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CRITICAL EXPONENTS FOR THE REGGEON QUANTUM SPIN MODEL ^{*)}

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ABSTRACT

The Reggeon Quantum Spin (RQS) model on the transverse lattice in D dimensional impact parameter space has been conjectured to have the same critical behaviour as the Reggeon Field Theory (RFT). Thus from a high "temperature" series of ten ($D = 2$) and twenty ($D = 1$) terms for the RQS model we extrapolate to the critical temperature $T = T_c$ by Padé approximants to obtain the exponents $\eta = 0.238 \pm 0.008$, $z = 1.16 \pm 0.01$, $\nu = 1.271 \pm 0.007$ for $D = 2$ and $\eta = 0.317 \pm 0.002$, $z = 1.272 \pm 0.007$, $\nu = 1.736 \pm 0.001$, $\lambda = 0.57 \pm 0.03$ for $D = 1$. These exponents naturally interpolate between the $D = 0$ and $D = 4-\epsilon$ results for RFT as expected on the basis of the universality conjecture.

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Much effort has been devoted in recent years to abstracting from the Reggeon Field Theory (RFT) a description of rising cross-section in terms of a phase transition in the multiple Pomeron exchange processes¹⁾. This critical Pomeron transition may be more fundamental than any of the details of the RFT. For example, in analogy with second order phase transitions of ferromagnets, where the short distance details are lost near the critical temperature $T \approx T_c$, we might understand the critical Pomeron transition as a general phenomena present in a large class of models. Since such a universality class of models is determined by basic properties of symmetry and dimensionality we might also then hope that the critical Pomeron transition but not the details of RFT will eventually be derived from fundamental properties of the theory of strong interactions.

It behoves us to study the critical Pomeron in the simplest possible model of its universality class. Just as the Landau-Ginzburg theory for ferromagnets in d dimensions may be replaced by the more tractable Ising quantum spin model on a $D = d - 1$ dimensional transverse lattice²⁾ with identical power law singularity at the critical points, it has been shown that the Reggeon field theory¹⁾ for the Pomeron field $\psi(\vec{x}, y)$

$$H_{RFT} = \int d^D x \left[\Delta_0 \bar{\psi} \psi + i \frac{r_0}{2} \bar{\psi} (\bar{\psi} + \psi) \psi + \alpha_0' \vec{\nabla} \bar{\psi} \cdot \vec{\nabla} \psi \right] \quad (1)$$

can be replaced by the RQS model^{3),4)} on a D dimensional lattice in impact parameter space $\vec{x} = (x_1, x_2, \dots, x_D) = b \vec{n}$

$$H_{RQS} = \delta \sum_{\vec{n}} N_{\vec{n}} + A \sum_{\langle \vec{n}, \vec{m} \rangle} (\sigma_{\vec{n}}^{\bar{\psi}} - \sigma_{\vec{m}}^{\bar{\psi}})(\sigma_{\vec{n}}^{\psi} - \sigma_{\vec{m}}^{\psi}) \quad (2)$$

Here δ is the single site gap (replacing $\Delta_0 = 1 - \alpha_0$) and A is the inter-site coupling (replacing α_0' the Pomeron slope and r_0 the triple Pomeron coupling). The spin matrix representations of the fields ψ and $\bar{\psi}$ are:

$$\sigma_{\vec{n}}^{\psi} = C_{\vec{n}} + N_{\vec{n}}, \quad \sigma_{\vec{n}}^{\bar{\psi}} = C_{\vec{n}}^+ - N_{\vec{n}} \quad (3a)$$

where

$$C_{\vec{n}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_{\vec{n}}, \quad C_{\vec{n}}^{\dagger} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_{\vec{n}}, \quad N_{\vec{n}} = C_{\vec{n}}^{\dagger} C_{\vec{n}}$$

