p+q}(M)$$

and an exterior derivative

$$d: \Omega^p(M) \rightarrow \Omega^{p+1}(M)$$

In terms of local coordinates a p form looks like

$$\omega = \frac{1}{p!} f_{\mu_1 \dots \mu_p}(x) dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}, \quad (2.1)$$

where $f_{\mu_1 \dots \mu_p}$ is antisymmetric in its indices, and one has

$$d\omega = \frac{1}{(p+1)!} \sum_{k=1}^{p+1} (-1)^{k+1} \frac{\partial}{\partial x^{\mu_k}} f_{\mu_1 \dots \mu_{k-1} \mu_{k+1} \dots \mu_{p+1}}(x) dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{p+1}} \quad (2.2)$$

On orientable Riemannian manifolds one can define the familiar duality isomorphism

$$*: \Omega^p(M) \rightarrow \Omega^{n-p}(M)$$

by

$$* \omega = \frac{1}{(n-p)!} \frac{1}{p!} g^{-\frac{1}{2}} \varepsilon^{\mu_1 \dots \mu_p \nu_1 \dots \nu_{n-p}} f_{\mu_1 \dots \mu_p}(x) dx_{\nu_1} \wedge \dots \wedge dx_{\nu_{n-p}} \quad (2.3)$$

(with g the determinant of the metric tensor $g_{\mu\nu}$). The duality operation allows the introduction of a symmetric bilinear pairing of differential forms

$$\langle \cdot, \cdot \rangle : \Omega^p(M) \times \Omega^p(M) \rightarrow \Omega^n(M)$$

by

$$\langle \alpha, \beta \rangle = \alpha \wedge \beta \quad (2.4)$$

Defining

$$\langle \alpha, \beta \rangle = 0 \text{ for degree } \alpha \neq \text{degree } \beta \quad (2.5)$$

we get by linear extension a bilinear pairing

$$\langle \cdot, \cdot \rangle : \Omega^p(M) \times \Omega^p(M) \rightarrow \Omega^n(M)$$

If M is compact, as will be assumed in the sequel, this gives rise to a symmetric positive definite bilinear form on $\Omega(M)$, defined by

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle \quad (2.6)$$

For

$$\alpha = \frac{1}{p!} f_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}, \beta = \frac{1}{p!} g_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$$

one has, for example

$$(\alpha, \beta) = \frac{1}{p!} \int_M f_{\mu_1 \dots \mu_p} g^{\mu_1 \dots \mu_p} \sqrt{g} d^n x \quad (2.7)$$

We denote the adjoint of the exterior derivative d by δ . So

$$\delta : \Omega^p(M) \rightarrow \Omega^{p-1}(M)$$

One easily computes

$$\delta \omega = (-1)^{n-p+n+1} * d * \omega \text{ for } \omega \in \Omega^p(M) \quad (2.8)$$

which in terms of local coordinates is

$$\delta \omega = - \frac{1}{(p-1)!} \frac{1}{\sqrt{g}} \partial_\lambda \sqrt{g} f^{\lambda \mu_1 \dots \mu_{p-1}}(x) dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{p-1}} \quad (2.9)$$

Thus δ acts like a generalized divergence operator. The Hodge differential operator

$$D = d + \delta : \Omega(M) \rightarrow \Omega(M)$$

is evidently symmetric with respect to the scalar product defined above, and it turns out to be elliptic¹¹⁾.

Now we define an operator $\Gamma_S^E : \Omega(M) \rightarrow \Omega(M)$ by

$$\Gamma_S^E \omega = (-1)^p \omega \text{ for } \omega \in \Omega^p(M) \quad (2.10)$$

Evidently $(\Gamma_S^E)^2 = 1$, and the eigenspaces of Γ_S^E with eigenvalues $+1$ and -1 are the spaces $\Omega^e(M) \subset \Omega(M)$ and $\Omega^o(M) \subset \Omega(M)$, which are generated by the differential forms of even and odd degree, respectively. Γ_S is symmetric with respect to the form $(.,.)$. Because of

$$D \Gamma_S^E + \Gamma_S^E D = 0$$

one can define the restriction D_E of D to $\Omega^e(M)$:

$$D_E : \Omega^e(M) \rightarrow \Omega^o(M)$$

which is also elliptic but no longer symmetric. Rather the adjoint D_E^+ is an elliptic operator

$$D_E^+ : \Omega^o(M) \rightarrow \Omega^e(M)$$

The index of the operator D_E is defined by

$$\text{ind } D_E = \dim \text{kernel } D_E - \dim \text{kernel } D_E^+ \quad (2.11)$$

It is equal to the Euler characteristic of M .

This follows from the definition of the Euler characteristic $\chi(M)$

$$\chi(M) = \sum_q (-1)^q b_q \quad (2.12)$$

where b_q is the q^{th} Betti number, which is the dimension of the q^{th} cohomology group $H^q(M)$

$$H^q(M) = Z^q(M)/B^q(M)$$

Here $Z^q(M)$ is the set of closed q forms and $B^q(M)$ is the set of exact q forms. Hodge's theorem⁽¹¹⁾ states that

$$b_q = \dim \mathcal{H}^q(M) \quad (2.13)$$

where $\mathcal{H}^q(M)$ denotes the space of harmonic differential q forms (a differential form ω is called harmonic if $D\omega = 0$). Inserting (2.13) into (2.12) one is clearly led to

$$\chi(M) = \text{ind } D_E \quad (2.14)$$

Analogously, one constructs an operator D_H , the index of which is the signature of the manifold M , to be introduced below. Let $\dim M = 4\ell$ be a multiple of 4. One verifies that the operator $\Gamma_5^H : \Omega(M) \rightarrow \Omega(M)$

$$\Gamma_5^H \omega = (-1)^{\frac{n(n+1)}{2} + \ell} * \omega \quad (\omega \in \Omega^r(M)) \quad (2.15)$$

fulfils

$$(\Gamma_5^H)^2 = 1; \quad D\Gamma_5^H + \Gamma_5^H D = 0$$

and Γ_5^H is symmetric. Denoting the eigenspaces of Γ_5^H belonging to eigenvalues ± 1 by $\Omega^\pm(M)$ [$\Omega^+(M)$ and $\Omega^-(M)$ are orthogonal], one can again define by restriction elliptic operators:

$$D_H : \Omega^+(M) \rightarrow \Omega^-(M); \quad D_H^+ : \Omega^-(M) \rightarrow \Omega^+(M)$$

The index of D_H is given by

$$\text{ind } D_H = \dim \text{kernel } D_H - \dim \text{kernel } D_H^+ \quad (2.16)$$

This expression can be simplified by the observation that any harmonic form $\omega \in \mathcal{H}^p(M)$ leads to the two forms

$$\eta = \frac{1}{\sqrt{2}} (\omega + \Gamma_5^H \omega) \in \text{kernel } D_H \subset \Omega^+(M)$$

and

$$\tilde{\eta} = \frac{1}{\sqrt{2}} (\omega - \Gamma_5^H \omega) \in \text{kernel } D_H^+ \subset \Omega^-(M)$$

For $p \neq 2\ell$ this means that the spaces $\text{kernel } D_H \cap \Omega^p(M)$ and $\text{kernel } D_H^+ \cap \Omega^p(M)$ are isomorphic. Hence $\text{ind } D_H$ only gets contributions from 2ℓ forms. On $\Omega^{2\ell}(M)$ we have

$$\Gamma_5^H = * \quad (2.17)$$

The signature of M $\text{sign } M$ is introduced by means of the symmetric bilinear form $\tilde{P}$ on $\Omega^{2\ell}(M)$ defined by

$$P = \int_M \omega_1 \wedge \omega_2 \quad (2.18)$$

For closed ω_1 and ω_2 the value of $\tilde{P}$ is unchanged if one replaces $\omega_1 \rightarrow \omega_1 + d\eta_1$, $\omega_2 \rightarrow \omega_2 + d\eta_2$. Thus $\tilde{P}$ induces a symmetric bilinear form P on $H^{2\ell}(M)$, which by Poincaré's duality theorem¹¹⁾ is also non-degenerate. The signature of this form P (number of positive minus number of negative diagonal elements of P after diagonalization) is $\text{sign } M$.

