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RECONSTRUCTION OF THE DEEP INELASTIC
STRUCTURE FUNCTIONS FROM THEIR MOMENTS

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ABSTRACT

A mathematically rigorous and extremely easy to handle method for inverting moments is presented. The method is exemplified by reconstructing the structure function νW_2^{ep} of deep inelastic scattering from its moments as given by asymptotic freedom.

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As is well known, the property of asymptotic freedom¹⁾ enjoyed by the colour quark-gluon theory (QCD) of strong interactions allows us, among other things, to calculate the (logarithmic) corrections to naïve Bjorken scaling for the deep inelastic scattering structure functions. When comparing theory with experiment, however, a difficulty appears: for the first only gives direct predictions for the moments of the structure functions²⁾. For example, if $F(x, Q^2) = F_2 = vW_2$, and we define its moments

$$\mu_n(Q^2) = \int_0^1 dx F(x, Q^2) x^n, \quad (1)$$

then, to a precision $O(\alpha_c^2(Q^2))$, QCD tells us that (for the valence part)

$$\mu_n(Q^2) = \mu_n(Q_0^2) \times \frac{3\pi + (H_{n+2} + P_{n+2} + L_{n+2}(Q^2))\alpha_c(Q^2)}{3\pi + (H_{n+2} + P_{n+2} + L_{n+2}(Q_0^2))\alpha_c(Q_0^2)} \left(\frac{\log Q^2/\Lambda^2}{\log Q_0^2/\Lambda^2} \right)^{-\gamma_{0n+2}/2\beta_0}. \quad (2)$$

Here Q_0^2 is a reference momentum and $\alpha_c(Q^2)$ the effective gluon coupling constant¹⁾ [$\alpha_c = \bar{g}^2/4\pi$ with the notation of Refs. 1)]. Λ is an energy parameter; experimentally^{3), 4), 5)} $\Lambda \sim 0.15$ to 0.45 GeV. Explicit expressions for the H, P, L, γ, β may be found in Ref. 5). Finally, $\mu_n(Q_0^2)$ is supposed to be taken from experiment. Since what we require are the values of $F_2(x, Q^2)$ at specific values of x, Q^2 , we are led to a problem of inverting moments.

In the existing literature this problem has been tackled in two different ways. Method (I): One may invert (1) and (2) by brute force^{2), 6)}. Replacing Eq. (2) by its lowest order approximation²⁾, we have

$$F(x, Q^2) = \int_x^1 dx' F\left(\frac{x}{x'}, Q_0^2\right) T\left(\frac{\log Q^2/\Lambda^2}{\log Q_0^2/\Lambda^2}, x'\right). \quad (3a)$$

The kernel T is given by

$$T(\lambda, x) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \lambda^{-A_z} x^{-1-z}, \quad (3b)$$

where A_z is the analytic continuation of $\gamma_{0n}/2\beta_0$ to complex $n \rightarrow z$. This method presents a number of serious drawbacks: To obtain T one requires knowledge of all the moments; Eqs. (3) are quite unstable; it is impossible to find, other than numerically, the value of T beyond the lowest order approximation. This has led people to use a second type of approach^{5),7)}. Method (II): Assume a definite functional form, depending on certain parameters, for different contributions to $F(x, Q^2)$ and fix these parameters by fitting the moments. Now, while this method simplifies the calculations, these still remain complicated and moreover, not only the problem of unstabilities is not solved (just bypassed) but choice of specific functional forms implies potentially dangerous biases^{*)}.

In view of this we think it useful to introduce a third method which possesses the advantage of being at the same time model independent, mathematically rigorous, and extremely simple from a computational point of view. It presents the drawback that it is unable to represent the singularities of F and, therefore, its behaviour at the endpoints $x = 0, 1$ for which you have to fall back on previous methods (I) and (II). Our method has been described from a mathematical point of view elsewhere [Ref. 8]; see also Ref. 9)] and applied successfully to other branches of high energy⁸⁾ or solid state physics¹⁰⁾ but we believe it is the first time it is used for the problem at hand.

We will assume that using, say, Eq. (2), we have calculated the moments $\mu_0, \mu_1, \dots, \mu_N$ and will then show how to reconstruct F . This we will do in several steps, according to the increasing assumptions we make about this function. To begin with, errors in the μ_n shall be neglected; a general discussion about errors will be found at the end of this note.

*) For example, with the functional form of, e.g., xV_8 in Ref. 7),

$$x V_8(x, Q^2) = (\text{constant}) x^{\eta_1} (1-x)^{\eta_2},$$

one links artificially the behaviour of xV_8 at the boundaries, $x^{\eta_1}, (1-x)^{\eta_2}$ with the fact that it necessarily is maximum at $\bar{x} = \eta_1/(\eta_1 + \eta_2)$. On the other hand, I am aware that this works both ways, and Method (II) is a good way for pumping in/extracting useful physical information on the different contributions (valence quarks, ocean, gluons) to $F(x, Q^2)$.