Occasionally it is also convenient to generalize this representation of RQS by introducing an irrelevant coupling θ [corresponding to adding $(\psi^{\dagger}\psi)^2$ in (1)] through generalized spin matrices

$$\begin{aligned} \sigma_{\vec{n}}^{\psi}(\theta) &= \sqrt{2} \cos(\frac{\theta}{2}) C_{\vec{n}} + \sqrt{2} \sin(\frac{\theta}{2}) N_{\vec{n}} \\ \sigma_{\vec{n}}^{\bar{\psi}}(\theta) &= \sqrt{2} \cos(\frac{\theta}{2}) C_{\vec{n}}^{\dagger} - \sqrt{2} \sin(\frac{\theta}{2}) N_{\vec{n}} \end{aligned} \quad (3b)$$

so that

$$N_{\vec{n}} = \sigma_{\vec{n}}^{\bar{\psi}} \sigma_{\vec{n}}^{\psi} / 2 \cos \theta$$

At $\theta = 0$ we have a Hermitean model corresponding to a pure quadratic coupling ($r_0 = 0$). However, the RFT model is always regarded as taking $\theta = \pi/2$.

The basic structure of the two models of Eqs. (1) and (2) are very similar. The "Hamiltonian" H generates boosts in rapidity ($y = \log s$) and has a spectrum that gives the j plane singularities. At high $T = \delta/A$ ($\alpha_0 < \alpha_{0c}$) they both have a subcritical (disordered) phase where the leading singularity is a Regge pole at $E = 1 - j = \Delta > 0$,

$$G^{1,1}(E, k) = \frac{i}{E - \Delta - \alpha'_0 k^2} \quad (4)$$

and the cross-section (i.e., correlation function) goes to zero exponentially as $y = \log s \rightarrow \infty$

$$\sigma(y) \sim \int_{C-i\infty}^{C+i\infty} \frac{dE}{2\pi} e^{-yE} G^{1,1}(E, 0) \sim e^{-y\Delta} \quad (5)$$

Near the critical surface ($T \gtrsim T_c$) the correlation length $\xi = \Delta^{-1} \sim (T - T_c)^{-\nu}$ becomes large and the amplitude has the scaling form

$$G^{(1)}(y, \vec{x}, \tau) = y^{2-\frac{D}{2}} F(\vec{x}^2 y^{-2}, (\tau - \tau_c) y) \quad (6)$$

which at $T = T_c$ gives increasing cross-section ($\eta > 0$)

$$\sigma(y) \propto y^2 \quad (7)$$

Combining this result and the scaling behaviour of higher Green's functions $G^{n,m}$ many predictions have been made for the asymptotic behaviour of elastic and in-elastic differential cross-sections¹⁾.

Both models in Eqs. (1) and (2) have a specific "Reggeon non-Hermitean" property demanded by t channel unitarity

$$H^\dagger = U H U^\dagger \quad (8)$$

where

$$U = \exp(-i\pi \int d^D x \bar{\Psi} \Psi) \quad (RFT) \quad (9)$$

$$U = \exp(-i\pi \sum_{\vec{n}} N_{\vec{n}}) \quad (RQS)$$

In RFT perturbative Reggeon unitarity is manifest, and we believe that both models possess non perturbative unitarity for the subcritical domain [$\alpha(0) < 1$ or $T > T_c$] as a consequence of (8) and the lack of renormalization of the bare vacuum $H|0\rangle = |0\rangle$.

In this letter we present a precise evaluation of the critical exponents for the RQS model ($\theta = \pi/2$) and compare them with exponents calculated in RFT. We use the most reliable and accurate technique of statistical mechanics⁵⁾, namely a high order perturbation expansion in the intersite coupling A/δ (or high "temperature" $T = \delta/A$ expansion). We expect⁶⁾ the series to diverge at $T \rightarrow T_c \sim \sim 0(1)$ so we extrapolate back to that point.

