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CLASSICAL SOLUTIONS OF GENERALLY INVARIANT
GAUGE THEORIES

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ABSTRACT

A new class of multimeron solutions is obtained
in the framework of a generally invariant gauge
theory.

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1. The important role that Yang-Mills gauge models are expected to play for a description of particle interactions has encouraged a detailed study of classical solutions for those theories¹⁾. A further reason of appeal is represented by their high character of symmetry, usual symmetries being combined with invariance under local gauge transformations and conformal transformations. Many interesting mathematical and physical features have thus emerged, like the topological classification of the solutions and their possible relevance to some problems of hadron physics. In particular, the importance has been recently stressed of the so-called "meron" configurations for an understanding of confinement²⁾. Merons are localized field configurations with one half unit of topological charge and in previous works^{3),4)} we have investigated their properties both in the Minkowski and Euclidean domains. The meron solution is of the general form

$$A_{\mu}(x) = \frac{1}{2} G^{-1}(x) \partial_{\mu} G(x) \quad (1)$$

This representation has several important features. First of all, as we shall discuss again in the following, its expression (1) is preserved by conformal transformations and $G(x)$ is transformed as a conformal scalar of zero dimensionality. Secondly, some interesting observable quantities become particularly simple in the meron case. Thus one finds easily that, as a consequence of Eq. (1), the strength tensor becomes

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + [A_{\mu}, A_{\nu}] = -[A_{\mu}, A_{\nu}] \quad (2)$$

If we further introduce the pseudoscalar density $D(x)$, which is the divergence of the topological current

$$D(x) = \frac{1}{4} \varepsilon^{\mu\nu\rho\sigma} \text{Tr. } F_{\mu\nu}(x) F_{\rho\sigma}(x) \quad (3)$$

it then follows from (2) that $D(x)$ is vanishing everywhere, apart, of course, from the points of singularity of the solution.

The most elementary expression of the meron solutions is given by

$$A_\mu = - \frac{\sigma_{\mu\nu} x_\nu}{x^2} \quad (4)$$

and corresponds to the choice

$$G(x) = \frac{\bar{s} \cdot x}{|x|}, \quad G^{-1}(x) = \frac{s \cdot x}{|x|} \quad (5)$$

where the matrices $s_\mu \equiv (1, i\vec{\sigma})$, $\bar{s}_\mu \equiv (1, -i\vec{\sigma})$ and $\sigma_{\mu\nu} = \frac{1}{4} (s_\mu \bar{s}_\nu - s_\nu \bar{s}_\mu)$ are related to the $SU(2)$ degrees of freedom. (For convenience, we work in this paper with a Euclidean metric.) The single antimeron solution corresponds to $G(x) \leftrightarrow \leftrightarrow G^{-1}(x)$.

This simple form can be improved and a new class of solutions generated by applying to (4) appropriate conformal transformations. It is a remarkable fact that the form of A_μ , given in Eq. (1), is preserved, all new parameters being embodied in the "superpotential" $G(x)$. In particular, the final form of $G(x)$ obtained in this way turns out to be

$$G(x) = \frac{\bar{s} \cdot (x-a)}{|x-a|} \frac{s \cdot (x-b)}{|x-b|} \frac{\bar{s} \cdot (a-b)}{|a-b|} \quad (6)$$

and it is easily shown to correspond to a meron at $x = a$ and to an antimeron at $x = b$ ^{*)}. The pseudoscalar density $D(x)$ receives contributions from these two singularities and the result is:

$$D(x) = \frac{1}{2} \left[\delta(x-a) - \delta(x-b) \right] \quad (7)$$

*) While (4) describes a meron located at the origin $x = 0$ with the antimeron moved to infinity.

The natural generalization of this result would be represented by a multiple meron and antimeron configuration.

The knowledge of these more general solutions would no doubt represent interesting information which seems, on the other hand, necessary for an understanding of the dynamics of confinement⁵⁾. In view of the form invariance of the representation (1) for the meron it is tempting to guess that an enlarged set of solutions might be obtained by allowing a more general expression for $G(x)$. One would thus determine such a class of functions by applying to the elementary solution (5) a set of co-ordinate transformations more general than the conformal ones. The validity of such a procedure requires, in principle, invariance of the underlying theory under the larger group of general co-ordinate representations:

$$X'_\mu = Z_\mu(x) \quad (8)$$

This unfortunately is not the case for the usual Yang-Mills theories.

