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MULTI-POMERON COUPLINGS TO EXTERNAL PARTICLES
IN REGGEON FIELD THEORY

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A B S T R A C T

We show that, contrary to previous assertions, multi-Pomeron couplings to the external particles are as equally important as single couplings in the asymptotic behaviour of the amplitude in the Reggeon calculus. Our arguments require a careful discussion of the relation of the Reggeon calculus to a renormalizable field theory and of the renormalization of composite operators in Reggeon field theory.

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1. INTRODUCTION

The Reggeon calculus was introduced by Gribov¹⁾ as a means of elucidating the complicated structure in the angular momentum plane which results when the leading vacuum singularity has an intercept near one. More recently, the similarity of the theory to an infra-red problem in quantum field theory has led to a solution based on the renormalization group approach^{2),3)}. There are many different ways to use the renormalization group in this problem⁴⁾. However, the most successful from a calculational point of view is based on the close relationship of the Reggeon calculus to a renormalizable quantum field theory, whose infra-red behaviour can be inferred from the Callan-Symanzik equation²⁾.

In this way the contribution to the elastic scattering amplitude from all diagrams where only one Pomeron couples to each external particle was obtained^{2),3)}. When the Pomeron intercept is exactly one, this gives a term proportional at high energy to $(\ln s)^{-\gamma}$ in the total cross-section, where $-\gamma > 0$ is a critical exponent. It was also claimed [e.g., Ref. 5)] that other contributions, where more than one Pomeron couples to the external particles, are down by powers of $\ln s$.

Our main result is that this claim is erroneous, namely that multi-Pomeron couplings to the external particles are equally important in determining the asymptotic behaviour of the total cross-section. The previous results were based on a misinterpretation of the physical relevance of the renormalized Green functions of the related renormalizable field theory which resulted also in a multiple counting of several contributions. We shall show that the relationship is more subtle than previously assumed, giving a different result for the physical quantities.

Because this relationship of the Reggeon calculus to a renormalizable quantum field theory has not been adequately discussed before in the literature, we shall give a careful account in the next section. Basically our main point is that the physical quantities of the Reggeon calculus correspond more nearly to the unrenormalized Green functions of a cut-off field theory. The introduction of renormalized Green functions, which have a finite perturbation expansion in the renormalized coupling constant as the cut-off goes to infinity, is solely a mathematical artifice. Since they have one less dimensional parameter, the Callan-Symanzik equation determines their behaviour at large $\ln s$. From this the large $\ln s$ behaviour of the unrenormalized Green functions can be inferred. For multiplicatively renormalizable quantities, such as the two and three-point functions, this is of course a trivial exercise. However, the composite operators which correspond to multi-Pomeron couplings are not multiplicatively renormalizable, and disentangling the high energy behaviour of their unrenormalized Green functions is

consequently more complicated. In fact, if O_i and $\bar{O}_j$ are composite local operators, in order to make finite the unrenormalized Green function $\langle 0 | O_i(y, \vec{b}) \bar{O}_j(0, 0) | 0 \rangle$ it is necessary to subtract off pieces proportional to all Green functions of the form $\langle 0 | O_k(y, \vec{b}) \bar{O}_\ell(0, 0) | 0 \rangle$, where the canonical dimensions of O_k and $\bar{O}_\ell$ are less than or equal to those of O_i and $\bar{O}_j$, respectively. These will naturally include the two-point function $\langle 0 | \psi(y, \vec{b}) \bar{\psi}(0, 0) | 0 \rangle$. Only then, after making these subtractions, does one have a multiplicatively renormalizable quantity, which obeys a simple scaling law as $\ln s \rightarrow \infty$, which can be derived from a Callan-Symanzik equation. Explicit calculation then shows that this renormalized Green function is asymptotically negligible compared with the two-point function. As a result, each unrenormalized Green function $\langle 0 | O_i(y, \vec{b}) \bar{O}_j(0, 0) | 0 \rangle$ is dominated as $\ln s \rightarrow \infty$ by its subtraction proportional to the two-point function. In the case when O_i and $\bar{O}_j$ contain multiple powers of the Pomeron fields (ψ and $\bar{\psi}$, respectively) and so correspond to multi-Pomeron couplings to the external particles, this gives our main result.