To calculate the exponents η , z , ν for $D = 1$ and $D = 2$ we consider the "generalized susceptibilities"

$$\chi_{pq}(\tau) = \frac{\partial^q}{\partial E^q} \int_0^\infty dy e^{Ey} \sum_{\vec{n}} (\vec{n}^2 + 1)^{\frac{p}{2}} G^{1,1}(y, \vec{n}, \tau) \Big|_{E=0} \quad (10)$$

which upon substitution of the scaling form of $G(y, \vec{n}, T)$ from Eq. (6) must diverge for $T \rightarrow T_c$ like

$$\chi_{pq}(\tau) \propto (T - T_c)^{-\gamma(p,q)} \quad (11a)$$

where

$$\gamma(p,q) = \nu(\eta + 1 + q + p \frac{z}{2}) \quad (11b)$$

By studying the singularity of $\chi_{pq}(T)$ for different values of p, q we can calculate T_c , η , z and ν . The high temperature expansion is simply the power series expansion of $\chi_{pq}(T)$ about $K = T^{-1} = 0$

$$\chi_{pq}(\tau) = \sum_{\ell=0}^N T^{-\ell} a_{\ell}(p,q) \quad (12)$$

which has to be extrapolated to $T = T_c$ in order to determine $\gamma(p,q)$.

The propagator in Eq. (10)

$$G(E, \vec{n}, \tau) = \int_0^\infty dy e^{Ey} G(y, \vec{n}, \tau) = \langle 0 | \sigma_{\vec{n}}^{\psi} \frac{1}{E - H_0 - KV} \sigma_{\vec{0}}^{\bar{\psi}} | 0 \rangle \quad (13)$$

(with $H_0 = \sum_{\vec{n}} N_{\vec{n}}$ and $KV = H/\delta - H_0$) is expanded in power series in $K = T^{-1}$. The lattice Feynman diagrams for each order are enumerated and evaluated by a computer program. Details of the calculations will be presented in a forthcoming article⁷⁾.

For $D = 2$ and 1 the calculations of χ_{pq} have been performed up to $N = 10$ and $N = 20$ terms for $q = 0, 1, 2$ and $p = 0, \pm 0.2, \pm 0.4, \pm 0.5$, respectively. For example, for $p = q = 0$ we have the series ($K = T^{-1}$)

$$\begin{aligned} -\chi_{00}^{D=2}(K) = & 1 + 4K + 12K^2 + 36K^3 + 96K^4 + \frac{2444}{9}K^5 \\ & + 682.9629K^6 + 1927.6790K^7 + 4544.5113K^8 \\ & + 13397.7337K^9 + 28438.5927K^{10} + \dots \end{aligned}$$

similarly for $D = 1$ we have series for $\chi_{pq}^{D=1}(K)$ up to order K^{20} .

A preliminary method for finding the location ($K_c = T_c^{-1}$) and degree [exponent $\gamma(p, q)$] of the singularity which causes the power series to diverge is the ratio test for large ℓ :

$$R_\ell(p, q) \equiv \frac{a_\ell(p, q)}{a_{\ell-1}(p, q)} \longrightarrow \frac{1}{K_c} \left[1 + \frac{1}{\ell} (\gamma(p, q) - 1) \right] \quad (14)$$

We find, however, in our case (see Fig. 1a) that an unexpected singularity at $T_1 \approx -2D$ controls the divergences since it is closer to $K = 0$ than is T_c^{-1} . This difficulty can be overcome by transforming to a better definition of temperature

$$T_B = T + B \quad \left(K_B = \frac{K}{1 + KB} \right)$$

The new expansion coefficients $b_\ell(p, q, B)$ can be calculated directly from the $a_\ell(p, q)$'s

$$\chi_{pq}(T_B) = \sum_{\ell=0}^N T_B^{-\ell} b_\ell(p, q, B) \quad (15)$$

and for a range of $B \sim 0(1)$ the new ratio $R_\ell(p, q, B) = b_\ell(p, q, B)/b_{\ell-1}(p, q, B)$ is indeed linear in $1/\ell$ for large ℓ and thus $K_c = T_c^{-1}$ as well as $\gamma(p, q)$ can be determined (Fig. 1b).