Now let $(\omega_1, \dots, \omega_r, \omega_{r+1}, \dots, \omega_{r+s})$ be a basis of $\mathcal{H}^{2\ell}(M)$ such that

$$\Gamma_5^H \omega_i = * \omega_i = \begin{cases} \omega_i & \text{for } 1 \leq i \leq r \\ -\omega_i & \text{for } r+1 \leq i \leq r+s \end{cases} \quad (2.19)$$

We can assume $(\omega_i, \omega_k) = 0$ for $i \neq k$. Furthermore,

$$\int_M \omega_i \wedge \omega_i = (\omega_i, * \omega_i) = \begin{cases} > 0 & \text{for } 1 \leq i \leq r \\ < 0 & \text{for } r+1 \leq i \leq r+s \end{cases} \quad (2.20)$$

Together with Hodge's theorem¹¹⁾ this means that

$$\text{ind } D_H = \text{sign } M \quad (2.21)$$

It is known from the Atiyah-Singer theorem^{8,11)} that the index of an elliptic operator is expressible by the Pontrjagin class and the Euler class of the underlying manifold and the Chern classes of the bundle under consideration; these are expressible by polynomials in the curvature forms derived from any connection on the bundle and a Riemannian connection on the tangent bundle of M . For instance, for $\dim M = 4$:

$$\text{ind } D_E = \frac{1}{128\pi^2} \int d^4x \frac{1}{\sqrt{g(x)}} \epsilon^{\tau\lambda\lambda\theta} \epsilon^{\mu\nu\zeta\eta} R_{\tau\lambda\mu\nu} R_{\lambda\theta\zeta\eta}(x) \quad (2.22)$$

$$\text{ind } D_H = \frac{1}{96\pi^2} \int d^4x \varepsilon^{\mu\nu\lambda\rho} R_{\mu\nu}{}^{\xi\eta} R_{\lambda\rho}{}^{\xi\eta}(x) \quad (2.23)$$

Our next task is to express the integrands on the right-hand sides of Eqs. (2.22) and (2.23) as anomalies of appropriately defined currents.

For $\dim M = n$ the quantity $\omega \in \Omega(M)$ is a 2^n component field and $\Gamma_5^{E,H}$ are real $2^n \times 2^n$ matrices. Admitting now the forms ω to be complex valued we can introduce a Lagrangian

$$\mathcal{L} = \langle \bar{\omega}, D\omega \rangle \quad (2.24)$$

leading to the field equations

$$D\omega = D\bar{\omega} = 0 \quad (2.25)$$

These equations can also be expressed by matrices of the Duffin-Kemmer type⁷⁾ (see Appendix D). The Lagrangian $\mathcal{L}$ is invariant under the transformations

$$\omega \rightarrow e^{\alpha \Gamma_5^{E,H}} \omega \quad (\alpha \in \mathbb{R}) \quad (2.26)$$

Hence we are led to two formally conserved Noether currents, the Euler current

$$j_\mu^E = -g^{\frac{1}{2}} \sum_{p=0}^{n-1} \frac{(-1)^p}{p!} \left(\bar{f}_{\mu\mu_1\cdots\mu_p} f^{\mu_1\cdots\mu_p} + \bar{f}_{\mu_1\cdots\mu_p} f_\mu^{\mu_1\cdots\mu_p} \right) \quad (2.27)$$

and the Hirzebruch current

$$j_\mu^H = -g^{\frac{1}{2}} \sum_{p=0}^{n-1} \frac{(-1)^{\frac{p(p+1)}{2}}}{p!} \left(\bar{f}_{\mu\mu_1\cdots\mu_p} * f^{\mu_1\cdots\mu_p} + (-1)^p \bar{f}_{\mu_1\cdots\mu_p} * f_\mu^{\mu_1\cdots\mu_p} \right) \quad (2.28)$$

It is advantageous to reduce the number of field components occurring in the currents wherever possible by making the transition to dual fields; this leads, for $\dim M = 4$, to the expressions

$$j_\mu^E = g^{\frac{1}{2}} (-\bar{f}_\mu f - \bar{f} f_\mu + \bar{f}_{\mu\nu} f^\nu + \bar{f}^\nu f_{\mu\nu} - * \bar{f}^\nu * f_{\mu\nu} - * \bar{f}_{\mu\nu} * f^\nu - * \bar{f} * f_\mu - * \bar{f}_\mu * f) \quad (2.29)$$

$$j_\mu^H = g^{\frac{1}{2}} (\bar{f}_\mu * f + \bar{f} * f_\mu - \bar{f}_{\mu\nu} * f^\nu + \bar{f}^\nu * f_{\mu\nu} - * \bar{f}^\nu * f_{\mu\nu} + * \bar{f}_{\mu\nu} * f^\nu + * \bar{f} f_\mu + * \bar{f}_\mu f) \quad (2.30)$$

When the products of fields occurring in Eqs. (2.27) to (2.30) are replaced by the corresponding Green's functions of the Hodge operator D , a regularization has to be introduced. In the divergence equations for $j_\mu^{E,H}$ anomalies will then occur which are twice the corresponding index densities. This will be shown explicitly for $\dim M = 4$ in the next section.

3. CURRENT ANOMALIES

3.1 Spinor axial-vector current

The first thing to be demonstrated in this section is that the index theorem for the Dirac operator in an arbitrary Riemannian manifold mentioned in Ref. 6 can be related to the axial-vector current anomaly of Ref. 4.

The spinor axial-vector current, constructed according to the covariant point-separation method¹²⁾, is found from the quantity

$$j_\mu^5(x, x') = i g^{\frac{1}{2}}(x) \text{tr } \gamma_\mu(x) \gamma_5 G(x, x') I(x', x) \quad (3.1)$$

by letting $x' \rightarrow x$. Here $G(x, x')$ is the spinor Green's function defined by Eq. (A.1), and $I(x', x)$ is the bispinor function¹³⁾. *A priori* one expects both singular and directional-dependent terms to arise for $x' \rightarrow x$ in (3.1): such terms must then be subtracted.

A differentiation of (3.1) is performed by subjecting both x' and x to an infinitesimal variation and connecting the variations at x' and x by means of the bivector function¹³⁾ $g_{\nu\mu}(x', x)$. In this way we get from (3.1)

$$\begin{aligned} \partial^\mu j_\mu^5(x, x') &\simeq i g^{\frac{1}{2}}(x) \text{tr } \gamma_\mu(x) \gamma_5 G(x, x') (I_{;\mu}^\mu(x', x) + g^{\nu\mu}(x', x) I_{;\nu}^\mu(x', x)) \\ &\quad + 2 g^{\frac{1}{2}}(x) \sum_k \psi_{0k}^+(x) \gamma_5 \psi_{0k}(x) \end{aligned} \quad (3.2)$$

In deriving Eq. (3.2) we used (A.3) and the equation for parallel transport of γ -matrices

$$g^{\nu\mu}(x', x) I(x', x) \gamma^\mu(x) = \gamma^{\nu'}(x') I(x', x) \quad (3.3)$$

The sum over zero modes in Eq. (3.2) is clearly twice the index density; the remaining task is the evaluation of the terms involving the Green's function at short distances.

