Ref.TH.2409-CERN

NUMBER OF PARAMETERS FOR INSTANTON SOLUTIONS
ON ARBITRARY FOUR - DIMENSIONAL COMPACT MANIFOLDS

H. Römer *)
CERN -- Geneva

A B S T R A C T

By explicitly evaluating the topological index of an appropriately chosen elliptic complex we obtain a formula which gives the number of free parameters for self-dual solutions of the Yang-Mills equations with arbitrary simple group on an arbitrary compact orientable four-dimensional Riemannian manifold.

*) On leave of absence from the Physikalisches Institut der Universität Bonn, Germany.

The problem of determining the number of parameters on which self-dual instanton solutions of the Euclidean $SU(2)$ Yang-Mills equations on the four-dimensional sphere S^4 are dependent (disregarding pure gauge freedom) has been solved by various authors ¹⁾⁻³⁾. Also complete results for arbitrary simple gauge groups are now available ⁴⁾.

In many situations in relativity theory and probably also in ordinary quantum field theory one may be interested in instanton solutions on more general Riemannian manifolds M . Self-duality can only be formulated if M is four-dimensional and orientable, and we shall further assume it to be compact (without boundary). Partial results [signature of M equal to zero and group $SU(2)$] have been obtained for this more general problem, too ¹⁾. Our task is to determine the number of instanton solutions on an arbitrary compact orientable four-dimensional Riemannian manifold M with arbitrary simple gauge group G .

The problem can be cast into the following form. Denote by $\Omega_g^p(M)$ the set of p -forms on M with values in the complexified Lie algebra $\mathfrak{g}$ of G , and also the corresponding vector bundles. A Yang-Mills field on M is given by a (locally defined) one-form

$$A \in \Omega_g^1(M)$$

The corresponding field strength is the two-form

$$F = dA + \frac{1}{2} A \wedge A \quad (\in \Omega_g^2(M)) \quad (1)$$

All the forms are $\mathfrak{g}$ -valued and a Lie algebra multiplication is implied on the right-hand side. We always take A such that $*F = F$ (self-duality). This also guarantees that A solves the Yang-Mills equation. Introducing a covariant exterior derivative D of $\mathfrak{g}$ -valued forms by

$$D\omega = d\omega + A \wedge \omega \quad (2)$$

one observes that

$$D^2\omega = F \wedge \omega \quad (3)$$

rather than zero.

Now the change δF of F due to an infinitesimal change δA of A can be written as

$$\delta F = D \delta A \quad (4a)$$

and if δA is due to an infinitesimal gauge transformation one has

$$\delta A = D \delta V \quad (\delta V \in \Omega_g^0(M)) \quad (4b)$$

Let P_- be the projection onto the anti-self-dual forms in Ω_g^2 and define $\Omega_{-g}^2(M) = P_- \Omega_g^2(M)$ and $D_- = P_- D$.

Then one has ³⁾ an elliptic complex

$$0 \longrightarrow \Omega_g^0(M) \xrightarrow{D} \Omega_g^1(M) \xrightarrow{D_-} \Omega_{-g}^2(M) \longrightarrow 0 \quad (4c)$$

For instance, $D_- D = 0$ is guaranteed by Eq. (3) and self-duality. The index of this complex is given by

$$ind = h_0 - h_1 + h_2 \quad , \quad (5)$$

where

$$h_0 = \dim \text{kernel } D$$

$$h_1 = \dim \text{kernel } D_- - \dim \text{image } D$$

$$h_2 = \dim \Omega_{-g}^2 - \dim \text{image } D_- = \dim \text{kernel } D_-^*.$$

h_1 is the number of linear independent changes δA which keep F self-dual minus the number of such changes which arise by gauge transformations, i.e., gives just the quantity to be determined.

h_0 is the number of linear independent covariantly constant vector fields V with values in $\mathfrak{g}$ ($DV=0$). Such a field has to commute with F at every point. If $h_0 \neq 0$, the holonomy group of A is of smaller dimension than G . In this case, A can be considered as actually belonging to a smaller group $H \subset G$, and to be a G Yang-Mills potential only by embedding H into G . So, we can restrict ourselves to $h_0 = 0$. For the more general case one has to consider subgroups of G and their embeddings into G ⁴⁾.