The ratio test in Eq. (14) and Figs. 1a and 1b served us as an indication for the quality of our calculations. A more systematic and accurate way to work

past the radius of convergence $R = |T_1|^{-1}$ as well as to analyze the rest of the singularity structure in the complex T plane is to use Padé approximants. [Indeed $[L,L]$ Padé approximants are invariant⁸⁾ under our transformation $K \rightarrow K_B = K(1 + BK)^{-1}$]. Very stable results are obtained for the value of T_c and $\gamma(p,q)$ by applying the $[L,M]$ Padé approximants to

$$\frac{d}{dk} \log \chi_{pq}(k) \xrightarrow[k \rightarrow k_c]{} - \frac{\gamma(p,q)}{k - k_c} \quad (16)$$

Our final exponents given in the Table are calculated by a best fit (in terms of η , z and ν) of the relations for $\gamma(p,q)$ in Eq. (11b) for an overdetermined set of values of p, q . The errors quoted are the root mean square deviation of $\gamma(p,q)$ from its linear dependence on p and q .

These calculations were extended to include the determination of the approach to scaling exponent¹⁰⁾ λ (see Table) by applying the Padé approximants to

$$\frac{d}{dk} \log [(k_c - k) \frac{d}{dk} \log \chi_{pq}(k) - \gamma(p,q)] \xrightarrow{} - \frac{\lambda \nu}{k - k_c} \quad (17)$$

In Fig. 2 we compare our results with: (1) exact values given for $D = 0, 4$; (2) the results of the $\epsilon = 4 - D$ expansion; (3) Cardy's calculations¹¹⁾ using perturbation theory at large order for the RFT. These results give significant support to the hypothesis that the RQS and RFT models belong to the same universality class.

It is interesting to note that our results for η and ν deviate substantially from the ϵ expansion results because these exponents must interpolate smoothly to their exact values at $D = 0$. The exponent z however, is not constrained in this way (indeed it is not even defined at $D = 0$) and the second order ϵ expansion gives very accurate results for $D = 2$ and 1 .

From our results at $D = 2$ we can obtain an estimate of $\alpha_{0c} \equiv 1 - \Delta_{0c}$ from the known expressions^{3),4)} for A and δ . Assuming¹⁾ $\alpha_0^* \approx 0.3 \text{ GeV}^{-2}$ and $r_0 \approx 0.6 \text{ GeV}^{-1}$ we obtain $1.03 \geq \alpha_{0c} \geq 1.01$ depending on what value we choose for the lattice spacing b , which we assume to be equal to π/p_{Tmax} where p_{Tmax}

is the transverse momentum cut-off in the interval $0.2 \text{ GeV} \leq p_{T\text{max}} \leq 0.5 \text{ GeV}$. These values for α_{0c} are in qualitative agreement with its estimates in previous works¹²⁾ but it is difficult to be more precise because the expression for δ derived from $D = 0$ theory is in principle reliable only for $\exp(2\Delta_0^2/b^2r_0^2)$ large.

Moreover it may be better to consider the lattice spacing as a free parameter corresponding to a new scale within the hadron. In this case the RQS and RFT models have the same number of parameters (b, δ, A and r_0, Δ_0, α' , respectively) to adjust in phenomenological applications.

It is useful to consider our calculation in the context of earlier calculations¹³⁾⁻¹⁵⁾. If we look at the phase plane in $T = \delta/A$ and θ at $y = \infty$ (Fig. 3) we see that our expansion is taken along the line at $\theta = \pi/2$ and $T \rightarrow \infty$. Note that the RQS model ($\theta = \pi/2$) is precisely half way between two Hermitean models: at $\theta = 0$ an XY model in a magnetic field along the Z axis and at $\theta = \pi$ a classical antiferromagnetic Ising model in a magnetic field. Hence any detailed comparison of the critical Pomeron transition with ferromagnetic transition is likely to be misleading. For any D we have exactly computed in the Hermitean models to get $T_c(\theta = 0) = 0$ and $T_c(\theta = \pi) = 4D$, so that our values of $T_c(\theta = \pi/2)$ in the RQS model at $D = 2$ and 1 are reasonable interpolations. In Fig. 3 we have drawn the critical line $T_c(\theta)$ estimated by our high temperature series for θ less than about $2/3\pi$ where the approximation becomes unreliable.