This can be carried out by means of the formulae listed in the Appendices. If the Green's function were regular at short distances it would mean that the

first term on the right-hand side of (3.2) would vanish for $x' \rightarrow x$. The singularity structure displayed in Eq. (A.10) makes a non-vanishing result possible. In this particular case only the leading singularity of the Green's function contributes, and no infinities arise in (3.2) owing to the limiting procedure. The result, obtained by Eqs. (A.10), (B.1) and (B.2), is

$$i g^{\frac{1}{2}}(x) \text{tr} \gamma_{\mu}(x) \gamma_5 G(x, x') [I;^{\mu}(x', x) + g^{\nu\mu}(x', x) I;^{\nu}(x', x)] \simeq \\ - \frac{1}{96\pi^2 6^2} \varepsilon^{\mu\lambda\Xi\eta} R^{\xi}_{\tau\mu\nu} R^{\Xi\eta}_{\xi\kappa} \sigma_{\lambda} \sigma^{\tau} \sigma^{\nu} \sigma^{\kappa} \quad (3.4)$$

which by means of (C.3) and (C.8) reduces to

$$- \frac{1}{384\pi^2} \varepsilon^{\mu\lambda\Xi\eta} R_{\mu\lambda\rho\tau} R^{\tau}_{\Xi\eta} \sigma^{\tau} + \frac{1}{48\pi^2 6} \varepsilon^{\mu\lambda\Xi\eta} R_{\mu\rho;\eta\Xi} \sigma_{\lambda} \sigma^{\rho} \quad (3.5)$$

The second term in (3.5) shows that $j^5_{\mu}(x, x')$ for $x' \rightarrow x$ must contain the directional-dependent term

$$\frac{1}{48\pi^2 6} \varepsilon^{\tau\lambda}_{\mu} R_{\tau\rho;\eta\Xi} \sigma_{\lambda} \sigma^{\rho} \quad (3.6)$$

This can, of course, also be verified directly. The directional-dependent term can be removed by subtraction, or by the averaging procedure

$$\sigma_{\lambda} \sigma^{\rho} \rightarrow \frac{1}{4} g_{\lambda}^{\rho} \quad (3.7)$$

In any case we are left over with the divergence equation for the finite directional-independent axial-vector current in an arbitrary background metric

$$\partial^{\mu} j^5_{\mu}(x) = - \frac{1}{384\pi^2} \varepsilon^{\mu\lambda\Xi\eta} R_{\mu\lambda\rho\tau} R^{\tau}_{\Xi\eta} \sigma^{\tau}(x) + 2g^{\frac{1}{2}}(x) \sum_k \psi_{0k}^+(x) \gamma_5 \psi_{0k}(x) \quad (3.8)$$

It should be emphasized that this result does not depend on the particular procedure used for going to the limit $x' \rightarrow x$; it can be considered a consequence of the algebraic identity (C.3).

The anomalous term which thus has turned up in the divergence equation of the axial-vector current is twice the density of the Hirzebruch $\hat{A}$ genus.

3.2 Euler current

Motivated by Eq. (2.29) we can introduce a point-split Euler current in the same way as (3.1) was obtained from the formal expression for the spinor axial-vector current. The outcome is

$$\begin{aligned} j_{\mu}^E(x, x') = & g^{\frac{1}{2}}(x) (-G_{\nu'}(x, x') g^{\nu'}_{\mu}(x', x) - G_{\mu}(x, x') + G_{\mu\nu p'}(x, x') g^{\nu p'}_{\mu}(x', x) \\ & + G_{\nu\lambda' \xi'}(x, x') g^{\lambda' \mu}(x', x) g^{\xi' \nu}(x', x) - *G^*_{\nu\lambda' \xi'}(x, x') g^{\lambda' \mu}(x', x) g^{\xi' \nu}(x', x) \quad (3.9) \\ & - *G^*_{\mu\nu p'}(x, x') g^{\nu p'}_{\mu}(x', x) - *G^*_{\mu}(x, x') - *G^*_{\nu'}(x, x') g^{\nu'}_{\mu}(x', x)) \end{aligned}$$

The Green's functions occurring here are Green's functions of the Hodge operator D; they are defined in Eq. (A.13). The divergence equation corresponding to (3.2) is then found to be

$$\begin{aligned} \partial^{\mu} j_{\mu}^E(x, x') \simeq & 2g^{\frac{1}{2}}(x) (-G_{\nu'}(x, x') g^{\nu' \mu}_{i\mu}(x', x) + G_{\mu\nu p'}(x, x') g^{\nu p'}_{i\mu}(x', x) \\ & + G_{\nu\lambda' \xi'}(x, x') g^{\lambda' \mu}_{i\mu}(x', x) g^{\xi' \nu}(x', x) + G_{\nu\lambda' \xi'}(x, x') g^{\lambda' \mu}(x', x) g^{\xi' \nu}_{i\mu}(x', x)) \\ & + 2g^{\frac{1}{2}}(x) g^{\xi' \mu}(x', x) (-G_{\nu'}(x, x') g^{\nu' \mu}_{i\xi'}(x', x) + G_{\mu\nu p'}(x, x') g^{\nu p'}_{i\xi'}(x', x) \quad (3.10) \\ & + G_{\nu\lambda' \xi'}(x, x') g^{\lambda' \mu}_{i\xi'}(x', x) g^{\xi' \nu}(x', x) + G_{\nu\lambda' \xi'}(x, x') g^{\lambda' \mu}(x', x) g^{\xi' \nu}_{i\xi'}(x', x)) \\ & - 2g^{\frac{1}{2}}(x) \sum_k (\bar{f}_{0k}(x) f_{0k}(x) - \bar{f}_{0k\mu}(x) f_{0k}^{\mu}(x) + \frac{1}{2} \bar{f}_{0k\mu\nu}(x) f_{0k}^{\mu\nu}(x) \\ & - * \bar{f}_{0k\mu}(x) * f_{0k}^{\mu}(x) + * \bar{f}_{0k}(x) f_{0k}(x)) \end{aligned}$$

where the sum runs over all orthonormal zero modes of the Hodge operator. Here terms involving *G* functions have been eliminated by Eq. (A.17).

The terms in Eq. (3.10) involving Green's functions at short distances are evaluated in the same way as in the spinor case, namely the singularities of the Green's functions are balanced against the power series expansions of the bi-vectors in such a way that something finite and non-vanishing for $x' \rightarrow x$ results. The relevant formulae are now (A.17), (B.3) and (B.4). In the present case, all contributions from leading singularities cancel each other and there is also an extensive cancellation between the contributions from the next leading terms.