The layout of this paper is as follows. In the next section we give a careful discussion of the role of the renormalized Green functions and the Callan-Symanzik equation in determining the high energy behaviour of the Reggeon calculus. Section 3 is devoted to the renormalization of composite operators and the general form of the renormalization group equations which they satisfy. In Section 4 we calculate to $O(\epsilon)$ the scaling behaviour of the correlation functions of the operators $\psi^n, \bar{\psi}^m$, as an example of the general formalism. In Section 5 we describe two cases where our results can be independently verified. For the case of a theory with only a constant triple Pomeron coupling, the Green functions $\langle 0 | \psi(y, \vec{b}) \bar{\psi}(0, 0)^2 | 0 \rangle$ and $\langle 0 | \psi(y, \vec{b})^2 \bar{\psi}(0, 0)^2 | 0 \rangle$ can be directly related to the two-point function using the equations of motion. This relationship agrees with our general result. In the case of zero transverse dimensions⁶⁾, all Green functions are calculable in the limit $\alpha(0) \rightarrow 1$, and agree with our result. Apart from these examples, we shall work throughout in the massless ($\alpha(0) = 1$) theory defined in $4-\epsilon$ transverse dimensions. We expect the same results to emerge from other approaches, and also to be relevant when the Pomeron intercept is slightly below one⁷⁾.

The question of the renormalization of composite operators in the analogous ϕ^4 theory and its application to the problem of second order phase transitions has been discussed thoroughly by Brézin, Le Guillou and Zinn-Justin⁸⁾, and we rely heavily on their formalism.

2. ROLE OF THE RENORMALIZED THEORY

The Reggeon calculus was originally introduced by Gribov¹⁾ as a set of diagrammatic rules, which account for all the possible singularities in the angular momentum plane arising from a Pomeron pole with intercept near one, and its attendant cuts implied by t channel unitarity. The elements of the calculus are: 1) a propagator, which in momentum space has the form $i[E + \alpha_0(-k^2) - 1]^{-1}$, where $E = 1-j$ and $\alpha_0(t)$ is the bare trajectory function; 2) couplings of arbitrary numbers of Pomerons to themselves, which contain an arbitrary dependence on the transverse momenta $\vec{k}$; 3) couplings of arbitrary numbers of Pomerons to the external particles, again with an arbitrary dependence on the transverse momenta. This latter dependence corresponds to the fact that a hadron has a finite size in impact parameter as seen by the Pomeron.

The only requirement placed on the trajectory function and the coupling is that they should be non-singular near $k^2 = 0$; we thus expand $\alpha_0(-k^2) = 1 - \Delta_0 - \alpha'_0 k^2 + O(k^4)$. Agreement with low energy data implies a strong cut-off of order m_π in all couplings. We then assume that this momentum dependence of the coupling can be largely absorbed into a factor $e^{-\alpha'_0 k^2/\Lambda}$ in the propagator. From a calculational point of view we can obtain all our results using any other regularization method (e.g., Pauli-Villars etc.). The particular cut-off factor $e^{-\alpha'_0 k^2/\Lambda}$ has been chosen here because of its somewhat more direct physical meaning⁹⁾, though a detailed relation between this cut-off and low energy phenomenology needs further investigation. This procedure is necessary in order to work within a local field theory. Having done this, the diagrams can be obtained from a Lagrangian density of the form

$$\mathcal{L} = \bar{\Psi} \left(\frac{1}{2} \vec{\partial}_y^2 - \alpha'_0 \nabla^2 + \Delta_0 \right) e^{-\alpha'_0 \nabla^2/\Lambda} \Psi + \frac{i}{2} r_0 \bar{\Psi} (\Psi + \psi) \Psi + \mathcal{L}' \quad (2.1)$$

where $\psi(y, \vec{b})$ and $\bar{\Psi}(y, \vec{b})$ are local scalar fields in rapidity (y) and impact parameter ($\vec{b}$) space. $\mathcal{L}'$ contains an infinite number of terms which originate from: 1) self-couplings of more than three Pomerons; 2) residual momentum dependence of the triple Pomeron coupling, leading to high order derivative couplings of the fields; 3) higher order terms in the trajectory function. If we observe that $\mathcal{L}$ has dimension $E^{1+D/2}$ in D transverse dimensions (we take k to have dimension $E^{1/2}$) then r_0 has dimension $E^{1-D/4}$, and all the other

couplings in $\mathcal{L}'$ have lower dimension. The renormalization group then shows (a posteriori) that we can ignore $\mathcal{L}'$ term by term in evaluating the high energy behaviour¹⁰⁾. We shall return to this point and state it more carefully at the end.

