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AN INFINITE SET OF CONSERVATION LAWS
OF THE SUPERSYMMETRIC SINE-GORDON THEORY

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A B S T R A C T

We find an ansatz for the explicit expression for an infinite set of bosonic conserved currents of the supersymmetric sine-Gordon theory. The corresponding charges all commute with the supersymmetric generators and reduce to those of the conventional sine-Gordon theory for vanishing fermionic field. Our result supports a previous conjecture used for computing the exact S matrix of this model.

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Much progress has been obtained in the quantization of two-dimensional field theories which correspond, at the classical level, to integrable systems, such as the sine-Gordon model, the $O(N)$ non-linear σ model, the Gross-Neveu model and the massive Thirring model ¹⁾.

A common striking feature is the factorization ^{2),3)} of the multiparticle S matrix into a product of exactly computable ²⁾ two-body S matrices. This property can be derived ⁴⁾ in the framework of the general axiomatic S matrix theory, provided there is no particle production and conservation of the particle momenta.

These two requirements are met indeed by the mentioned models because of the existence, for each of them, of infinitely many conservation laws which survive quantization ^{3),5)}.

Prompted by close analogies with the four-dimensional Yang-Mills theory (instanton solutions and asymptotic freedom), much attention has been drawn to the $O(3)$ non-linear σ model ⁶⁾, among the others. The addition of fermions, then, should make it closer to the believed more physically relevant QCD.

Now a natural way of incorporating fermions is via supersymmetric extensions. Such extensions have been worked out for the non-linear σ model ^{7),8)} as well as for the sine-Gordon model ^{7),9)}. It is a remarkable fact that, in doing that, not only the instanton and soliton solutions persist but also the (well-behaved) fermionic solutions in the field of the instantons and solitons can be easily built up ⁷⁾. A very important question is whether infinitely-many conservation laws exist in such supersymmetrized models. No example of a non-trivial conserved current (besides the supersymmetric energy-momentum tensor) has been exhibited up to now.

Quite recently, Shankar and Witten ¹⁰⁾ have been able to write exact S matrices which, under the hypothesis that such conservation laws exist, would be associated with the supersymmetric form of the non-linear σ model and the sine-Gordon model. As a matter of fact the S matrix for the supersymmetric sine-Gordon model appears as a factor of the S matrix for the supersymmetric non-linear σ model ¹⁰⁾. Once again the sine-Gordon model seems to play a central rôle.

The aim of the present note is to suggest a solution of the problem of the infinite set of conservation laws for the supersymmetric sine-Gordon theory.

Namely, we give, using superfield techniques, a recursive formula that provides the expression of the n^{th} conserved current in terms of the $(n-1)$ previous ones ; as the first current is known, this formula generates explicitly the infinite set of conserved currents. Needless to say, for a vanishing fermion field, our expression of these currents consistently reduces to the known formulae of the bosonic sine-Gordon model ³⁾.

Before exhibiting the explicit expression of the ansatz for the infinite set of conserved currents, we wish to prove, on general grounds, that all the relevant charges commute with the supersymmetry generators Q_α ^{*)}.

The local conserved bosonic current density ^{**) $j_{\nu a}^{(n)}(x)$} ($n=2,3,\dots$), a being a set of additional Lorentz (tensorial) indices, is embedded in a spinor superfield

$$J_{\alpha a}^{(n-\frac{1}{2})}(x, \theta) = \Psi_{\alpha a}^{(n-\frac{1}{2})}(x) + i(\gamma^\nu \theta)_\alpha j_{\nu a}^{(n)}(x) + i \left[(A_a^{(n)}(x) + B_a^{(n)}(x) \gamma_5) \theta \right]_\alpha + \frac{i}{2} \bar{\theta} \theta \chi_{\alpha a}^{(n+\frac{1}{2})}(x). \quad (1)$$

The conservation of the bosonic current $j_{\nu a}^{(n)}$ with respect to the index ν is assured by the supersymmetric covariant conservation law :

$$D^\alpha J_{\alpha a}^{(n-\frac{1}{2})}(x, \theta) = 0 \quad (2)$$

where

$$D^\alpha = \bar{D}_\alpha = (D\gamma^0)_\alpha = \left(\frac{\partial}{\partial \theta} + i \bar{\theta} \not{\partial} \right)_\alpha$$

is the supersymmetric covariant derivative.