What remains is

$$\begin{aligned} -\frac{g^{\frac{1}{2}}}{4\pi^2\epsilon} [& -\frac{1}{2} R_{\mu\nu\tau\xi} R^{\mu\nu\tau}{}_{\rho} \sigma^{\xi\rho} + R_{\mu\tau} R^{\mu}{}_{\rho} \sigma^{\tau\rho} - R_{\mu\tau} R^{\nu\tau\mu\rho} \epsilon_{\nu\rho} \\ & - \frac{1}{2} R R^{\nu\rho} \epsilon_{\nu\rho} \sigma_{\rho}^{\rho}] \quad (3.11) \end{aligned}$$

which by Eq. (C.1) becomes

$$\frac{1}{64\pi^2} g^{-\frac{1}{2}} \epsilon^{\mu\nu\lambda\rho} \epsilon^{\tau\kappa\xi\eta} R_{\mu\nu\tau\kappa} R_{\lambda\rho\xi\eta} \quad (3.12)$$

We conclude that the finite directional-independent Euler current fulfils the equation

$$\begin{aligned} \delta^\mu j_\mu^E(x) = & \frac{1}{64\pi^2} g^{-\frac{1}{2}}(x) \varepsilon^{\mu\nu\lambda\rho} \varepsilon^{\tau\kappa\zeta\eta} R_{\mu\nu\tau\kappa} R_{\lambda\rho\zeta\eta}(x) \\ & - 2g^{\frac{1}{2}}(x) \sum_k (\bar{f}_{0k}(x) f_{0k}(x) - \bar{f}_{0k\mu}(x) f_{0k}^\mu(x) + \frac{1}{2} \bar{f}_{0k\mu\nu}(x) f_{0k}^{\mu\nu}(x) \\ & - * \bar{f}_{0k\mu}(x) * f_{0k}^\mu(x) + * \bar{f}_{0k}(x) * f_{0k}(x)) \end{aligned} \quad (3.13)$$

The anomaly occurring here is obviously twice the density of the Euler characteristic. Again the anomaly is independent of the limiting procedure and is due to an algebraic identity, namely Eq. (C.1).

3.3 Hirzebruch current

The treatment of the Hirzebruch current runs parallel to that of the other two currents. Motivated by Eq. (2.30) we introduce the point-split expression

$$\begin{aligned} j_\mu^H(x, x') = & \varepsilon^{\lambda\tau\rho\sigma} G_{\lambda\tau\rho\sigma\nu'}(x, x') g_{\mu}^{\nu'}(x', x) - \varepsilon_\mu^{\tau\rho\sigma} G_{\tau\rho\sigma}(x, x') \\ & + \frac{1}{2} \varepsilon_\mu^{\nu\lambda\rho} G_{\nu\xi\eta'}(x, x') g_{\lambda}^{\xi'}(x', x) g_{\rho}^{\eta'}(x', x) \\ & + \frac{1}{2} \varepsilon_\mu^{\nu\lambda\rho} G_{\lambda\rho\xi'}(x, x') g_{\nu}^{\xi'}(x', x) \\ & - \frac{1}{2} \varepsilon_\mu^{\nu\lambda\rho} * G_{\lambda\rho\xi'}^*(x, x') g_{\nu}^{\xi'}(x', x) \\ & - \frac{1}{2} \varepsilon_\mu^{\nu\lambda\rho} * G_{\nu\xi\eta'}^*(x, x') g_{\lambda}^{\xi'}(x', x) g_{\rho}^{\eta'}(x', x) \\ & - \varepsilon_\mu^{\tau\rho\sigma} * G_{\tau\rho\sigma}^*(x, x') + \varepsilon^{\lambda\tau\rho\sigma} * G_{\lambda\tau\rho\sigma\nu'}^*(x, x') g_{\mu}^{\nu'}(x', x) \end{aligned} \quad (3.14)$$

with divergence

$$\begin{aligned} \delta^\mu j_\mu^H(x, x') = & \varepsilon^{\mu\nu\lambda\rho} (2G_{\nu\xi\eta'}(x, x') g_{\lambda;\mu}^{\xi'}(x', x) g_{\rho}^{\eta'}(x', x) + G_{\lambda\rho\xi'}(x, x') g_{\nu;\mu}^{\xi'}(x', x)) \\ & + \varepsilon^{\mu\nu\lambda\rho} g_{\tau;\mu}^{\tau'}(x', x) (2G_{\nu\xi\eta'}(x, x') g_{\lambda;\tau'}^{\xi'}(x', x) g_{\rho}^{\eta'}(x', x) \\ & + G_{\lambda\rho\xi'}(x, x') g_{\nu;\tau'}^{\xi'}(x', x)) - 2g^{\frac{1}{2}}(x) \sum_k (- * \bar{f}_{0k}(x) f_{0k}(x) \\ & + * \bar{f}_{0k\mu}(x) f_{0k}^\mu(x) + \frac{1}{2} * \bar{f}_{0k\mu\nu}(x) f_{0k}^{\mu\nu}(x) + \bar{f}_{0k\mu}(x) f_{0k}^\mu(x) - \bar{f}_{0k}(x) * f_{0k}(x)) \end{aligned} \quad (3.15)$$

where the sum again runs over all orthonormal zero modes of the Hodge operator, and terms involving $*G^*$ functions were eliminated by (A.17).

The evaluation of the anomaly runs as before, but in contrast to the previous cases there will now be a contribution from both the leading and the next leading singularities of Green's functions. By combination of (3.15) with (A.17), (B.3), and (B.4) one finds the following contribution from the leading singularity

$$-\frac{1}{24\pi^2\epsilon^2} \epsilon^{\mu\lambda\xi\eta} R_{\tau\mu\nu}^{\kappa} R_{\xi\eta\kappa\rho}(x) \sigma_{\lambda} \sigma^{\tau} \sigma^{\nu} \sigma^{\rho} \quad (3.16)$$

i.e. four times the result (3.4) for the spinor axial-vector current, and the identities (C.3) and (C.8) reduce this result to

$$-\frac{1}{96\pi^2} \epsilon^{\mu\lambda\xi\eta} R_{\mu\lambda\rho\tau} R_{\xi\eta}^{\rho\tau} + \frac{1}{12\pi^2\epsilon} \epsilon^{\mu\lambda\xi\eta} R_{\mu\rho;\eta\xi} \sigma_{\lambda} \sigma^{\rho} \quad (3.17)$$

There is again cancellation between terms arising from next leading singularities in the Green's functions; the surviving terms originate from the same parts of the Green's functions as in the Euler case and are

$$\frac{1}{16\pi^2\epsilon} \epsilon^{\mu\nu\lambda\rho} R_{\lambda\rho}^{\xi\tau} R_{\mu\nu\xi}^{\tau} \sigma_{\tau} \sigma^{\eta} - \frac{1}{4\pi^2\epsilon} \epsilon^{\mu\nu\lambda\rho} \sigma_{\lambda} R_{\rho\xi} R_{\nu}^{\xi\mu\eta} \sigma^{\eta} \quad (3.18)$$

which by means of the identities (C.2) and (C.8) can be written

$$\frac{1}{32\pi^2} \epsilon^{\mu\lambda\xi\eta} R_{\mu\lambda\rho\tau} R_{\xi\eta}^{\rho\tau} + \frac{1}{2\pi^2\epsilon} \epsilon^{\mu\nu\lambda\rho} R_{\rho\eta;\mu\nu} \sigma_{\lambda} \sigma^{\eta} \quad (3.19)$$

After disposing of the directional-dependent term in Eqs. (3.17) and (3.19) either by subtracting it or by performing the averaging indicated in (3.7) we conclude that the finite directional-independent Hirzebruch current fulfils the equation

$$\begin{aligned} \partial^{\mu} j_{\mu}^H(x) &= \frac{1}{48\pi^2} \epsilon^{\mu\lambda\xi\eta} R_{\mu\lambda\rho\tau} R_{\xi\eta}^{\rho\tau}(x) \\ &- 2g^{\frac{1}{2}}(x) \sum_k (- * \bar{f}_{0k}(x) f_{0k}(x) + * \bar{f}_{0k\mu}(x) f_{0k}^{\mu}(x) \\ &+ \frac{1}{2} * \bar{f}_{0k\mu\nu}(x) f_{0k}^{\mu\nu}(x) + \bar{f}_{0k\mu}(x) f_{0k}^{\mu}(x) - \bar{f}_{0k}(x) * f_{0k}(x)) \end{aligned} \quad (3.20)$$

The anomaly occurring here is twice the Hirzebruch signature density of the manifold.