In order to get confirmed in this point of view, it can be inspiring to look at two-dimensional models which exhibit such larger symmetry properties. Indeed, exact solutions of the multimeron type have first been explicitly obtained for the $O(3)$ non-linear σ model in two dimensions⁵⁾. Furthermore, in a very recent paper by Glimm and Jaffe⁶⁾, new classical solutions for $SU(2)$ Yang-Mills field equations in the Euclidean space are obtained. These solutions are shown to correspond to a set of merons and antimérons localized on the imaginary time axis. For this particular configuration the problem can be reduced to the case of an Abelian Higgs model for a two-dimensional curved space⁷⁾, where one can again take advantage of a larger group of symmetry. It is, in fact, well known that conformally invariant field theories in two dimensions possess a larger symmetry which is a consequence of the invariance under the general co-ordinate transformations (8).

The use of the group of general co-ordinate transformations in four-dimensional space-time is at the root of the method presented in this paper. Our work is based on a reformulation of gauge theories in an appropriate new framework, invariant under both local gauge and generalized co-ordinate transformations. We obtain a set of multiple meron and antimeron configurations, arising as classical solutions of such a generalized gauge theory. For a particular location of the merons and antimérons our solutions will also be solutions of the usual Yang-Mills equations and will coincide with those obtained by Glimm and Jaffe.

2. Our aim is the formulation of a gauge theory which is invariant under both general co-ordinate reparametrizations and local gauge transformations. In this undertaking we shall follow the pioneering work by Wilczek⁸⁾, where it was shown how a Riemann structure gives rise to a $O(4)$ gauge structure.

It is thus convenient to base our treatment on the so-called tetrad or vierbein formalism⁹⁾. These are vector fields $e^a(x)$ and are the simplest entities carrying internal symmetry indices "a" ($a = 1 \dots 4$) and co-ordinate indices μ (we work Euclidean, i.e., $\mu = 1 \dots 4$). Under co-ordinate transformations $x \rightarrow x' = Z(x)$, the vierbeine transform covariantly, i.e.,

$$e'^a_\mu(x') = \frac{\partial x'^\nu}{\partial x'^\mu} e^a_\nu(x) \quad (9)$$

while for local gauge transformations the rule is

$$\begin{aligned} e'^a_\mu(x) &= L^a_b(x) e^b_\mu(x) \\ L^a_b(x) L^b_c(x) &= \delta^a_c \end{aligned} \quad (10)$$

The internal symmetry indices "a" are raised and lowered by the Kronecker symbol δ_{ab} , while for the co-ordinate indices "μ" we have to introduce a metric tensor $g_{\mu\nu}(x)$, according to the known rules of tensor analysis. In analogy with the gravitational case we define:

$$g_{\mu\nu}(x) = \delta_{ab} e^a_\mu(x) e^b_\nu(x) \quad (11)$$

according to this definition $g^{\mu\nu}(x)$ is a covariant tensor and a gauge scalar. For its inverse $g^{\mu\nu}(x)$ we have $g_{\mu\nu}(x) g^{\nu\lambda}(x) = \delta^\lambda_\mu$ and so on.

The next step is the introduction of the $O(4)$ gauge potentials, whose definition is, as usual, related to the requirement of having gauge covariant derivatives. The appropriate expression turns out to be:

$$\begin{aligned} A_{\mu ab}(x) &= \left\{ e^c_\mu (\partial_a e_{bv} - \partial_b e_{av}) + \frac{1}{2} e^c_\mu e^d_\nu (\partial_a e_{cd} - \partial_d e_{ac}) e_{bv} \right\} \\ &\quad - (a \rightleftharpoons b) \end{aligned} \quad (12)$$

The gauge fields $A_{\mu ab}$ are covariant vectors^{*)} which behave in the familiar way under gauge transformations.

Let us finally define the covariant field tensor

$$F_{\mu\nu}^{ab} = \partial_\mu A_\nu^{ab} - \partial_\nu A_\mu^{ab} + A_\mu^{ac} A_\nu^b - A_\nu^{ac} A_\mu^b \quad (13)$$

Next, for convenience, we shall introduce the operator notation

$$A_\mu = \frac{1}{2} \sigma^{ab} A_{\mu ab} \quad (14)$$

where σ^{ab} is the matrix generator of the gauge transformation. Then (13) reads

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad (15)$$

We now move to establish the equations of motion of the theory. The precise form of these equations depends on the invariance properties we want to impose on the theory. The simple requirements of Lorentz (and conformal) invariance plus gauge invariance leads to the familiar Yang-Mills expression. The departure from the conventional theory occurs if we further demand the covariance of the equations of motion under general reparametrizations: the usual prescriptions of tensor calculus then yield

$$\partial_\mu (\sqrt{g} F^{\mu\nu}(x)) = \sqrt{g} [F^{\mu\nu}(x), A_\mu(x)] \quad (16)$$

where the metric tensor $g_{\mu\nu}$ has been used to build the contravariant quantity