The quantity h_2 can be shown to vanish in many interesting cases. For instance, one can convince oneself that DD^* is strictly positive so that h_2 vanishes if the curvature of M is everywhere positive and if the Weyl part of the curvature tensor of M is self-dual^{3),4)}. Moreover, for $h_2=0$ the infinitesimal variations can really be integrated to independent global self-dual solutions.

To compute

$$h_1 = h_0 + h_2 - \text{ind}$$

we have to evaluate the index, which by the Atiyah-Singer index theorem⁵⁾ is given by

$$\text{ind} = \frac{(\text{ch } \Omega_{\mathbb{R}}^0 - \text{ch } \Omega_{\mathbb{R}}^1 + \text{ch } \Omega_{\mathbb{R}}^2) \text{td } \Omega_{\mathbb{R}}^1 e}{e(\Omega_{\mathbb{R}}^1)} [M] \quad (6)$$

The meaning of the entering quantities is the following.

- $\Omega_{\mathbb{R}}^1$ and $\Omega_{\mathbb{C}}^1$ are the real and complexified cotangent bundle of M , respectively.
- $e(\Omega_{\mathbb{R}}^1)$ is the Euler class of M , representable by a closed four-form on M which integrated over M yields the Euler characteristic $\chi(M)$ of M .
- ch and td are the Chern character and the Todd class, which are defined in the following way :

Let E be a complex vector bundle over M , which is the direct sum of complex one-dimensional vector bundles E_i . Denote the first Chern class of E_i (expressible as a closed two-form on M) by y_i . Then

$$\text{td } E = \prod_i \frac{y_i}{1 - e^{-y_i}} = 1 + \frac{1}{2} C_1(E) + \frac{1}{12} (C_2(E) + C_1^2(E)) + \dots \quad (7)$$

$$\text{ch } E = \sum_i e^{y_i} = \dim E + C_1(E) + \frac{1}{2} (C_1^2(E) - 2C_2(E)) + \dots \quad (8)$$

and the Chern classes $c_k(E)$ are given by

$$\sum_k \alpha^k C_k(E) = \prod_i (1 + \alpha y_i) \quad (\alpha \in \mathbb{C}) \quad (9)$$

The Chern character is additive and multiplicative, i.e.,

$$\begin{aligned} \text{ch}(E \oplus F) &= \text{ch} E + \text{ch} F \\ \text{ch}(E \otimes F) &= \text{ch} E \text{ch} F, \end{aligned} \quad (10)$$

which helps to evaluate the index in our problem. One has

$$\begin{aligned} \Omega_g^0 &= V_g \\ \Omega_g^1 &= V_g \otimes \Omega_{\mathbb{C}}^1 \\ \Omega_{-g}^2 &= V_g \otimes \Omega_{\mathbb{C}}^2, \end{aligned} \quad (11)$$

where V_g is a complex vector bundle over M with structural group G and fibre g . So,

$$\text{ind} = \frac{\text{ch} V_g (\text{ch} \Omega_{\mathbb{C}}^0 - \text{ch} \Omega_{\mathbb{C}}^1 + \text{ch} \Omega_{\mathbb{C}}^2)}{e(\Omega_{\mathbb{R}}^1)} \text{td} \Omega_{\mathbb{C}}^1 [M] \quad (12)$$

Now there is a formal splitting of $\Omega_{\mathbb{C}}^1$ such that the $(y_i)_{i=1,2,3,4}$ are pairwise opposite in sign $y_1 = x_2, y_2 = -x_1, y_3 = x_2, y_4 = -x_2$. Thus

$$\begin{aligned} \text{ch} \Omega_{\mathbb{C}}^0 &= 1 \\ \text{ch} \Omega_{\mathbb{C}}^1 &= \sum_{i=1}^2 (e^{x_i} + e^{-x_i}) \\ \text{td} \Omega_{\mathbb{C}}^1 &= \prod_{i=1}^2 \frac{x_i}{1 - e^{-x_i}} \frac{(-x_i)}{1 - e^{x_i}}; \end{aligned} \quad (13)$$

in addition, one can achieve

$$e(\Omega_{\mathbb{R}}^1) = x_1 x_2. \quad (14)$$

It is now a matter of representation theory of $SO(4)$ to determine the Chern characters of the exterior powers of $\Omega_{\mathbb{C}}^1$ and of the anti-self-dual part of the second exterior power. The problem is just to express the group theoretical characters of exterior powers and of the anti-self-dual second exterior power of the four-dimensional representation ρ_4 of $SO(4)$ in terms of the characters of ρ_4 .