In the supercritical region ($T < T_c$) Amati et al.¹³⁾ find the disc solution for $T = 0$ (i.e., $\delta = 0$) on the RFT line ($\theta = \pi/2$) and the spin model of Bronzan and Sugar¹⁵⁾ is also at $T = 0$ but with pure imaginary θ . It is quite plausible as emphasized by Cardy¹⁴⁾ that the supercritical phase for $D = 1$ ($T < T_c(\theta)$) is controlled by the fixed point at $T = 0, \theta = \pi/2$ studied by Amati et al.¹³⁾. However, we feel that the nature of the supercritical theory is still an open question -- particularly for $D = 2$.

The RQS model is a convenient theory in which to do precise calculations for the critical Pomeron transition. If, as we believe, this model exhibits t channel Reggeon unitarity for the disordered phase ($T > T_c$) it should be considered on an equal footing with RFT to investigate the transition to the "ordered" domain ($T < T_c$). We hope to be able to explore in the near future the nature of this phase starting from $T > T_c$ and analytically continue beyond

$T = T_c$ singularity. The location of the singularity $E_s(T)$ [$E_s(T_c) = 0$] above and below T_c will teach us about the nature of the phase transition. In contrast to the earlier investigations by Amati et al.¹³⁾ at $T = 0$, this should teach us about the "ordered" theory for the physically interesting range of the parameter T near T_c . Also since this region is close to the fixed point which also by universality must appear in the RFT we feel it is a better indication of the nature of the transition common to both models.

A final note on the implications of our results. It had been claimed¹⁶⁾ that the Reggeon Quantum Spin model has the same critical behaviour at $T = T_c$ as the RFT and it differs away from $T = T_c$. In other words, the RQS model and RFT have identical exponents η, z as indeed is seen comparing our results with Ref. 11) in Fig. 2 but they differ in the exponent ν . Since ν was not calculated in Ref. 11) nor in any other work to a significant accuracy we are unable to compare our result $\nu(D = 2) = 1.271 \pm 0.007$ to anything. Thus, a reliable calculation of ν in RFT can be of major importance in the controversy between Refs. 13) and 16), at least from a numerical point of view. The same has to be said about calculating λ at $D = 2$ in the RQS model¹⁷⁾.

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Table 1

Critical exponents for Reggeon Field Theory
calculated in the Reggeon Quantum Spin Model.

D	η	z	ν	λ	T_c
1	0.317 ± 0.002	1.272 ± 0.007	1.736 ± 0.001	0.57 ± 0.03	0.60628 ± 0.00004
2	0.238 ± 0.008	1.16 ± 0.01	1.271 ± 0.007		2.428 ± 0.003

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- 2) P. Pfeuty, Ann. Phys. (N.Y.) 57 (1970) 79.

B. Stoeckly and D.J. Scalapino, Phys. Rev. B11 (1975) 205.

- 3) D. Amati, M. Ciafaloni, M. Le Bellac and G. Marchesini, Nuclear Phys. B112 (1976) 107.

- 4) R.C. Brower, M.A. Furman and K. Subarao, Phys. Rev. D15 (1977) 1956.

Note that here A is redefined to be twice as big so that in Eq. (2)

$$A = 4\alpha' \left(\frac{\Delta_0}{g_0 \ell} \right)^2 = 2 A_{BFS}$$

- 5) See, for example, H.E. Stanley, "Introduction to phase transition and critical phenomena", Oxford Univ. Press, New York and Oxford (1971).

The use of high-temperature expansion in RFT was suggested by G. Parisi, Phys. Letters 56B (1975) 470.

Using an early formulation of the theory on the lattice, J. Ellis and R. Savit, Nuclear Phys. B94 (1975) 477, had calculated only the first three terms in a series. Such a low order series enabled them barely to estimate the order of magnitude of the critical exponents as is suggested in Fig. 1.

