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EXCHANGE EFFECTS AND THE TRACE OF THE ENERGY MOMENTUM TENSOR
IN WEAKLY BOUND STATES

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A B S T R A C T

A method for the computation of relativistic exchange effects in weakly bound states is proposed which also takes into account self-energy-type contributions. As an application the trace of the energy momentum tensor is computed on a nucleus.

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Matrix elements of observable operators on weakly bound states are usually computed within the so-called impulse approximation. Exchange corrections ¹⁾, however, may turn out to be essential when the expression is constrained by general principles.

A very nice example is provided by the trace of the energy-momentum tensor $\Theta_{\mu\nu}$. It is well known that ^{*)} :

$$\langle \underline{P} | \Theta_{\mu}^{\mu} | \underline{P} \rangle = \langle \underline{P} | \Theta_{00} - \Theta_{ii} | \underline{P} \rangle = \frac{E_P^2 - \underline{P}^2}{E_P} = \frac{M^2}{E_P} \quad (1)$$

where $|\underline{P}\rangle$ is a single particle of momentum $\underline{P}$ and energy E_P and Θ_{ii} is the spatial trace of $\Theta_{\mu\nu}$. Equation (1) implies, of course :

$$\langle \underline{P}=0 | \Theta_{ii} | \underline{P}=0 \rangle = 0 \quad (2)$$

If we now think of a weakly bound two-particle state such as for example the deuteron, its most natural description is like a superposition of two free nucleon states :

$$|D\rangle = \int d^3k_1 d^3k_2 \delta^3(\underline{P} - \underline{k}_1 - \underline{k}_2) f_0(k_1, k_2) \frac{(p_{k_1}^{+\uparrow} n_{k_2}^{+\uparrow} - n_{k_1}^{+\uparrow} p_{k_2}^{+\uparrow})}{2} |0\rangle \quad (3)$$

where $p_k^{+\uparrow}$ and $n_k^{+\uparrow}$ are the creation operators of spin-up proton and neutron. In order to be definite, the spin and isospin quantum numbers have been chosen to be those of the deuteron. The wave function $f_0(k_1, k_2)$ is assumed to be a pure s state, normalized, according to the footnote, as :

$$\int d^3k_1 d^3k_2 |f_0(k_1, k_2)|^2 \delta^3(\underline{P} - \underline{k}_1 - \underline{k}_2) = 1 \quad (4)$$

*) States are normalized according to :

$$\langle \underline{P} | \underline{P}' \rangle = (2\pi)^3 \delta^3(\underline{P} - \underline{P}')$$

In view of Eq. (1), we have :

$$\langle D_{\text{rest}} | \Theta_{ii} | D_{\text{rest}} \rangle = 2 \int d^3k \left| f(k, -k) \right|^2 \frac{k^2}{E} \neq 0 \quad (5)$$

in contradiction with (2).

It is clear that this result comes from the naïve two-body treatment of the deuteron, while relativistically it is to be taken into account that a bound state cannot be described in terms of a fixed number of constituents. In order to see how this feature may solve the paradox (5), let us take into account the influence of an isospin-one-pionic field coupled to nucleons through the usual γ_5 interaction

$$H_I = i g \int d^3x \bar{N} \gamma_5 \tau_i N \pi_i \quad (6)$$

To describe a bound state in such a theory we would have to solve, for the equation

$$(H_0 + H_I) |D\rangle = E_D |D\rangle \quad (7)$$

where

$$H_0 = \int d^3x \left\{ \bar{N} (-i \underline{\gamma} \cdot \underline{\nabla} + M_0) N + \frac{1}{2} [\dot{\pi}^2 + (\underline{\nabla} \pi)^2 + \mu_0^2 \pi^2] \right\}$$

is the free part of the Hamiltonian, $|D\rangle$ is the physical deuteron state of mass M_D and momentum $\underline{P}$ and $E_D = \sqrt{M_D^2 + \underline{P}^2}$ is its energy.

We are looking for a solution of (7) of the form :

$$|D\rangle = \sqrt{Z_D} \left[\int d^3k_1 d^3k_2 \delta^3(\underline{P} - \underline{k}_1 - \underline{k}_2) f_0(k_1, k_2) \frac{(P_{k_1}^{+\uparrow} n_{k_2}^{+\uparrow} - n_{k_1}^{+\uparrow} P_{k_2}^{+\uparrow})}{2} |0\rangle \right. \\ \left. + \int f_1 \begin{array}{c} \rightarrow \\ \cdots \\ \rightarrow \end{array} + \int f_2 \begin{array}{c} \rightarrow \\ \cdots \\ \rightarrow \\ \leftarrow \end{array} + \int f_3 \begin{array}{c} \rightarrow \\ \cdots \\ \rightarrow \end{array} + \int f_4 \begin{array}{c} \rightarrow \\ \cdots \\ \rightarrow \\ \leftarrow \end{array} + \dots \right] \quad (8)$$

The function f_0 is the usual two-body wave function of the deuteron and we have symbolically indicated it by $\rightarrow$, $\leftarrow$, $----$ a nucleon, an antinucleon and a pion, respectively; so that Eq. (8) shows that, besides the two-nucleon state, states with two nucleons and a pion, two nucleons, a pion and a nucleon-antinucleon pair, two nucleons and two pions, two nucleons and a nucleon-antinucleon pair with amplitudes f_1, f_2, f_3, f_4 , respectively, are taken into account.