*) Owing to the curl character of Eq. (12) the derivatives which appear do actually transform covariantly under reparametrizations.

$$F^{\mu\nu}(x) = g^{\mu\rho}(x) g^{\nu\sigma}(x) F_{\rho\sigma}(x) \quad (17)$$

and

$$g = \det g_{\mu\nu} \quad (18)$$

Equation (17) defines our model we shall call a generally invariant gauge theory. It is formally identical to a Yang-Mills theory in a curved space but the physical content is different. The quantities $g_{\mu\nu}(x)$ and $A_\mu(x)$ (or $F_{\mu\nu}$), one can consider as the effective fields, are not actually independent and according to Eqs. (11) and (12) they are entirely expressed in terms of the vierbeine $e_\mu^a(x)$, which are thus the fundamental dynamical variables of the theory.

The usual Yang-Mills theory corresponds then to the particularly simple case in which we insert in Eqs. (16) and (17)

$$g_{\mu\nu} \longrightarrow g_{\mu\nu}^0 = \delta_{\mu\nu} \quad (19)$$

Once the new framework is fixed, all physically interesting quantities can easily be rewritten according to these rules. $D(x)$ will still be given by Eq. (3).

We will now concentrate on the determination of an extensive set of classical solutions of Eq. (16).

3. The symmetry properties of the above theoretical scheme allow us to derive a full class of solutions of Eq. (16), once an elementary one is known, just by applying general co-ordinate transformations. Since the simplest solution available is the single meron one [Eq. (4)], the first step in this construction is to verify that it does indeed satisfy the covariant equation (16).

This is easily done, noticing that a vierbein of the simple form

$$e_{\mu}^a(x) = (x^2)^{-\frac{1}{2}} \delta_{\mu a} \quad (20)$$

gives

$$g_{\mu\nu} = \frac{1}{x^2} \delta_{\mu\nu} \quad (21)$$

$$A_{\mu} = - \sigma_{\mu\nu} x_{\nu} / x^2 \quad (22)$$

It is easy to verify that both the value of $g_{\mu\nu}$ given in the previous equation and the value $g_{\mu\nu}^0 = \delta_{\mu\nu}$ lead to the same field equations for A_{μ} , which then coincide with the familiar Yang-Mills equations^{*)}. As a consequence, the single meron solution (4) is also a solution of our generalized gauge theory.

Next, it is immediate to see that under a general reparametrization $x_{\mu} \rightarrow z_{\mu}(x)$, $A_{\mu} = \frac{1}{2} G^{-1} \partial_{\mu} G$ keeps its form unchanged, $G(x)$ transforming as a scalar, i.e., $G(x) \rightarrow G[z(x)]$. Therefore $A_{\mu}(x)$ as given by Eq. (1) is a solution of the generalized equation.

In order to have a physically meaningful characterization of a class of functions $G(x)$ we will take into account the properties of the topological current. Since the general result is that for our solution $D(x) = 0$ in the domain of regularity, the non-vanishing contributions will be concentrated at the singularities of $A_{\mu}(x)$ which derive from the zeros and the poles of $G(x)$. Following the hint provided by Eqs. (6) and (7) we shall concentrate on fractional functions $G(x)$. This class of functions corresponds of course to a given set of transformations $x \rightarrow z(x)$. On the other hand, the nature of the problem is such that, to dispose of the Lorentz and gauge degrees of freedom, one has to resort to the matrix formulation based on the "quaternions" s_{μ} and $\bar{s}_{\mu}$. Thus the final expressions of $G[z(x)]$ will be in terms of $s \cdot z$, leading therefore to an effective correspondence $s \cdot x \rightarrow s \cdot z(x)$.

The general form of $s \cdot z$ we will consider is expressed in terms of zeros and poles, namely $[(s \cdot a)^{-1} = \bar{s} \cdot a / a^2]$:

*) This is true for any $g_{\mu\nu} = \rho(x) \delta_{\mu\nu}$, in particular to $\rho(x) = 1/(1+x^2)$, leading to the elementary instanton solution.

$$S \cdot z = S \cdot (x-a_1) [S \cdot (x-b_1)]^{-1} S \cdot (x-a_2) \cdot \dots \quad (23)$$

Use of the simple properties of the $s_\mu, \bar{s}_\mu$ matrices, like $s_\mu \bar{s}_\nu + \bar{s}_\nu s_\mu = 2\delta_{\mu\nu}$ and similar ones, will lead to the explicit function $z(x)$ ^{*)}. [We omit, for simplicity, terms which do not bear any x dependence and which can easily be introduced by appropriate limiting processes ^{**) .}]