Step 1: No assumptions. Reconstruction in the average

It is well known^{8),9)} that, in general, point-like reconstruction of F is impossible. Fortunately what experimentalists actually measure is not $F(x)$ ^{*}) at exactly the point x , but (because of uncertainties in v, Q) an average of F around the point x : this is what is then required of the theoretical prediction.

Posed this way the problem ceases to be unstable. It is possible to reconstruct averages of F with any reasonable weight function; but we will choose a particular set, optimal for the problem at hand⁸⁾. This set will be the (normalized) Bernstein polynomials^{8),9)}

$$b^{(N,k)}(x) = \frac{(N+1)!}{k!} \sum_{l=0}^{N-k} \frac{(-1)^l}{l! (N-k-l)!} x^{k+l}, \quad k = 0, 1, \dots, N. \quad (4a)$$

The polynomial $b^{(N,k)}(x)$ acts as weight over the point

$$x_{N,k} = \int_0^1 dx \, b^{(N,k)}(x) x = \frac{k+1}{N+2}; \quad (4b)$$

so we may define the average of F around $x_{N,k}$ as

$$\tilde{F}(x_{N,k}) = \int_0^1 dx \, b^{(N,k)}(x) F(x). \quad (5a)$$

Using now (4a), we see that this may be written simply as

$$\tilde{F}(x_{N,k}) = \frac{(N+1)!}{k!} \sum_{l=0}^{N-k} \frac{(-1)^l}{l! (N-k-l)!} \mu_{k+l}, \quad (5b)$$

i.e., $\tilde{F}$ is immediately obtained from the moments.

^{*}) We will drop the Q^2 dependence whenever it be irrelevant.

The proof that the $b^{(N,k)}$ are suitable weight functions may be found in a number of references^{8),9)}. Intuitively it is obvious if we realize that they may be rewritten as

$$b^{(N,k)}(x) = \frac{x^k (1-x)^{N-k}}{\int_0^1 dx \cdot x^k (1-x)^{N-k}}.$$

Therefore the b are positive, normalized to $\int b = 1$ and with a unique maximum near^{*)} $x_{N,k}$. The "focusing power" of $b^{(N,k)}$ is given by the inverse of the dispersion, $\Delta_{N,k}$,

$$\begin{aligned} \Delta_{N,k}^2 &= \left| \int_0^1 dx \cdot b^{(N,k)}(x) x^2 - \left\{ \int_0^1 dx \cdot b^{(N,k)}(x) x \right\}^2 \right| \\ &= \frac{N-k+1}{(N+2)^2 (N+3)}. \end{aligned} \quad (6)$$

The points at which the average is taken, $x_{N,k}$, may be varied by varying N and k to cover the interval $(0,1)$ in $\sim 1/2N$ to $\sim 1/3N$ bins with precision given by (6). An example of reconstruction of $F_2^V(x, Q^2) = W_2^{ep}(u_V \& d_V)$ (this means that only the valence quark contribution is taken into account) with $N = 7, 8$, and 9 , the moments given by Eq. (2) [as calculated in Ref. 5)] is shown in Fig. 1 for fixed Q^2 , varying x , and Fig. 2 for fixed x , varying Q^2 . Once the moments were known, the whole reconstruction took an afternoon with a pocket calculator! With $N \sim 50$, which any decent computer can handle, reconstruction with dispersion inferior to the experimental one¹¹⁾ ($\sim 1\%$) is easily obtained.

It may be proved⁸⁾ that this is the best you can do without extra assumptions. It is a lot, but with reasonable hypotheses we can still go further.

*) Actually the maximum of $b^{(N,k)}(x)$ lies at $x_{\max} = k/N$. This distortion ($x_{\max} \neq x_{N,k}$) is indeed necessary since any reasonable weight should not extend beyond the endpoints $x = 0, 1$.

Step 2: Semi-bounded $\partial F/\partial x$. Upper and lower bounds for F_2 .

We assume that there exists a finite constant^{*)} L such that either

$$\frac{\partial F(x)}{\partial x} + L \geq 0 \quad \text{or} \quad -\frac{\partial F(x)}{\partial x} + L \geq 0,$$

then it is possible to obtain very tight upper and lower bounds to $F(x)$ itself. Suppose, e.g., the case is that $-\partial F/\partial x \geq L$; and let $\rho = L - \partial F/\partial x \geq 0$. By partial integration we obtain

$$\int_0^1 dx \, \rho(x) x^n = n \mu_{n+1} + \frac{L}{n}, \quad n \geq 1. \quad (7)$$

On the other hand

$$Lx - F(x) = \int_0^1 dx' \, \rho(x') \theta(x-x'); \quad (8)$$

to obtain upper and lower bounds for the left-hand side of (8), and hence for F is a standard problem in the theory of moments⁹⁾. What one essentially does is find polynomials $p_{\pm}(x')$ of degree $\leq N-1$ with

$$p_{-}(x') = \sum C_{-}^{(n)} x'^n \leq \theta(x-x') \leq p_{+}(x') = \sum C_{+}^{(n)} x'^n,$$

so then, since ρ is positive, we may multiply it by this inequality and integrate to obtain the desired bounds in terms of the μ_n . Of course the polynomials $p_{\pm}$ have to be optimized^{9),10)}. This has been worked out in detail for the densities of states of solids¹⁰⁾ (functions with shapes very similar to F_2): no example of application to our case is presented here because, except from a conceptual point of view, such bounds are less useful than the results just discussed or those we will discuss in the following. (The reason is that, due to computational difficulties, it is hard to go beyond $N \sim 12$ moments.)