APPENDIX A

The Green's function of the Dirac operator is defined by

$$G(x, x') = \sum_k \frac{\psi_k(x) \psi_k^+(x')}{\lambda_k} \quad (\text{A.1})$$

$$i \gamma_\mu(x) \psi_{k; \mu}(x) = \lambda_k \psi_k(x) \quad (\text{A.2})$$

where the sum in (A.1) runs over all orthonormal eigenfunctions of the Dirac operator with non-vanishing eigenvalue (we are assuming here as always that the manifold is compact and the metric positive definite), so

$$i \gamma_\mu(x) G;^\mu(x, x') = g^{-\frac{1}{2}}(x) \delta(x, x') - \sum_k \psi_{0k}^+(x) \psi_{0k}(x') \quad (\text{A.3})$$

We write¹³⁾

$$G(x, x') = i \gamma_\mu(x) \mathcal{G};^\mu(x, x') \quad (\text{A.4})$$

with

$$-\mathcal{G}_{;\mu}{}^\mu(x, x') + \frac{1}{4} R \mathcal{G}(x, x') = g^{-\frac{1}{2}}(x) \delta(x, x') - \sum_k \psi_{0k}(x) \psi_{0k}^+(x') \quad (\text{A.5})$$

The solution has the form¹⁴⁾

$$\mathcal{G}(x, x') = \frac{\mu(x, x')}{8\pi^2 \sigma} + \frac{v(x, x')}{16\pi^2 \sigma} \ln \sigma + w(x, x') \quad (\text{A.6})$$

where $\sigma = \sigma(x, x')$ is the geodetic interval¹³⁾ and μ , v , and w all are regular for $x' \rightarrow x$, with

$$\mu(x, x') = \Delta^{\frac{1}{2}} I(x, x') \quad (\text{A.7a})$$

$$\Delta = g^{-\frac{1}{2}}(x) \det(-\sigma_{;\mu\nu}) g^{-\frac{1}{2}}(x') \quad (\text{A.7b})$$

and (with $\sigma^\mu = \sigma^\mu_j$)

$$2\sigma^\mu v_{;\mu} + v \sigma_{;\mu}{}^\mu = 2v - 2\mu_{;\mu}{}^\mu + \frac{1}{2} R \mu \quad (\text{A.8})$$

leading to the short-distance behaviour of v [cf. Ref. 13]

$$v(x, x') \simeq \frac{R}{12} I(x, x') \quad (\text{A.9})$$

The leading and the next leading singularity of the Green's function G itself now follows by combination of (A.4), (A.6), (A.7), and (A.9)

$$\begin{aligned} G(x, x') \simeq & -i \gamma^\mu(x) \sigma_\mu \frac{\Delta^{\frac{1}{2}} I(x, x')}{8\pi^2 \sigma^2} + i \gamma^\mu(x) \frac{(\Delta^{\frac{1}{2}} I(x, x'))_{;\mu}}{8\pi^2 \sigma} \\ & + i \gamma^\mu(x) \sigma_\mu \frac{R}{12} \frac{I(x, x')}{16\pi^2 \sigma} \end{aligned} \quad (\text{A.10})$$

where in the second term we can use the estimate

$$(\Delta^{\frac{1}{2}} I(x, x'))_{;\mu} \simeq I(x, x') \left(\frac{1}{6} R_{\mu\nu} \sigma^\nu - \frac{i}{4} \Sigma_{\xi\eta} R^{\xi\eta}{}_{\mu\nu} \sigma^\nu \right) \quad (\text{A.11})$$

following from Eqs. (17.90) and (17.91) of Ref. 13 as well as Eq. (B.1) given below.

The Green's functions of the Hodge operator can be treated in the same way.

With

$$\phi = \begin{Bmatrix} 1 \\ \dagger_\mu \\ \dagger_{\mu\nu} \\ \dagger_{\mu\nu\lambda} \\ \dagger_{\mu\nu\lambda\rho} \end{Bmatrix}, \quad * \phi = \begin{Bmatrix} * 1 \\ * \dagger_\mu \\ * \dagger_{\mu\nu} \\ * \dagger_{\mu\nu\lambda} \\ * \dagger_{\mu\nu\lambda\rho} \end{Bmatrix} \quad (\text{A.12})$$

they are given by the expressions (with indices suppressed)

$$G(x, x') = \sum_k \frac{\phi_k(x) \bar{\phi}_k(x')}{\lambda_k}; \quad * G(x, x') = \sum_k \frac{* \phi_k(x) * \bar{\phi}_k(x')}{\lambda_k} \quad (\text{A.13})$$

and

$$\mathcal{G}(x, x') = \sum_k \frac{\phi_k(x) \bar{\phi}_k(x')}{\lambda_k^2}; \quad * \mathcal{G}(x, x') = \sum_k \frac{* \phi_k(x) * \bar{\phi}_k(x')}{\lambda_k} \quad (\text{A.14})$$

where the sums run over all orthonormal eigenfunctions of the Hodge operator with non-vanishing eigenvalue.

Squaring the Hodge operator one then finds the differential equations fulfilled by the G 's

$$- \mathcal{G}_{i\mu}{}^{\mu}(x, x') = g^{-\frac{1}{2}}(x) \delta(x, x') - \sum_k f_{ok}(x) \bar{f}_{ok}(x') \quad (\text{A.15a})$$

$$- \mathcal{G}_{\mu\nu; \lambda}{}^{\lambda}(x, x') + R_{\mu}{}^{\lambda} \mathcal{G}_{\lambda\nu}(x, x') = g^{-\frac{1}{2}}(x) g_{\mu\nu} \delta(x, x') - \sum_k f_{ok\mu}(x) \bar{f}_{ok\nu}(x') \quad (\text{A.15b})$$

$$\begin{aligned} - \mathcal{G}_{\mu\nu\lambda'p'; \bar{c}}{}^{\bar{c}}(x, x') + R_{\mu}{}^{\bar{c}}(x) \mathcal{G}_{\bar{c}\nu\lambda'p'}(x, x') + R_{\nu}{}^{\bar{c}}(x) \mathcal{G}_{\mu\bar{c}\lambda'p'}(x, x') \\ - R_{\mu\nu}{}^{\bar{c}\bar{c}}(x) \mathcal{G}_{\bar{c}\bar{c}\lambda'p'}(x, x') = g^{-\frac{1}{2}}(x) (g_{\mu\lambda'} g_{\nu p'} - g_{\mu p'} g_{\nu\lambda'}) \delta(x, x') \\ - \sum_k f_{ok\mu\nu}(x) \bar{f}_{ok\lambda'p'}(x') \end{aligned} \quad (\text{A.15c})$$

where the sums run over orthonormal zero modes of D. The equations fulfilled by the $\ast G$ functions only differ from those for the G 's by the terms involving sums over zero modes. Consequently, the singular parts are the same. They are found in the same way as in the spinor case and are

$$\mathcal{G}(x, x') \simeq \ast \mathcal{G}^{\ast}(x, x') \simeq \frac{\Delta^{\frac{1}{2}}}{8\pi^2\epsilon} - \frac{R}{6} \frac{1}{16\pi^2} \log \epsilon \quad (\text{A.16a})$$