Details of the Borel-Hirzebruch formalism for the computation of characteristic classes can be found in Ref. 6).

One gets right away

$$\text{ch } \Omega_c^2 = \text{ch } \Omega_{+c}^2 + \text{ch } \Omega_{-c}^2 = 2(1 + 2 \cosh x_1 \cosh x_2) \quad (15)$$

The computation of $\text{ch } \Omega_{-}^2$ is somewhat more involved. It can, for instance, be done by noticing that

$$\text{ch } \Omega_{+c}^2 - \text{ch } \Omega_{-c}^2 = \text{ch } \Lambda_{+} - \text{ch } \Lambda_{-} \quad , \quad (16)$$

where $\Lambda_{\pm}$ are those parts of $\Lambda = \oplus \Omega^p$ which are even and odd under the involutive operation $\alpha = i^{p(p-1)/2} *$.

This has to be the case, because for "non-central" dimension $p \neq 2$ there are isomorphisms between the even and odd parts, so that their contribution to the difference $\text{ch } \Lambda_{+} - \text{ch } \Lambda_{-}$ drops out. The quantity $\text{ch } \Lambda_{+} - \text{ch } \Lambda_{-}$ enters in the Hirzebruch signature theorem ^{5),6)}. One has

$$\text{ch } \Lambda_{+} - \text{ch } \Lambda_{-} = 4 \sinh x_1 \sinh x_2 \quad . \quad (17)$$

Combining Eqs. (12)-(17) one obtains

$$\begin{aligned} \text{ind} &= \left\{ 2 \text{ch } V_g \frac{(1 - \cosh x_1 - \cosh x_2 + \cosh(x_1 - x_2))}{x_1 x_2} \prod_{i=1}^2 \frac{x_i}{1 - e^{-x_i}} \frac{(x_i)}{1 - e^{x_i}} \right\} [M] \\ &= \left\{ -2 \text{ch } V_g \left(1 + \frac{x_1^2 + x_2^2}{6} - \frac{1}{4} x_1 x_2 + \dots \right) \left(1 - \frac{x_1^2 + x_2^2}{12} + \dots \right) \right\} [M] \\ &= \left\{ -2 \text{ch } V_g \left(1 + \frac{1}{12} p_1 - \frac{1}{4} e + \dots \right) \right\} [M] \quad , \quad (18) \end{aligned}$$

where we have inserted the definitions $p_1 = x_1^2 + x_2^2$ and $e = x_1 x_2$ for the Pontryagin class and the Euler class.

Their representation in terms of the geometric curvature tensor $K_j^i = \frac{1}{2} R_{jkl}^i dx^k \wedge dx^l$ is

$$r_1 = \frac{1}{8\pi^2} K_i^j \wedge K_j^i$$

and

$$e = \frac{1}{32\pi^2} \varepsilon_{ijkl} K^{ij} \wedge K^{kl} \quad (19)$$

Finally, $\text{ch } V_g$ has to be determined. One obtains from Eq. (8)

$$\text{ch } V_g = \dim G - c_2(V_g) + \dots \quad (20)$$

because the first Chern class of V_g has to vanish. This has to be inserted into Eq. (18). Defining an integer $K_G = -\int_M c_2(V_g)$ and taking into account that $\int_M e = \chi$, the Euler characteristic of M and $\frac{1}{2} \int_M p_1 = \tau$, the signature of M , and extracting from Eq. (18) the part which consists of four-forms on M , one arrives at the expression

$$\text{ind} = -2K_G + \frac{\dim G}{2} (\chi - \tau) \quad (21)$$

To compute K_G one has to find an expression for $c_2(V_g)$. Application of the Borel-Hirzebruch formalism⁶⁾ yields the result that

$$c_2(V_g) = 2C(G)a_2 \quad (22)$$

is an even integer multiple of another characteristic class a_2 which arises by induction from the classifying space B_T of the maximal torus T of G . An expression for a_2 in terms of the field strength F can also readily be obtained as well as the value of the integer $C(G)$. We do not give the derivation here, because it requires some representation theory of Lie groups and a deeper understanding of characteristic classes, and because the results coincide with the expression obtained in Ref. 4) by another method for the sphere S^4 : $C(G)$ is the Casimir operator of G and is given by