In writing the couplings of the Pomerons to the external particles in terms of the local fields $\psi, \bar{\psi}$ we have to take into account the residual momentum dependence, which will be expanded up in powers of the momenta. As a result the general form of the scattering amplitude will be

$$A(y, \vec{b}) = i \sum_{i,j} N_i N_j \langle 0 | O_i(y, \vec{b}) \bar{O}_j(0,0) | 0 \rangle_{\Lambda} \quad (2.2)$$

where the sum is over all local operators O_i and $\bar{O}_j$ formed out of the fields ψ and $\bar{\psi}$, respectively, and derivatives with respect to b . (They must, of course, be rotational scalars in b space.) We shall also have occasion to include arbitrary derivatives ∂_y in the $O_i, \bar{O}_j$.

The N_i are simply constants determined by the detailed structure of the hadronic vertices. We use the notation $\langle \rangle_{\Lambda}$ to emphasize that these are unrenormalized Green functions, evaluated with a cut-off Λ .

With the neglect of $\mathcal{L}'$, (2.1) defines a renormalizable field theory in the limit $\Lambda \rightarrow \infty$, provided $D \leq 4$. This (highly non-trivial) statement means that, given any (multiplicatively renormalizable) Green function $G(E_i, \alpha'_0 k_i^2, r_0, \Delta_0, \alpha'_0, \alpha'_0 \Lambda^2, E_N)$, there exists a function $Z_G(r_0, \Delta_0, \alpha'_0, \alpha'_0 \Lambda^2, E_N)$ such that the quantity

$$G_R(E_i, \alpha'_0 k_i^2, r, \Delta, \alpha', \Lambda, E_N) = Z_G(r_0, \Delta_0, \alpha'_0, \alpha'_0 \Lambda^2, E_N) \times G(E_i, \alpha'_0 k_i^2, r_0, \Delta_0, \alpha'_0, \alpha'_0 \Lambda^2) \quad (2.3)$$

has a finite limit as $\Lambda \rightarrow \infty$ when expressed in terms of (r, Δ, α') , which are related to the two and three-point functions at an arbitrary normalization point given by $E_N \neq 0$.

$$\Gamma = (2\pi)^{(D+1)/2} \left[\int_R \right]^{(1,2)} \Big|_{\text{NORM. PT.}} \quad (2.4)$$

$$\alpha' = - \frac{\partial}{\partial k^2} i \Gamma_R^{(1,1)} \Big|_{\text{NORM. PT.}} \quad (2.5)$$

$$\Delta = i \Gamma_R^{(1,1)} (E = k^2 = 0) \quad (2.6)$$

Once (2.3) is established we can derive the Callan-Symanzik equation for G_R by observing that

$$E_N \frac{\partial G}{\partial E_N} \Big|_{\alpha'_0, \gamma_0, \Delta_0, \Lambda \text{ fixed}} = 0 \quad (2.7)$$

When $\Delta = 0$ this leads to a scaling law for G_R as $E_N \rightarrow \infty$. Since G_R is independent of Λ as $\Lambda \rightarrow \infty$ this implies a scaling law for G_R as $E \rightarrow 0$, by dimensional analysis.

The $E \rightarrow 0$ behaviour of G at finite Λ (the quantity of physical interest) can now be inferred by a generalization of the following lemma:

If

$$Z(\Lambda/E_N) G(E/\Lambda) = G_R(E/E_N, \Lambda/E_N) \quad (2.8)$$

has a finite limit as $\Lambda \rightarrow \infty$, and $G_R(E/E_N, \infty) \sim (E/E_N)^\gamma$ as $E \rightarrow 0$, then $G(E/\Lambda) \sim (E/\Lambda)^\gamma$ as $E \rightarrow 0$ for all fixed Λ .