*) α is a (two-dimensional) spinorial index and takes values $\alpha = 1, 2$.

**) In our notations, the index n not only labels the set of currents, but also gives their scale dimension.

We give, for convenience of the reader, some useful properties of the operator D^α . In components we have ^{*}) $(D^1, D^2) = (-D_2, D_1)$ with

$$D_1 = -\frac{\partial}{\partial \theta_2} + i\theta_2 \partial_+ , \quad D_2 = \frac{\partial}{\partial \theta_1} + i\theta_1 \partial_-$$

which, in turn, imply

$$D_1 D_1 = -i\partial_+ , \quad D_2 D_2 = i\partial_- , \quad D_1 D_2 = -D_2 D_1 = \frac{1}{2} D^\alpha D_\alpha = \frac{1}{2} \bar{D} \bar{D}.$$

In terms of component fields given in Eq. (1), Eq. (2) means

$$\chi_{\alpha a}^{(n+\frac{1}{2})} = (\not{\partial} \Psi_a^{(n-\frac{1}{2})})_\alpha , \quad A_a^{(n)} = 0, \quad \partial^\nu j_{\nu a}^{(n)} = 0. \quad (3)$$

It is immediate to show that Eq. (2) implies

$$[Q_\alpha, j_{0a}^{(n)}(x)] \propto \partial_1 \Psi_{\alpha a}^{(n-\frac{1}{2})}(x) \quad (4)$$

which then implies the invariance of the charges

$$I_a^{(n)} = \int dx_1 j_{0a}^{(n)}(x) \quad (5)$$

under supersymmetry, i.e.,

$$[Q_\alpha, I_a^{(n)}] = 0 \quad (6)$$

^{*}) We use the light-cone notations : $x^\pm = (x^1 \pm x^0)/2$, $\partial_\pm = \partial/\partial x^\pm = \partial_1 \pm \partial_0$, moreover γ^0 is defined as follows : $\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Coming back to the supersymmetric sine-Gordon model, let us start from the action

$$\int d^2x d^2\theta \left(-\frac{i}{2} \Phi D_1 D_2 \Phi + m \cos \Phi \right) \quad (7)$$

which gives the superfield equation of motion

$$D_1 D_2 \Phi = i m \sin \Phi. \quad (8)$$

In Eqs. (7), (8), Φ stands for a scalar superfield :

$$\Phi(x, \theta) = \varphi(x) + i \bar{\theta} \psi(x) + \frac{i}{2} \bar{\theta} \theta F(x). \quad (9)$$

In terms of component fields, Eq. (8) reads

$$\begin{aligned} F &= -m \sin \varphi \\ \not{D} \psi &= -m \psi \cos \varphi \\ \square \varphi &= -m \left(F \cos \varphi + \frac{i}{2} \bar{\psi} \psi \sin \varphi \right) = \\ &= \frac{1}{2} \left(m^2 \sin 2\varphi - i m \bar{\psi} \psi \sin \varphi \right). \end{aligned} \quad (10)$$