$$\mathcal{G}_{\mu\nu}(x, x') \simeq \ast \mathcal{G}_{\mu\nu}^{\ast}(x, x') \simeq \frac{\Delta^{\frac{1}{2}} g_{\mu\nu}}{8\pi^2\epsilon} + \left(-\frac{R}{6} g_{\mu\nu} + R_{\mu}{}^{\lambda} g_{\lambda\nu}\right) \frac{1}{16\pi^2} \log \epsilon \quad (\text{A.16b})$$

$$\begin{aligned} \mathcal{G}_{\mu\nu\lambda'p'}(x, x') \simeq \ast \mathcal{G}_{\mu\nu\lambda'p'}^{\ast}(x, x') \simeq \frac{\Delta^{\frac{1}{2}}}{8\pi^2\epsilon} (g_{\mu\lambda'} g_{\nu p'} - g_{\mu p'} g_{\nu\lambda'}) \\ + \left[-\frac{R}{6} (g_{\mu\lambda'} g_{\nu p'} - g_{\mu p'} g_{\nu\lambda'}) + R_{\mu}{}^{\bar{c}} (g_{\bar{c}\lambda'} g_{\nu p'} - g_{\bar{c}p'} g_{\nu\lambda'}) \right. \\ \left. + R_{\nu}{}^{\bar{c}} (g_{\mu\lambda'} g_{\bar{c}p'} - g_{\mu p'} g_{\bar{c}\lambda'}) - R_{\mu\nu}{}^{\bar{c}\bar{c}} (g_{\bar{c}\lambda'} g_{\bar{c}p'} - g_{\bar{c}p'} g_{\bar{c}\lambda'})\right] \frac{1}{16\pi^2} \log \epsilon \end{aligned} \quad (\text{A.16c})$$

From these results it follows that

$$\begin{aligned} G_{\nu}{}^{\mu}(x, x') \simeq \ast G_{\nu}{}^{\mu\ast}(x, x') \simeq - \mathcal{G}_{\mu\nu; \lambda}{}^{\lambda}(x, x') \simeq \\ \frac{\Delta^{\frac{1}{2}}}{8\pi^2\epsilon^2} \delta^{\mu}_{\nu} + \frac{1}{8\pi^2\epsilon} \left(\frac{R}{12} \delta^{\mu}_{\nu} - \frac{1}{6} R^{\mu\lambda} \delta_{\lambda\nu}\right) \end{aligned} \quad (\text{A.17a})$$

$$\begin{aligned} G_{\mu}(x, x') \simeq \ast G_{\mu}^{\ast}(x, x') \simeq \mathcal{G}_{i\mu}(x, x') \simeq \\ - \frac{\Delta^{\frac{1}{2}}}{8\pi^2\epsilon^2} \delta_{\mu} - \frac{1}{8\pi^2\epsilon} \left(\frac{R}{12} \delta_{\mu} - \frac{1}{6} R_{\mu\nu} \delta^{\nu}{}_{\mu}\right) \end{aligned} \quad (\text{A.17b})$$

$$\begin{aligned}
 G_{\mu\nu\lambda}(x, x') \simeq - * G_{\mu\nu\lambda}^*(x, x') \simeq g_{\nu\lambda';\mu}(x, x') - g_{\mu\lambda';\nu}(x, x') \simeq \\
 \cdot \frac{\Delta^{\frac{1}{2}}}{8\pi^2\epsilon^2} (g_{\mu\nu\lambda'} - g_{\nu\mu\lambda'}) - \frac{1}{8\pi^2\epsilon} \left(\frac{R}{12} (g_{\mu\nu\lambda'} - g_{\nu\mu\lambda'}) \right. \\
 \left. - \frac{1}{6} R_{\mu\tau} \sigma^{\tau} g_{\nu\lambda'} + \frac{1}{6} R_{\nu\tau} \sigma^{\tau} g_{\mu\lambda'} - \frac{1}{2} R_{\mu\nu\tau\xi} \sigma^{\xi} g^{\tau}_{\lambda'} \right. \\
 \left. - \frac{1}{2} g_{\mu} R_{\nu}^{\tau} g_{\tau\lambda'} + \frac{1}{2} g_{\nu} R_{\mu}^{\tau} g_{\tau\lambda'} \right)
 \end{aligned}
 \tag{A.17c}$$

$$\begin{aligned}
 G_{\mu\nu'\lambda}(x, x') \simeq - * G_{\mu\nu'\lambda}^*(x, x') \simeq - g_{\tau\mu\nu'\lambda';}^{\tau}(x, x') \simeq \\
 \frac{\Delta^{\frac{1}{2}}}{8\pi^2\epsilon^2} \sigma^{\tau} (g_{\tau\nu'} g_{\mu\lambda'} - g_{\tau\lambda'} g_{\mu\nu'}) + \frac{1}{8\pi^2\epsilon} \left(\frac{R}{12} \sigma^{\tau} (g_{\tau\nu'} g_{\mu\lambda'} - g_{\tau\lambda'} g_{\mu\nu'}) \right. \\
 \left. - \frac{1}{6} R^{\tau\xi} \sigma_{\xi} (g_{\tau\nu'} g_{\mu\lambda'} - g_{\tau\lambda'} g_{\mu\nu'}) - \frac{1}{2} R_{\tau\mu}^{\xi\tau} \sigma^{\tau} g_{\xi\lambda'} g_{\tau\nu'} \right. \\
 \left. - \frac{1}{2} \sigma^{\tau} R_{\mu}^{\xi} (g_{\tau\nu'} g_{\xi\lambda'} - g_{\tau\lambda'} g_{\xi\nu'}) \right),
 \end{aligned}
 \tag{A.17d}$$

where also an estimate analogous to (A.11) was used. All other Green's functions are regular at short distances.

APPENDIX B

The short distance expansions of derivatives of the bispinor $I(x', x)$ and the bivector $g_{\mu\nu}(x', x)$ can be found by the technique of coincidence limits^{13,15,16}. The results needed here are straightforward to obtain; they are for the bispinor

$$I_{;\mu}(x', x) = I(x', x) \sum_{\xi\eta} \left(\frac{1}{4} R^{\xi\eta}_{\mu\nu} \sigma^\nu - \frac{1}{12} R^{\xi\eta}_{\mu\nu;\tau} \sigma^\nu \sigma^\tau - \frac{1}{48} R^\lambda_{\tau\mu\nu} R^{\xi\eta}_{\lambda\rho} \sigma^\tau \sigma^\nu \sigma^\rho + \frac{1}{48} R^{\xi\eta}_{\mu\nu;\tau\rho} \sigma^\nu \sigma^\tau \sigma^\rho + \dots \right) \quad (B.1)$$

$$g^{\nu\mu}(x', x) I_{;\nu}(x', x) = I(x', x) \sum_{\xi\eta} \left(\frac{1}{4} R^{\xi\eta}_{\mu\nu} \sigma^\nu - \frac{1}{6} R^{\xi\eta}_{\mu\nu;\tau} \sigma^\nu \sigma^\tau - \frac{1}{48} R^\lambda_{\tau\mu\nu} R^{\xi\eta}_{\lambda\rho} \sigma^\tau \sigma^\nu \sigma^\rho + \frac{1}{16} R^{\xi\eta}_{\mu\nu;\tau\rho} \sigma^\nu \sigma^\tau \sigma^\rho + \dots \right) \quad (B.2)$$

where

$$\sum_{\xi\eta} = \frac{1}{4i} [\sigma_\xi, \sigma_\eta]$$

and for the bivector

$$g_{\mu\nu;\lambda}(x', x) = g_{\mu\xi}(x', x) \left(-\frac{1}{2} R_\nu^{\xi\lambda} \sigma^\lambda + \frac{1}{2} R_\nu^{\xi\lambda;\tau} \sigma^\lambda \sigma^\tau + \frac{1}{24} R_\nu^{\xi\sigma} R_{\sigma\tau\lambda\rho} \sigma^\lambda \sigma^\tau \sigma^\rho - \frac{1}{24} R_\nu^{\xi\lambda;\tau\rho} \sigma^\lambda \sigma^\tau \sigma^\rho + \dots \right) \quad (B.3)$$

$$g^{\rho\lambda}(x', x) g_{\mu\nu;\rho}(x', x) = g_{\mu\xi}(x', x) \left(-\frac{1}{2} R_\nu^{\xi\lambda} \sigma^\lambda + \frac{1}{3} R_\nu^{\xi\lambda;\tau} \sigma^\lambda \sigma^\tau + \frac{1}{24} R_\nu^{\xi\sigma} R_{\sigma\tau\lambda\rho} \sigma^\lambda \sigma^\tau \sigma^\rho - \frac{1}{8} R_\nu^{\xi\lambda;\tau\rho} \sigma^\lambda \sigma^\tau \sigma^\rho + \dots \right) \quad (B.4)$$