*) The case in which L is not constant, but a known function $L(x)$ is quite similar.

Step 3: Twice differentiable F. Point-like reconstruction ^{*}.

Under the assumption that F is twice differentiable except perhaps at the endpoints ^{**}) we can obtain very approximate point-wise reconstruction. In fact, in the neighbourhood $|x - x_{N,k}| \leq \Delta_{N,k}$, we may write, with an error $\sim \Delta_{N,k}^3$,

$$F(x) \simeq F(x_{N,k}) + (x - x_{N,k}) \frac{\partial F(x_{N,k})}{\partial x} + \frac{(x - x_{N,k})^2}{2} \frac{\partial^2 F(x_{N,k})}{\partial x^2}. \quad (9)$$

Integrating with $b^{(N,k)}(x)$ and using (4b), (5a) and (6) we get at once

$$F(x_{N,k}) \simeq \tilde{F}(x_{N,k}) - \frac{1}{2} \Delta_{N,k}^2 \frac{\partial^2 F(x_{N,k})}{\partial x^2}.$$

We may, to an acceptable $O(\Delta_{N,k}^3)$ error, replace $\partial^2 F(x_{N,k})/\partial x^2$ by $\partial^2 \tilde{F}(x_{N,k})/\partial x^2$. The last can be calculated (for $k = 2, 3, \dots, N$) by the same method as $\tilde{F}$ before: in fact, integrating by parts,

$$\int_0^1 dx \frac{\partial^2 F(x)}{\partial x^2} x^n = n(n-1) \mu_{n-2},$$

so finally, with $\tilde{F}(x_{N,k})$ given by Eq. (5b)

$$F(x_{N,k}) = \tilde{F}(x_{N,k}) + \delta(x_{N,k}),$$

$$\delta(x_{N,k}) = \frac{1}{2} \frac{(N+1)! (N-k+1)}{k! (N+2)^2 (N+3)} \sum_{l=0}^{N-k} \frac{(-1)^l (k+l)(k+l-1)}{l! (N-k-l)!} \mu_{k+l-2} \quad (10)$$

^{*}) This method is presented here for the first time.

^{**}) This assumption fails in general for the densities of states of solids (Van Hove singularities), but is very probably valid for our case.

The results for the case $F = \sqrt{W_2}$ are shown in Figs. 1a and 2b. (For 2a δ is negligible; for 1b the result would be as for 2b.) We see that even with as few as ten moments the method is extremely effective.

Errors:

- i) First of all we have rounding errors. These are of the order

$$(N-k)! / \left[\frac{N-k}{2}! \right]^2 \sim 2^{N-k}$$

times the relative error in the μ_n . This is what limits to ~ 50 the number of moments that may be used.

- ii) Errors in the input $\mu_n(Q_0^2)$ and in the theoretical expression for the μ_n should be transmitted stably⁸⁾ to the reconstructed $F, \tilde{F}$. However, no useful explicit formula may be given; the best practical procedure¹⁰⁾ is to effect random variations of the μ_n inside their error bars, and see how this affects $F, \tilde{F}$.
- iii) It is obvious that none of the methods proposed are able to reproduce singularities; in particular the endpoint behaviour has to be obtained by Methods (I) and (II) described at the beginning.
- iv) The error in Eq. (10) is hard to obtain without more assumptions. A reasonable guess, however, is to take it $\sim \tilde{F}(x_{N,k}) [\delta(x_{N,k}) / \tilde{F}(x_{N,k})]^{3/2}$.

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FIGURE CAPTIONS

Fig. 1 : Comparison of the reconstructed F_2^V with its exact or fitted value.

a) Continuous line: exact $F_2^V(x, Q_0^2)$ (input). Black dots: reconstructed averages. Triangles: point-like reconstruction Horizontal bars: dispersion of the averages. $N = 8$.

b) Continuous line: extrapolated $F_2^V(x, Q^2)$ using Method II (Ref. 7)). Small circles: experimental points from Ref. 12). Broken lines: fits to the whole νW_2 (Ref. 7)). Black triangles: reconstructed averages, $N = 7$; black dots: id., $N = 8$; black squares: id., $N = 9$. In both cases only the valence quark contributions (u_v, d_v) have been taken into account for the reconstruction and for F_2^V itself.

Fig. 2 : Comparison of the reconstructed F_2^V as a function of Q^2 . Black points: experimental data from Ref. 11). Broken line: fitted F_2^V (Ref. 5)). Solid lines: a) reconstructed averages. b) reconstructed averages and point-like values. $N = 8$.

FIG. 1 (a)

FIG.1 (b)

FIG.2