In order to normalize the two-body wave function f_0 as in Eq. (4), an over-all normalization factor $\sqrt{Z_D}$ has been introduced which allows $|D\rangle$ to be normalized according to footnote of p. 1. Of course, series (8) does not come to an end. However, in order to be able to proceed, we limit ourselves to consider only the components explicitly shown in (8): this is usually known as the Tamm-Dancoff approximation²⁾. Equation (7) projected on the various states appearing in (8) gives rise to an infinite set of homogeneous coupled equations. The structure of such a system is such that, once truncated as we did, every f can be expressed in terms of f_0 and E_D which are then determined by a complicated homogeneous integral equation. We are not interested in the computation of the two-body wave function and energy eigenvalue because in the realistic situations it is already known that the pion is not enough to explain the nuclear binding.

Rather, in the spirit of exchange effect calculations once E_D and f_0 are given, Eqs. (7) and (8) allow us to compute the distribution of mesons, pairs, etc., inside the bound state. Of course in the same way, if other objects are coupled to nucleons, it is possible to compute their distribution inside the deuteron through analogous equations. In order to renormalize the infinities which inevitably arise within a relativistic approach, we will also expand f_1, f_2, f_3, f_4 in powers of the coupling constant g . It is easy to convince oneself that the states retained in (8) allow the computation up to the second order. In this approximation, we get, for example:

$$f_1(k_1, k_2, k_3) = \frac{g}{(2\pi)^{3/2}} \delta^3(\underline{p} - \underline{k}_1 - \underline{k}_2 - \underline{k}_3) \frac{1}{\sqrt{2}\omega_{k_3}} \left\{ \sqrt{\frac{m^2}{E_{k_1} E_{p-k_2}}} \right. \\ \left. \cdot \bar{u}_{k_1}^\dagger \gamma_5 \tau^i \frac{p_{-k_2}^\dagger f(p-k_2, k_2) + \text{antisymmetrization}}{(E_D - E_{k_1} - E_{k_2} - \omega_{k_3})} \right\} \\ \omega_{k_3} = \sqrt{\mu_0^2 + k_3^2} \quad (9)$$

In the first term inside the parentheses it must be taken into account that the particle with momentum k_2 is a neutron and this information must be used in the antisymmetrization. In general, it is easy to convince oneself that the expression for the f_i 's is obtained by the old perturbation theory for stationary states ³⁾ except for the fact that the energy of the initial state must be put equal to E_D and that they must be mediated with amplitude f_0 . Once the f_i 's are known, Z_D may be simply computed by the normalization condition as a function of E_D and f_0 . From now on, we take the deuteron at rest and therefore $E_D = M_D$. With this in mind we are ready to evaluate the matrix element of Θ_μ^μ between deuteron states at rest and compare it with M_0 . The calculation is performed with the help of ^{4),*)} :

$$\Theta_\mu^\mu = M_0 \bar{N} N + \mu_0^2 \pi_i \pi_i \quad (10)$$

and sandwiching it between the state (8).

The procedure is straightforward, but very lengthy, and the full analytic expression will be given elsewhere ⁵⁾. As already stated, the result is marred by logarithmic divergences. The most natural way to treat them is to realize that such divergences are those that arise when we try to compute the nucleon physical matrix elements in terms of the bare ones.

In order to eliminate the infinities, we then write for the nucleons an expansion similar to (8) by which we can write a (divergent) expression for $\langle N | \Theta_\mu^\mu | N \rangle$ ($|N\rangle$ denotes the physical nucleon state). We know that this divergent expression must represent the quantity M^2/E . This means that, in the same spirit as the Lamb shift calculation, we can subtract the divergent expression mediated with $2f_0^2$ and add the quantity $2\int f_0^2 (M^2/E) d^3k$ which it represents. With some precautions, this procedure leads to a convergent expression. The reason for this, let us say it again, is that the infinite part of the original expression is the same as the infinite part of the corresponding expression evaluated on free nucleons superposed with amplitude f_0 . The finite parts are, of course, different and they represent the computable difference between the bound nucleon and the physical free nucleon. Since now the expansion is around physical nucleons and not the bare ones, it is not surprising that the effective coupling

*) The presence of anomalies does not affect the result to this order.

constant in this expansion is not $g^2/(2\pi)^3$ anymore, but rather $(g^2/(2\pi)^3)(B/M_D)$ where B is the binding energy of the deuteron. This means that for weakly bound states a perturbation approach may still make sense.