APPENDIX C

The algebraic identities needed for the derivations carried out in Section 3 are

$$\begin{aligned} R_{\mu\sigma\tau\rho} R_{\nu}{}^{\sigma\tau\rho} &= \frac{1}{4} g_{\mu\nu} (R_{\lambda\sigma\tau\rho} R^{\lambda\sigma\tau\rho} + R^2 - 4R_{\lambda\sigma} R^{\lambda\sigma}) \\ &\quad + 2R_{\mu\sigma\nu\rho} R^{\sigma\rho} - R R_{\mu\nu} + 2R_{\mu\lambda} R_{\nu}{}^{\lambda} = \\ &\quad \frac{1}{4} g_{\mu\nu} R_{\lambda\sigma\tau\rho} * R^{\lambda\sigma\tau\rho} + 2R_{\mu\sigma\nu\rho} R^{\sigma\rho} \end{aligned} \quad (C.1)$$

$$\begin{aligned} \varepsilon_{\xi\eta\lambda\rho} R^{\lambda\rho\sigma}{}_{\mu} R_{\xi\eta\sigma\nu} &= \frac{1}{4} g_{\mu\nu} \varepsilon_{\xi\eta\lambda\rho} R^{\lambda\rho\sigma\tau} R^{\xi\eta}{}_{\sigma\tau} \\ &\quad + \varepsilon_{\xi\mu\lambda\rho} R^{\lambda\rho\sigma\tau}{}_{\nu} R_{\tau}{}^{\xi} + \varepsilon_{\xi\nu\lambda\rho} R^{\lambda\rho\sigma\tau}{}_{\mu} R_{\tau}{}^{\xi} \end{aligned} \quad (C.2)$$

$$\begin{aligned} \varepsilon_{\tau\xi\mu\eta} R^{\xi\eta}{}_{\sigma\nu} R^{\sigma}{}_{\lambda}{}^{\tau}{}_{\rho} &= \frac{1}{16} g_{\mu\rho} g_{\lambda\nu} \varepsilon_{\sigma\tau\xi\eta} R^{\lambda\kappa\sigma\tau} R^{\xi\eta}{}_{\kappa\lambda} \\ &\quad + \frac{1}{2} (\varepsilon_{\tau\xi\mu\eta} R^{\xi\eta}{}_{\sigma\nu} R^{\sigma}{}_{\lambda}{}^{\tau}{}_{\rho} + \varepsilon_{\tau\lambda\xi\eta} R^{\xi\eta}{}_{\sigma\mu} R^{\sigma}{}_{\rho}{}^{\tau}{}_{\nu}) \\ &\quad + \frac{1}{2} \varepsilon_{\tau\xi\mu\eta} R^{\xi\eta}{}_{\sigma\nu} R^{\sigma\tau}{}_{\lambda\rho} + \frac{1}{2} \varepsilon_{\lambda\xi\mu\eta} (R^{\xi\eta}{}_{\rho\nu} R - R^{\xi\eta}{}_{\sigma\nu} R^{\sigma}{}_{\rho}) \end{aligned} \quad (C.3)$$

with

$$*R^{\mu\nu\lambda\rho} = \frac{1}{4g} \varepsilon^{\mu\nu\sigma\tau} \varepsilon^{\lambda\rho\xi\eta} R_{\sigma\xi\tau\eta} \quad (C.4)$$

being the double-dual of the Riemann tensor. The identities all can be derived from¹⁷⁾

$$R^{\mu\nu\lambda\rho} = *R^{\mu\nu\lambda\rho} + B^{\mu\lambda} g^{\nu\rho} - B^{\nu\lambda} g^{\mu\rho} + B^{\nu\rho} g^{\mu\lambda} - B^{\mu\rho} g^{\nu\lambda} \quad (C.5a)$$

$$B^{\mu\nu} = R^{\mu\nu} - \frac{1}{4} g^{\mu\nu} R \quad (C.5b)$$

As an example we derive (C.2)

$$\begin{aligned} \varepsilon_{\xi\eta\lambda\rho} R^{\lambda\rho\sigma}{}_{\mu} R^{\xi\eta}{}_{\sigma\nu} &= \frac{1}{2} \varepsilon_{\xi\eta\lambda\rho} R^{\lambda\rho\sigma}{}_{\mu} R^{\xi\eta}{}_{\sigma\mu} \\ &\quad + \frac{1}{2} \varepsilon_{\xi\eta\lambda\rho} (*R^{\lambda\rho\sigma}{}_{\mu} + B^{\lambda\sigma} g^{\rho}{}_{\mu} - B^{\rho\sigma} g^{\lambda}{}_{\mu} + B^{\rho}{}_{\mu} g^{\lambda\sigma} - B^{\lambda}{}_{\mu} g^{\rho\sigma}) \times \\ &\quad \times (*R^{\xi\eta}{}_{\sigma\nu} + B^{\xi\sigma} g^{\eta}{}_{\nu} - B^{\eta\sigma} g^{\xi}{}_{\nu} + B^{\eta}{}_{\nu} g^{\xi\sigma} - B^{\xi}{}_{\nu} g^{\eta\sigma}) = \\ &\quad \frac{1}{4} g_{\mu\nu} \varepsilon_{\xi\eta\lambda\rho} R^{\xi\eta\tau\sigma} R^{\lambda\rho}{}_{\tau\sigma} + \frac{1}{2} \varepsilon_{\sigma\mu\lambda\rho} R_{\xi\eta}{}^{\lambda\rho} (B^{\xi\sigma} g^{\eta}{}_{\nu} - B^{\eta\sigma} g^{\xi}{}_{\nu} + B^{\eta}{}_{\nu} g^{\xi\sigma} - B^{\xi}{}_{\nu} g^{\eta\sigma}) \\ &\quad + \frac{1}{2} \varepsilon_{\sigma\nu\lambda\rho} R_{\xi\eta}{}^{\lambda\rho} (B^{\xi\sigma} g^{\eta}{}_{\mu} - B^{\eta\sigma} g^{\xi}{}_{\mu} + B^{\eta}{}_{\mu} g^{\xi\sigma} - B^{\xi}{}_{\mu} g^{\eta\sigma}) = \\ &\quad \frac{1}{4} g_{\mu\nu} \varepsilon_{\xi\eta\lambda\rho} R^{\xi\eta\tau\sigma} R^{\lambda\rho}{}_{\tau\sigma} + \varepsilon_{\sigma\mu\lambda\rho} R_{\xi\nu}{}^{\lambda\rho} R^{\xi\sigma} + \varepsilon_{\sigma\nu\lambda\rho} R_{\xi\mu}{}^{\lambda\rho} R^{\xi\sigma} \end{aligned} \quad (C.6)$$

Here we used

$$\frac{1}{2} \epsilon_{\xi\eta\lambda\rho} R^{\lambda\rho\sigma}{}_{\mu} R^{\xi\eta}{}_{\sigma\nu} = \frac{1}{4} g_{\mu\nu} \epsilon_{\xi\eta\lambda\rho} R^{\lambda\rho\sigma\tau} R^{\xi\eta}{}_{\sigma\tau} + \frac{1}{2} \epsilon_{\xi\eta\lambda\rho} R^{\lambda\rho\sigma}{}_{\mu} R^{\xi\eta}{}_{\nu\sigma} \quad (C.7)$$

and several of the familiar symmetry properties of the Riemann tensor.