We can now define an infinite set of supercurrents $J_\alpha^{(n+\frac{1}{2})}(x, \theta)$ by the following ansatz that we have explicitly checked to satisfy Eq. (2) up to $J_\alpha^{(11/2)}$:

$$\begin{aligned} J_1^{(n+\frac{1}{2})} &= m \sin \Phi \gamma^{(n-\frac{1}{2})} \\ J_2^{(n+\frac{1}{2})} &= -i D_2 \Phi \cdot D_2 \gamma^{(n-\frac{1}{2})} \end{aligned} \quad (n=1, 2, 3, \dots) \quad (11)$$

in which the spinorial superfields $\gamma^{(n-\frac{1}{2})}$ are defined by the recursion formula which follows :

$$\begin{aligned}
 Y^{(n+\frac{1}{2})} &= \partial_- Y^{(n-\frac{1}{2})} + \partial_- \Phi \cdot \sum_{\ell=1}^{n-1} D_2 Y^{(\ell-\frac{1}{2})} \cdot Y^{(n-\ell-\frac{1}{2})} \\
 Y^{(\frac{1}{2})} &= -i D_2 \Phi.
 \end{aligned} \tag{12}$$

Another set of conserved supercurrents can be obtained by the interchange $\partial_+ \rightleftharpoons \partial_-$ and $1 \rightleftharpoons 2$ corresponding to a different set of the over-all Lorentz label a , as defined in Eq. (1).

We will now prove that if the spinor field $\psi(x)$ in Eq. (9) vanishes then Eqs. (12), (13) reproduce the well-known infinite set of conservation laws of the bosonic sine-Gordon model ^{11),3)}.

In fact, by putting $\psi(x)=0$ in Eq. (9), one gets from Eq. (12) the truncated form

$$\begin{aligned}
 Y^{(n-\frac{1}{2})}(x, \theta) &= \theta_1 f^{(n)}(x) + \theta_2 g^{(n)}(x), \\
 f^{(n+1)}(x) &= \partial_- f^{(n)}(x) + \partial_- \varphi(x) \cdot \sum_{\ell=1}^{n-1} f^{(\ell)}(x) \cdot f^{(n-\ell)}(x), \\
 g^{(n+1)}(x) &= \partial_- g^{(n)}(x) + \partial_- \varphi(x) \sum_{\ell=1}^{n-1} f^{(\ell)}(x) g^{(n-\ell)}(x), \\
 f^{(1)}(x) &= \partial_- \varphi(x), \quad g_1(x) = F(x).
 \end{aligned} \tag{13}$$

Because of the first of Eqs. (10), it follows

$$g^{(n+1)}(x) = -m \cos \varphi(x) f^{(n)}(x) \quad (n=1, 2, \dots) \tag{14}$$

Insertion of Eqs. (13) and (14) in Eq. (11) gives the following expressions for the truncated supercurrents

$$\begin{aligned}
 J_1^{(n+\frac{1}{2})} &= \theta_1 m \sin \varphi f^{(n)} - \theta_2 m^2 \frac{\sin 2\varphi}{2} f^{(n-1)}, \\
 J_2^{(n+\frac{1}{2})} &= \theta_1 \partial_- \varphi f^{(n)} - \theta_2 m \sin \varphi f^{(n)}, \quad (f^{(0)} = \tan \varphi).
 \end{aligned} \tag{15}$$

Finally the supersymmetric conservation law applied to Eq. (15) implies

$$\partial^\mu j_\mu^{(n)}(x) = 0 \quad (16)$$

where the conserved bosonic currents are given by

$$\begin{aligned} j_+^{(n)} &= -\frac{m^2 \sin 2\varphi}{2} f^{(n-2)} \\ j_-^{(n)} &= \partial_- \varphi f^{(n-1)}. \end{aligned} \quad (n=2,3,\dots) \quad (17)$$

These expressions, together with the recursion formula in Eq. (13), coincide with the form ³⁾ of the conserved currents of the conventional sine-Gordon model obtained through the trace identities of the inverse scattering method ¹¹⁾.