From this point of view, the introduction of the renormalized Green functions G_R is only a mathematical device to eliminate an unwanted parameter in the scaling law. Nevertheless, the existence of a renormalizable theory is deeply connected with the existence of scaling laws for the cut-off theory.

Since the Z 's are non-universal constants which depend on all parameters in $\mathcal{L}'$, insofar as we deal with multiplicatively renormalizable quantities it makes no difference whether we treat G or G_R as physical. The set of multiplicatively renormalizable quantities includes all Green functions of single

fields whose arguments do not coincide. It does not include the Green functions in (2.2) corresponding to multi-Pomeron couplings, and in these cases the difference between G and G_R .

3. RENORMALIZATION OF COMPOSITE OPERATORS

For the sake of definiteness, we shall consider the renormalization of the massless ($\Delta = 0$) theory in $D = 4$, since this is relevant to the ϵ expansion. Similar methods apply to the renormalization of the massive ($\Delta > 0$) theory in $D < 4$.

The renormalized one-particle irreducible truncated Green functions are defined by

$$\Gamma_R^{(n,m)} = Z^{(n+m)/2} \Gamma^{(n,m)} \quad (3.1)$$

where the wave function renormalization Z is determined by

$$\left. \frac{\partial}{\partial E} i \Gamma_R^{(1,1)} \right|_{E = -E_N, k^2 = 0} = 1 \quad (3.2)$$

In order to discuss composite operators O_i we first define the vertex $\Gamma_{O_i}^{(N)}(E_1, \vec{k}_1, \dots, E_N, \vec{k}_N)$. This is the one-particle irreducible truncated piece of the Fourier transform of the Green function

$$\langle 0 | O_i(0,0) \bar{\Psi}(y_1, \vec{L}_1) \bar{\Psi}(y_2, \vec{L}_2) \dots \bar{\Psi}(y_N, \vec{L}_N) | 0 \rangle_{\wedge} \quad (3.3)$$

Examples of the diagrammatic expansions of $\Gamma_{O_i}^{(N)}$ for low dimensional operators are shown in Fig. 1. We define analogous vertices $\Gamma_{\bar{O}_i}^{(N)}$ via the Green function

$$\langle 0 | \Psi(y_1, \vec{L}_1) \Psi(y_2, \vec{L}_2) \dots \Psi(y_N, \vec{L}_N) \bar{O}_i(0,0) | 0 \rangle_{\wedge} \quad (3.4)$$

In Reggeon field theory there exist two independent types of dimension, namely E and k . If the composite operator O_i contains n_i powers of the field, p_i derivatives ∂_y , and q_i derivatives ∇ , in D dimensions O_i has dimension

$$[O_i] = E^{p_i} k^{q_i + D n_i/2}$$

It is also useful to define the power δ_i which governs the degree of divergence of the insertions of O_i . In that case, k^2 is identified with E , and in $D = 4$

$$\delta_i = p_i + \frac{1}{2} q_i + n_i$$

The dimensions of some low order operators are shown in the Table.

If O_i has dimensions δ_i then $\Gamma_{O_i}^{(N)}$ will have dimension $(\delta_i - N)$. Thus, in general, vertices with $\delta_i \geq N$ will be divergent in perturbation theory, and must be subtracted. As an example, consider the vertex $\Gamma_{\psi^2}^{(1)}$, which is linearly divergent. On dimensional grounds, this divergence must have the form

$$\Gamma_{\psi^2}^{(1)}(E, k^2) = C_1(\Lambda) + E C_2(\Lambda) + k^2 C_3(\Lambda) + \text{finite part} \quad (3.5)$$

where $C_1(\Lambda)$ is linearly divergent, and C_2, C_3 are only logarithmically divergent (at any finite order in perturbation theory). These divergences can be removed from $\Gamma_{\psi^2}^{(1)}$ by subtractions proportional to $\Gamma_{\psi}^{(1)}$, $\Gamma_{\partial_y \psi}^{(1)}$ and $\Gamma_{\nabla^2 \psi}^{(1)}$. On the other hand, $\Gamma_{\psi^2}^{(2)}$ is only logarithmically divergent and it can be made finite by a multiplicative renormalization. As a general rule, any vertex $\Gamma_{O_i}^{(N)}$ can be made finite in renormalized perturbation theory by subtracting off vertices $\Gamma_{O_j}^{(N)}$ where $\delta_j \leq \delta_i$. The proof of this result lies deep in renormalization theory and we refer the reader to the literature¹¹⁾. We can then define renormalized vertices $(\Gamma_{O_i}^{(N)})_R$ by