Derivations of Eq. (C.1) not using (C.5a) can also be found in the literature^{13,17)}.

In (C.2) and (C.3) there occur terms on the right-hand side which can be reduced by means of

$$\epsilon_{\mu\nu\lambda\rho} R^{\mu\lambda}{}_{\tau\xi} R^{\rho\xi}{}_{\tau\eta} = 2 \epsilon_{\mu\nu\lambda\rho} R^{\rho}{}_{\tau\eta}{}^{\lambda\mu}, \quad (C.8)$$

which is a consequence of the Bianchi identity and the commutation relation for covariant derivatives.

APPENDIX D

In this Appendix we introduce Duffin-Kemmer-like Γ matrices, which allow us to write the Hodge Lagrangian, the Hodge operator, and the Euler and Hirzebruch currents in analogy to the Dirac case.

Let M be a compact orientable n -dimensional Riemannian manifold with Riemannian metric g .

On M we introduce locally-defined vector fields e_i ($i = 1, \dots, n$), whose values are pointwise linear independent, as well as their dual one-forms e^i , defined by

$$e^i(e_j) = \delta^i_j \quad (D.1)$$

We now observe that the exterior derivative on $\Omega(M)$ can be written as

$$d\omega = \sum_i e^i \wedge \nabla_{e_i} \omega \quad (\omega \in \Omega(M)) \quad (D.2)$$

where ∇ is the Riemannian connection which belongs to the Riemannian metric on M . To prove this formula one explicitly evaluates the right-hand side of Eq. (D.2), using the identity

$$\nabla_{e_i} e_k - \nabla_{e_k} e_i - [e_i, e_k] = 0 \quad (D.3)$$

which means that ∇ is torsionless ($[e_i, e_k]$ denotes the Lie bracket of the vector fields e_i and e_k), and compares the result with the well-known formula

$$\begin{aligned} d\varphi(e_1, \dots, e_p) &= \sum_{i=1}^{p+1} (-1)^{i+1} e_i \varphi(e_1, \dots, \hat{e}_i, \dots, e_{p+1}) \\ &\quad + \sum_{i < j} (-1)^{i+j} \varphi([e_i, e_j], e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{p+1}) \quad (\varphi \in \Omega^p(M)) \end{aligned} \quad (D.4)$$

Equation (D.3) allows us to write

$$\begin{aligned} D\varphi &= (d + \delta)\varphi = (d + (-1)^{np+n+1} * d *)\varphi = \\ &= \sum_i (e^i \wedge \nabla_{e_i} + (-1)^{np+n+1} * e^i \wedge * \nabla_{e_i})\varphi = \sum_i \Gamma^i \nabla_{e_i} \varphi \end{aligned} \quad (D.5)$$

where the linear operators Γ^i are defined by

$$\Gamma^i \varphi = (x^i \wedge + (-1)^{n+n+1} * x^i \wedge *) \varphi \quad \text{for } \varphi \in \Omega^n(M) \quad (\text{D.6})$$

(We have used the fact that ∇ commutes with $*$.)

One verifies that the Γ^i are antisymmetric with respect to the bilinear form $(.,.)$ of Eq. (2.6) and that

$$\{\Gamma^i, \Gamma_5^{E,H}\} = 0 \quad (\text{D.7})$$

with $\Gamma_5^{E,H}$ defined in Eqs. (2.10) and (2.15). For $\dim M = u$, the quantity $\omega \in \Omega(M)$ is a 2^n -component field and $\Gamma^i, \Gamma_5^{E,H}$ are representable by $2^n \times 2^n$ -matrices.

Admitting the forms ω to be complex valued we can introduce a Lagrangian

$$\mathcal{L} = \langle \bar{\omega}, \Gamma^i \nabla_i \omega \rangle \quad (\nabla_i \equiv \nabla_{e_i}) \quad (\text{D.8})$$

which leads to the equations of motion

$$\Gamma^i \nabla_i \omega = \Gamma^i \nabla_i \bar{\omega} = 0 \quad (\text{D.9})$$

($\Gamma^i \nabla_i = \nabla_i \Gamma^i$ because of hermiticity). Thus, harmonic forms appear as zero modes of equations of Duffin-Kemmer type.

The Lagrangian L is invariant under the transformations

$$\omega \rightarrow e^{\alpha \Gamma_5^{E,H}} \omega \quad (\alpha \in \mathbb{R}) \quad (\text{D.10})$$

and, hence, the currents

$$j^{i,E,H} = \langle \bar{\omega}, \Gamma^i \Gamma_5^{E,H} \omega \rangle \quad (\text{D.11})$$

are formally conserved. Their anomalies are related to the Euler characteristic and the L genus of M .

REFERENCES

- 1) J. Schwinger, Phys. Rev. 82, 664 (1951).
- 2) K. Johnson, Phys. Letters 5, 253 (1963).
- 3) S.L. Adler, Phys. Rev. 177, 2426 (1969).
J.S. Bell and R. Jackiw, Nuovo Cimento 60A, 47 (1969).
- 4) R. Delbourgo and A. Salam, Phys. Letters 30B, 381 (1972).
T. Eguchi and P.G.O. Freund, Phys. Rev. Letters 37, 1251 (1976).
- 5) S. Coleman, unpublished.
J. Kiskis, Phys. Rev. D15, 2329 (1977).
R. Jackiw and C. Rebbi, Phys. Rev. D16, 1052 (1977).
- 6) N.K. Nielsen and B. Schroer, Nuclear Phys. B127, 493 (1977).
- 7) N.K. Nielsen, H. Römer and B. Schroer, Phys. Letters 70B, 445 (1977).
- 8) M.F. Atiyah and I.M. Singer, Bull. Amer. Math. Soc. 69, 422 (1963); Ann. Math. 87, 596 (1968).
- 9) H. Römer and B. Schroer, Fractional winding number and surface effects, CERN preprint TH.2357 (1977).
- 10) S.W. Hawking, Phys. Letters 60A, 81 (1977).
- 11) R.S. Kulkarni, Index theorems of Atiyah-Bott-Patodi and curvature invariants (Presses de l'Univ. de Montréal, Montréal, 1975).
- 12) B.S. DeWitt, Phys. Rev. 19C, 295 (1975).
- 13) B.S. DeWitt, Dynamical theory of groups and fields (Blackie, London and Glasgow, 1965).
- 14) B.S. DeWitt and R.W. Brehme, Ann. Phys. (NY) 9, 220 (1960).
- 15) J.L. Synge, Relativity: the general theory (North-Holland, Amsterdam, 1960).
- 16) S.M. Christensen, Phys. Rev. D14, 2490 (1976).
- 17) C. Lanczos, Ann. Math. 39, 842 (1938).