G	SU(n)	SO(n)(n≥7)	Sp(2n)	G ₂	F ₄	E ₆	E ₇	E ₈
C(G)	n	n - 2	n + 1	4	9	12	18	30
dim G	n ² -1	$\frac{1}{2}n(n-1)$	n(2n+1)	14	52	78	133	248

For completeness, we have also listed dim G.

The integer

$$k = \frac{1}{16\pi^2} \int_M F^i \wedge F^i \quad (24)$$

with

$$F = T^i F^i \quad \text{and} \quad T^i T^j = C(G) \delta_{ij} \quad (25)$$

If M is the four-sphere S^4 , the integer k can be interpreted as a "winding number", in the more general case k classifies maps from M into the classifying space B_G . If one cuts M into four-dimensional pieces such that every piece lies entirely in one gauge patch, the k can be represented by an integral over the boundaries between the pieces with an integrand, which depends on the transition functions g_{ij} between the patches i and j.

Our final expression for the number of parameters is

$$n = 4k C(G) - \frac{\dim G}{2} (\chi - \tau) + h_0 + h_2 \quad (26)$$

For $G = \text{SU}(2)$ and $M = S^4$ we have $\chi = 2$, $\tau = 0$, and the well-known formula $n = 8k - 3$ is reproduced. The complex-two-dimensional projective space $P(2)$ (four real dimensions) would give $\chi = 3$ and $\tau = 1$. To obtain our formula one could also try a linear ansatz $\text{ind} = ak + b\chi + c\tau$ and determine a, b, c by evaluating h_0, h_1, h_2 for three independent cases like $k=0, M=S^4$; $k=0, M=P(2)$ and $k=1, M=S^4$. For proving that this ansatz is justified, however, one has to use a reasoning similar to the one outlined above.

It is reassuring that in fact $\chi - \tau$ is an even number for every compact orientable four-dimensional manifold M :

The signature τ of M is given by the signature of the symmetric bilinear form on cohomology classes in $\Omega_{\mathbb{R}}^2(M)$

$$(\alpha, \beta) = \int_M \alpha \wedge \beta \quad (27)$$

i.e., the number $\gamma_+ - \gamma_-$ of positive minus negative diagonal values. By Poincaré's duality the form is non-degenerate. Denoting the Betti numbers of M by b_i , one has

$$\begin{aligned} \chi &= b_0 - b_1 + b_2 - b_3 + b_4 \\ \tau &= \gamma_+ - \gamma_- \\ b_2 &= \gamma_+ + \gamma_- \end{aligned} \quad (28)$$

By Poincaré's duality

$$\begin{aligned} b_0 &= b_4 \\ \text{and} \\ b_1 &= b_3 \end{aligned} \quad (29)$$

Thus

$$\chi - \tau = 2(b_0 - b_1 + \gamma_-)$$

REFERENCES

- 1) A. Schwarz - Phys.Letters 67B, 172 (1977).
- 2) R. Jackiw and C. Rebbi - Phys.Letters 67B, 189 (1977).
- 3) M. Atiyah, N. Hitchin and I.M. Singer - Proc.Nat.Acad.Sci. U.S.A. (1977).
- 4) C.W. Bernard, N.H. Christ, A.H. Guth and E. Weinberg - Columbia Preprint CU-TP-105 (1977).
- 5) M. Atiyah and I.M. Singer - Bull.Ann.Math.Soc 69, 422 (1963) ; Ann.Math. 87, 484 (1968).
- 6) R.S. Palais, Ed., Seminar on the Atiyah-Singer Index Theorem, Annals of Mathematical Studies, Vol. 57, Princeton University Press (1965).