$$(\Gamma_{O_i}^{(N)})_R = Z^{N/2} \sum_j Z_{ij} \Gamma_{O_j}^{(N)} \quad (3.6)$$

where Z_{ij} vanishes if $\delta_j > \delta_i$. The $(\Gamma_{O_i}^{(N)})_R$ are finite in renormalized perturbation theory, although their exact value depends on the way the subtractions are performed. The factor $Z^{N/2}$ is inserted in Eq. (3.6) for convenience in later work. The Z_{ij} are determined by imposing normalization conditions on the $(\Gamma_{O_i}^{(N)})_R$. In general, we impose as many conditions on $(\Gamma_{O_i}^{(N)})_R$ as there are divergent pieces in $\Gamma_{O_i}^{(N)}$. For example, for the ψ^2 insertion we take

$$\frac{\partial}{\partial E} \left(\Gamma_{\psi^2}^{(1)}(E, k^2) \right)_R = \frac{\partial}{\partial k^2} \left(\Gamma_{\psi^2}^{(1)}(E, k^2) \right)_R = 0 \quad (3.7)$$

at $E = -E_N$, $k^2 = 0$, and

$$\Gamma_{\psi^2}^{(2)}(E_1, E_2, k_1^2, k_2^2) = 2 \quad (3.8)$$

at $E_1 = E_2 = -E_N$, $k_1^2 = k_2^2 = 0$.

Logarithmically divergent quantities must be subtracted away from zero energy and momentum, because of infra-red divergences arising in perturbation theory. However, linear and higher divergences can be subtracted at zero energy and momentum. Thus we can add the condition

$$\left(\Gamma_{\psi^2}^{(1)}(0, 0) \right)_R = 0 \quad (3.9)$$

Equations (3.7), (3.8) and (3.9) provide enough renormalization conditions for evaluating the four Z_{ij} in Eq. (3.6) (see Table). [Note that for this particular example in which $\Gamma_{0i}^{(1)}$ are related to $\Gamma^{1,1}$ through the equations of motion the renormalization conditions had to be chosen in such a way as not to contradict the renormalization conditions of $\Gamma^{1,1}$.]

Having defined renormalized vertices $(\Gamma_{0i}^{(N)})_R$ which are finite as $\Lambda \rightarrow \infty$, we can derive renormalization group equations for them. However, we omit this exercise here, as the Green functions (3.3) are not of great physical interest in the Reggeon calculus. We note that the subtraction scheme defined by (3.6), together with the usual subtractions of the two and three point functions, will render finite all Green functions like (3.3) and (3.4), if the operators $\psi, \bar{\psi}$ are replaced by their renormalized versions $Z^{-\frac{1}{2}}\psi, Z^{-\frac{1}{2}}\bar{\psi}$, respectively.

Having defined a subtraction scheme for single insertions of a composite operator, we consider their correlation functions $\langle 0 | O_i(y, \vec{b}) \bar{O}_j(0, 0) | 0 \rangle$, the quantities of physical interest. Any attempt to decompose this Green function into one-particle irreducible parts $\Gamma_{0i}^{(n)}$, $\Gamma^{(n,m)}$, and $\Gamma_{0j}^{(m)}$ along the lines of Ref. 5) necessarily runs into double-counting problems, as does the subsequent renormalization. Indeed, the amplitude defined in Eq. (2.2) will avoid any double

counting. For example, there is no contribution from $\langle \psi(y, \vec{b}) \bar{\psi}^2(0,0) \rangle$ to $\langle \psi^2(y, \vec{b}) \bar{\psi}^2(0,0) \rangle$ since any intermediate time integration will have zero measure for such a piece.