- 6) Indeed it is important to note that on the basis of earlier work^{2)-4), 12)} the existence of a critical point has been fairly well established. For example, the mean field theory of Ref. 4) gives a critical surface $T_c(\theta) = 4D \sin^2 \theta/2$.

- 7) R.C. Brower, M. Furman and M. Moshe - in preparation;
M. Furman, "Critical Behaviour of Reggeon field theory on a lattice", (Ph.D. thesis, UCSC 1977).

- 8) G.A. Baker Jr., "Essentials of Padé approximants", Academic Press, New York (1975).

- 9) Further improvement can be made by using Padé for

$$(k-k') \frac{d}{dk} \log \chi_{pq}$$

to calculate $\gamma(p,q)$ [see, e.g., Ref. 8)].

- 10) W.R. Frazer and M. Moshe, Phys. Letters 57B (1975) 475; Phys. Rev. D12 (1975) 2370.

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- 13) D. Amati, M. Ciafaloni, G. Marchesini and G. Parisi, Nuclear Phys. B114 (1976) 483, and Ref. 3).
- 14) J.L. Cardy, Nuclear Phys. B115 (1976) 141.
- 15) J.B. Bronzan and R.L. Sugar, Phys. Rev. D16 (1977) 466. The connection of their Hamiltonian H_{BS} to our Hamiltonian H_{RQS} is

$$H_{BS} \propto S H_{RQS}(\delta=0, \theta=i\phi) S^{\dagger}$$

where

$$S = \exp\left(i \frac{\pi}{2} \sum_{\vec{n}} N_{\vec{n}}\right)$$

It is difficult to understand how their model can be in the same universality class since the relation $UHU^{\dagger} = H^{\dagger}$ is not satisfied for $H = H_{BS}$.

- 16) A.R. White, CERN preprints TH.2259 (1977) and TH.2449 (1978).
- 17) Though the determination of the exponent λ in $D = 1$ is easy, we are unable to calculate λ at $D = 2$ since the Padé approximants to the expression of Eq. (17) do not show a stable pole at $K = K_C$. We believe that in order to calculate this exponent at $D = 2$ an even higher than ten orders expansion is needed. One of us (MM) would like to thank G. Parisi for a discussion on similar difficulties occurring in spin models in solid state physics. In Fig. 2 we have also added in black dots the values of λ at $D = 1$ and 3 reported to us, after these calculations were completed, by J. Cardy who calculated them in RFT using the method of Ref. 11). Our value for $\lambda(D = 1)$ agrees with his.

FIGURE CAPTIONS

- Fig. 1 : The ratio test at $D = 1$ (a) before and (b) after the mapping $K_B = K/(1 + KB)$ of the temperature plane. Note that K_{BC}^{-1} determined by the intercepts of the graphs as $\ell \rightarrow \infty$ [Eq. (14)] has different values for $B = 0.7$ and $B = 1.0$. However, $K_C^{-1} = T_C = T_{BC} - B = K_{BC}^{-1} - B$ is invariant under this change of B as explained in the text. Similarly $\gamma(p,q,B)$ is B -independent.
- Fig. 2 : Comparison of results for the critical exponents η , z , ν and λ as functions of the transverse dimensions: The $\varepsilon = 4-D$ expansion (dotted line), the exact calculations at $D = 0$ and 4 ($\otimes$). Cardy's results¹¹⁾ in RFT ($\bullet$) and the present calculation in the Reggeon Quantum Spin Model. ($\circ$).
- Fig. 3 : T, θ phase plane. RFT is half way between the two Hermitean models. At $D = 2$ and $0 \leq \theta \leq 2/3\pi$ we computed in the high temperature expansion the critical line $T_c(\theta)$ (solid line) and compare it with mean field theory⁴⁾ $T_c = 4D \sin^2 \theta/2$ (broken line) and renormalization group (at $D = 1$) results from Ref. 14) and M. Furman in Ref. 7).

FIG. 1a

FIG. 1b

FIG.2

FIG. 3a

FIG. 3b