The first current $J_\alpha^{(\frac{3}{2})}$ is well known because it is the multiplet to which the energy momentum tensor belongs. In fact from Eq. (11) with $n=1$ we get

$$J_\alpha^{(\frac{3}{2})} = \begin{pmatrix} -im \sin \Phi D_2 \Phi \\ -i D_2 \Phi \partial_- \Phi \end{pmatrix} = J_{\alpha,-} \quad (18)$$

The previous superfield with

$$J_{\alpha,+}^{(\frac{3}{2})} = \begin{pmatrix} -i D_1 \Phi \partial_+ \Phi \\ -im \sin \Phi D_1 \Phi \end{pmatrix} \quad (19)$$

obtained by Eqs. (11) and (12) by the substitution $1 \rightleftharpoons 2$, $\partial_+ \rightleftharpoons \partial_-$, can be recast in the Lorentz covariant form

$$J_{\alpha\mu}^{(\frac{3}{2})} = -i [(\not{\partial} \Phi + m \sin \Phi) \gamma_\mu D \Phi]_\alpha \quad (20)$$

The supercurrent $J_{\alpha\mu}^{(\frac{3}{2})}$ is related to the generators of supersymmetry Q_α and translations P_μ . It satisfies, beyond the constraint given by Eq. (2) the additional conservation

$$\partial^\mu J_{\alpha\mu}(x, \theta) = 0. \quad (21)$$

This equation implies that, besides the energy momentum tensor $j_{\mu\nu}^{(2)}(x)$, also a conserved spinor current exists.

The first non-trivial conserved current is $J_\alpha^{(7/2)}$ whose expression we explicitly exhibit

$$\begin{aligned} J_1^{(7/2)} &= -im \sin \Phi \left[D_2 \partial_{--} \Phi + D_2 \Phi (\partial_- \Phi)^2 \right] \\ J_2^{(7/2)} &= -i D_2 \Phi \left[\partial_{--} \Phi + (\partial_- \Phi)^3 \right]. \end{aligned} \quad (22)$$

Notice that, as in the pure bosonic case ³⁾, the current $J_\alpha^{(5/2)}$ is trivial. In fact, by using the invariance of Eq. (2) under the transformation

$$J_\alpha \rightarrow J_\alpha + (\gamma_5 D)_\alpha V \quad (23)$$

V being an arbitrary superfield, one gets $J_\alpha^{(5/2)} = \partial_- J_\alpha^{(3/2)}$. Therefore, the corresponding charge is vanishing.

We further note that a canonical formalism can be developed also for the supersymmetric theory by using standard Poisson brackets for the fundamental fields $\varphi(x)$ and $\psi(x)$. The infinitesimal transformations $\delta^n \varphi(x)$, $\delta^n \psi(x)$ generated by the charges $I^{(n)}$ can then be computed and hence the underlying symmetries of the Lagrangian, in analogy with the pure bosonic case ¹²⁾.

Some remarks are in order. Although we do not prove in general the ansatz in Eq. (12) to satisfy Eq. (2) for any n , the fact that it reproduces the bosonic formula (which is true for any n) together with the fact that it gives indeed the correct result for the first non-trivial currents up to $J_\alpha^{(11/2)}$, gives us enough confidence to believe that it is generally true.

One may wonder if the supercurrent $J_{\alpha}^{(n+\frac{1}{2})}$ does not only satisfy Eq. (2) but a more stringent equation like Eq. (21), this leading to conserved spinorial currents too, as it happens for $J_{\alpha}^{(\frac{3}{2})}$. The explicit form of $J_{\alpha}^{(7/2)}$ seems, to the best of our knowledge, to give a negative answer, spoiling then the possibility of a complete grading of the bosonic algebra generated by the $I^{(n)}$.

Next, at the light of our result, one can suspect the existence of a supersymmetric Bäcklund transformation. This would be a rigorous proof that our ansatz given by Eqs. (11) and (12) generates an infinite set of currents satisfying Eq. (2). At the same time, for the reasons given at the beginning, the supersymmetric non-linear σ model should be object of a similar investigation concerning the existence of higher order conservation laws.

Finally, our results being purely classical, we must investigate their quantum version. In the pure bosonic theory, the conservation laws survive quantization and it is unlikely they fail to do so in the supersymmetric extension. As a matter of fact, supersymmetry usually improves the quantum properties of a given bosonic theory.

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