We observe that the same renormalization constants Z_{ij} , used to render finite the single insertions, can also be used to define renormalized correlation functions

$$\begin{aligned} \langle 0 | O_i(y, \vec{b}) \bar{O}_j(0,0) | 0 \rangle_R &= \\ &= \sum_k \sum_l Z_{ik} Z_{jl} \langle 0 | O_k(y, \vec{b}) \bar{O}_l(0,0) | 0 \rangle_\Lambda \end{aligned} \quad (3.10)$$

which are infinite in the limit $\Lambda \rightarrow \infty$. This statement is not obviously true in the case of overlapping divergences, but we shall assume that it can be proved by renormalization theory methods. The only additional divergences of (3.10) come from short-distance behaviour $(y, \vec{b}) \rightarrow (0, \vec{0})$. Indeed it can be seen that quantities like $\langle 0 | \psi(y, \vec{b})^2 \bar{\psi}(0,0)^2 | 0 \rangle_\Lambda$ are divergent as $\Lambda \rightarrow \infty$ even in the free theory. However, these divergences are not relevant to the high energy ($y \rightarrow \infty$) behaviour. They may be removed by additional subtractions which are, however, analytic at $E = 0$. Using the notation

$$G_{ij}(E, k) = \int dy d^4b e^{E y + i \vec{k} \cdot \vec{b}} \langle 0 | O_i(y, \vec{b}) \bar{O}_j(0,0) | 0 \rangle_\Lambda \quad (3.11)$$

(3.10) can be written

$$(G_{ij})_R = \sum_k \sum_l Z_{ik} Z_{jl} G_{kl} \quad (3.12)$$

or, even more compactly, in a matrix form

$$\underline{G}_R = \underline{Z}^{(2)} \underline{G} \quad (3.13)$$

where

$$\underline{Z}^{(2)}_{ijke} = (\underline{Z} \otimes \underline{Z})_{ijke} = Z_{ik} Z_{jl}$$

We now derive renormalization group equations for $\underline{G}_R$ by observing that

$$E_N \frac{\partial}{\partial E_N} \underline{G} \Big|_{\text{bare quantities fixed}} = 0 \quad (3.14)$$

which gives

$$\left[\beta(g) \frac{\partial}{\partial g} + \gamma(\alpha', g) \frac{\partial}{\partial \alpha'} + E_N \frac{\partial}{\partial E_N} + \underline{Z}^{(2)} E_N \frac{\partial}{\partial E_N} \underline{Z}^{(2)-1} \right] \underline{G}_R = 0 \quad (3.15)$$

where as usual

$$\beta(g) = E_N \frac{\partial g}{\partial E_N} \Big|_{\text{bare quantities fixed}} \quad (3.16)$$

$$\gamma(\alpha', g) = E_N \frac{\partial \alpha'}{\partial E_N} \Big|_{\text{bare quantities fixed}} \quad (3.17)$$

(3.15) can be further simplified by observing that, by power counting, $\underline{Z}_{ij}$ must have the form

$$\underline{Z}_{ij} = \left(\frac{\Lambda}{E_N} \right)^{\delta_i - \delta_j} \underline{Z}_{ij} \quad (3.18)$$

where $\underline{Z}_{ij}$ contains only logarithmic divergences in perturbation theory, (3.18) of course implies that

$$\underline{Z}_{ijkl}^{(2)} = \left(\frac{\Lambda}{E_N} \right)^{\delta_i + \delta_j - \delta_k - \delta_l} \underline{Z}_{ijkl}^{(2)} \quad (3.19)$$

where

$$\underline{Z}^{(2)} = \underline{Z} \otimes \underline{Z}$$

and

$$(Z^{(2)-1})_{ijkl} = \left(\frac{\Lambda}{E_N}\right)^{\delta_i + \delta_j - \delta_k - \delta_l} (Z^{(2)-1})_{ijkl} \quad (3.20)$$

Thus the last term in (3.15) becomes

$$\left\{ Z^{(2)} E_N \frac{\partial}{\partial E_N} Z^{(2)-1} \right\}_{ijkl} = \left(\frac{\Lambda}{E_N}\right)^{\delta_i + \delta_j - \delta_k - \delta_l} \times$$

$$\times \sum_{m,n} \left\{ -(\delta_m + \delta_n - \delta_k - \delta_l) Z^{(2)}_{ijmn} (Z^{(2)-1})_{mnkl} \right. \\ \left. + Z^{(2)}_{ijmn} E_N \frac{\partial}{\partial E_N} (Z^{(2)-1})_{mnkl} \right\} \quad (3.21)$$