We can therefore conclude that the final form of G leading to a finite number of isolated singularities is:

$$G(x) = \frac{\bar{s} \cdot (x-a_1)}{|x-a_1|} \frac{s \cdot (x-b_1)}{|x-b_1|} \frac{\bar{s} \cdot (x-a_2)}{|x-a_2|} \dots \quad (24)$$

It is now easy to ascertain the behaviour of $A_\mu(x)$ around the singular points a_i, b_j . One finds that, apart from a constant gauge transformation (different from point to point)

$$\begin{aligned} \text{for } x \sim a_i, \quad A_\mu &\sim \frac{1}{2} \frac{s(x-a_i)}{|x-a_i|} \partial_\mu \frac{\bar{s}(x-a_i)}{|x-a_i|} \\ \text{for } x \sim b_i, \quad A_\mu &\sim \frac{1}{2} \frac{\bar{s}(x-b_i)}{|x-b_i|} \partial_\mu \frac{s(x-b_i)}{|x-b_i|} \end{aligned} \quad (25)$$

*) One can easily become convinced that the above choice of alternating s_μ and s_μ^{-1} matrices gives rise to a relativistically invariant relation $^\mu$ between x_μ and z_μ .

**) As a simple example, one can see that the transformation $s \cdot x \rightarrow s \cdot (x-a)$ $\{s \cdot (x-b)\}^{-1} s \cdot (a-b)$ corresponds to the conformal transformation

$$X_\mu \rightarrow \frac{(x-a)^2 (a-b)_\mu - (a-b)^2 (x-a)_\mu}{(x-b)^2}$$

which has been used to get from Eq. (5) the meron-antimeron solution of Eq. (6).

which confirms that the configuration based on $G(x)$ of Eq. (24) corresponds to having merons located at the points a_i and antimeron at the points b_j . This implies that

$$D(x) = \frac{1}{2} \left\{ \sum_i \delta(x - a_i) - \sum_j \delta(x - b_j) \right\} \quad (26)$$

Apart from this general characterization, the non-commutative nature of the matrices appearing in the formulae (23) or (24) makes the properties of the general solutions highly non-trivial and deserving further accurate investigation.

4. In the light of the previous remark, it is instructive to examine a simplified case in which all matrices appearing in $G(x)$ commute. This can be achieved when all singularities are arranged along the fourth axis. We then have

$$\bar{S} \cdot (x - a_i) = x_4 - a_i - i r \vec{\sigma} \vec{u} \quad ; \quad r = |\vec{x}| \quad ; \quad \vec{u} = \frac{\vec{x}}{r} \quad (27)$$

showing that all matrix factors are now commuting. Choosing cylindrical coordinates (in four dimensions) $t_{\pm} = x_4 \pm i r$, θ , ϕ , it is not hard to find the following results.

The transformation is identified by the relations

$$t_{\pm}(z) = f[t_{\pm}(x)] \quad ; \quad \{\theta, \varphi\}_z = \{\theta, \varphi\}_x \quad (28)$$

where

$$f(t_{\pm}) = \prod_i (t_{\pm} - a_i) / \prod_j (t_{\pm} - b_j) \quad (29)$$

Thus the gauge field takes on the beautiful form

$$A_\mu(x) = \frac{1}{2} e^{i\psi \vec{\sigma} \cdot \vec{u}} \partial_\mu e^{-i\psi \vec{\sigma} \cdot \vec{u}} \quad (30)$$

with

$$\psi = \frac{1}{2i} \ell_g \frac{f(t_+)}{f(t_-)} = \sum_i \arg(t_+ - a_i) - \sum_j \arg(t_+ - b_j) \quad (31)$$

and one can easily recognize that the time and radial components of A_μ in Eq. (30) coincide with those of Glimm and Jaffe, which have been however obtained as solutions of the original Yang-Mills equation and not of the present generally covariant scheme.

A detailed comparison between the two approaches will be very interesting.

5. The main conclusion which can be drawn from this paper is the great interest of theories which are invariant under general local space-time transformations. It is likely that the presence of a very general set of classical solutions might be the signal of a great simplification for the corresponding quantum mechanical problem. It is clear that the gauge theory discussed here (based on the fundamental use of vierbeine) can be viewed from a purely generally relativistic point of view. It does then coincide with the alternative theory of gravitation proposed by C.N. Yang and C.W. Kilmister¹⁰⁾. Our results indicate that a large set of classical solutions with interesting topological properties can be expected to exist for all generally invariant theories of gravitation.

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