The fact that we limit ourselves to second order terms in g^2 may be justified by thinking that for distant nucleons only the lightest states are likely to be disturbed by the nearby nucleon, while the heaviest states are only needed to dress the bare nucleon to become physical. The details of this procedure will be shown in Ref. 5).

In order to see how the presence of constituents in the nucleus other than nucleons themselves solve the contradiction found in Eq. (5), it is sufficient to consider the so-called static limit $M \rightarrow \infty$. In this limit any difference between the bound nucleons and the free ones may be neglected. This is reflected in the exact cancellation of all the self-energy-type effects and the resulting expression only comes from impulse and exchange-type contributions shown in the Figure, where

$$(Z_D - 1)_{\text{exch}} = -\frac{g^2}{(2\pi)^3} \int d^3k_1 d^3k_2 f_0(k_1, -k_1) f_0(k_2, -k_2) \frac{(k_1 + k_2)^2}{4M^2 \omega_{k_1+k_2}^3} \quad (11)$$

is the exchange contribution to $Z_D - 1$. The terms (a) and (b) in the Figure simply represent the impulse approximation to the mean value of ϕ_μ^μ :

$$\langle D | \phi_\mu^\mu | D \rangle_{\text{imp. approx.}} = 2 \int d^3k f_0^2(k, -k) \frac{M^2}{E_k} \approx 2M - 2 \frac{\langle k^2 \rangle}{2M} \quad (12)$$

where the expression on the right-hand side is the non-relativistic limit corresponding to the static approximation. The meaning of Eq. (12) is clear: it says that, in the absence of exchange corrections, the binding energy is exactly equal to the mean value of the kinetic energy of the nucleons. This is nothing else than the usual answer the virial theorem for a massless exchanged pion would give. Let us see the remaining contributions of the Figure. Diagrams (c), (d), (e) and (f) exactly cancel in the static limit. In the literature they are usually omitted because they represent a kind of double counting. In fact diagrams (e) and (f) are in part contained in the definition of the wave function. It is to be noted

that if we do not take the non-relativistic limit the cancellation is not exact and a certain correction is obtained ⁵⁾.

The contribution of (g), (h),... reads in the static limit

$$\begin{aligned} \langle D | \mathcal{O}_\mu^M | D \rangle_{\text{exch}} = & -\frac{g^2}{(2\pi)^3} \int d^3k_1 d^3k_2 f_0(k_1, -k_1) f_0(k_2, -k_2) \\ & \cdot \left[\frac{(k_1 + k_2)}{2M^2} \frac{\mu_0^2}{\omega_{k_1+k_2}^2} - \frac{(k_1 + k_2)^2}{2M^2 \omega_{k_1+k_2}^2} \right] \end{aligned} \quad (13)$$

In order to make clear the meaning of Eq. (13), it is useful to express it in terms of $\psi(x)$, the Fourier transform of f_0 :

$$\langle D | \mathcal{O}_\mu^M | D \rangle_{\text{exch}} = \frac{g^2}{4\pi} \frac{\mu_0^2}{4M^2} \int \mu_0 e^{-\mu_0|x|} |\psi(x)|^2 d^3x \quad (14)$$

Terms proportional to Dirac δ functions have been dropped for simplicity assuming the wave function $\psi(x)$ to be zero at the origin.

It is easy to recognize :

$$\langle D | \mathcal{O}_\mu^M | D \rangle_{\text{exch}} = \int |\psi(x)|^2 \left[\left(x \frac{\partial}{\partial x} + 1 \right) V(x) \right] d^3x \quad (15)$$

where

$$V = -\frac{g^2}{4\pi} \frac{\mu_0^2}{4M^2} \frac{e^{-\mu_0|x|}}{|x|} \quad (16)$$

is the usual form of the Yukawa potential due to one-pion exchange.

This means that the total binding energy has the form

$$B = M_D - 2M = -\langle T \rangle + \langle V \rangle + \left\langle x \frac{\partial}{\partial x} V \right\rangle \quad (17)$$

where T is the kinetic energy of the nucleons. Now in virtue of the virial theorem known from non-relativistic potential theory ³⁾ this expression is $\langle T \rangle + \langle V \rangle$, namely the total energy of the bound state counted from the mass of its constituents. This solves the problem of the self-stress of the deuteron because the average of the trace of the energy momentum tensor is the same as the mean value of Θ_{00} , i.e., the Hamiltonian.

This approach to exchange corrections, as already stated, is not limited to the non-relativistic limit, but rather shows its power in the fully relativistic situation, where intuitive concepts such as the deformation of the mesonic cloud of a bound nucleon, may be given a rigorous quantitative meaning. Applications to nuclear physics will be discussed in a forthcoming paper ⁵⁾.

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FIGURE CAPTION

The cross represents the action of θ_{μ}^{μ} .

+

+

$$+ (Z_D - 1)_{\text{exch}} \left(\begin{array}{c} \text{---} \times \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \times \text{---} \end{array} \right) +$$

(c) (d)

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+

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+

+

+

12
H