The renormalization group equation (3.15) therefore simplifies if we define

$$(\tilde{G}_R)_{kl} = \left(\frac{\Lambda}{E_N}\right)^{-\delta_k - \delta_l} (G_R)_{kl} \quad (3.22)$$

Then

$$\left\{ \left[\beta(g) \frac{\partial}{\partial g} + \gamma(\alpha', g) \frac{\partial}{\partial \alpha'} + E_N \frac{\partial}{\partial E_N} \right] \delta_{ik} \delta_{jl} \right. \\ \left. - \sum_{m,n} (\delta_m + \delta_n) Z_{im} Z_{jn} (Z^{-1})_{mk} (Z^{-1})_{nl} \right. \\ \left. + \sum_{m,n} Z_{im} Z_{jn} E_N \frac{\partial}{\partial E_N} (Z^{-1})_{mk} (Z^{-1})_{nl} \right\} (\tilde{G}_R)_{kl} = 0 \quad (3.23)$$

Since Z contains only logarithmic divergences, we expect the whole operator acting on $\tilde{G}_R$ in (3.23) to be finite in renormalized perturbation theory. Now we explicitly made the quantities G_R finite as $\Lambda \rightarrow \infty$, not $\tilde{G}_R$. Therefore there must be a decoupling of the renormalization group equations: if (3.23) is written as

$$\sum_{k,l} R_{ijkl} (\tilde{G}_R)_{kl} = 0 \quad (3.24)$$

then

$$R_{ijkl} = 0 \quad \text{unless} \quad \delta_i + \delta_j = \delta_k + \delta_l \quad (3.25)$$

Although this decoupling is not obvious from (3.25), we shall verify it (to first order) in the next section. Once this decoupling is observed, the form of R_{ijkl} implies that in fact it vanishes unless $\delta_i = \delta_k$ and $\delta_j = \delta_l$. Thus we can work in a subspace with δ_k and δ_l fixed. (3.23) back in terms of G_R then greatly simplifies to

$$\left\{ \left[\beta(g) \frac{\partial}{\partial g} + \gamma(\alpha', g) \frac{\partial}{\partial \alpha'} + E_N \frac{\partial}{\partial E_N} \right] \delta_{ik} \delta_{jl} + H_{ik} \delta_{jl} + H_{jl} \delta_{ik} \right\} (G_R)_{kl} = 0 \quad (3.26)$$

where

$$H_{ik} = \sum_m \gamma_{im} E_N \frac{\partial}{\partial E_N} (\gamma^{-1})_{mk} \quad (3.27)$$

For the free theory, (3.26) reduces to $E_N (\partial/\partial E_N) G_R = 0$, as it must.

So far, we have discussed the renormalization of the theory in $D = 4$. In order to discuss the theory in $4-\epsilon$ dimensions, it is convenient to fix the normalization by continuing back to $D = 4$. The only ϵ dependence then enters into the definition of the dimensionless renormalized coupling g . All our previous equations are true with $\beta(g)$ replaced by

$$\beta(g, \epsilon) = -(\epsilon/4)g + \beta(g, 0) \quad (3.28)$$

In $4-\epsilon$ dimensions $G_{ij}(E, \vec{k})$ has dimensions

$$[G_{ij}(E, \vec{k})] = E^{p_i + p_j - 1} k^{q_i + q_j + \frac{D}{2}(n_i + n_j) - D} \quad (3.29)$$

Therefore $(G_R)_{ij}$ has the functional form

$$(G_R)_{ij} = E_N^{p_i + p_j - 1} \left(\frac{E_N}{\alpha'} \right)^{\frac{1}{2}(q_i + q_j - D) + \frac{D}{4}(n_i + n_j)} \quad (3.30)$$

$$\times f_{ij} \left(\frac{E}{E_N}, \frac{\alpha' k^2}{E_N}, g \right)$$

and so satisfies

$$\left[E_N \frac{\partial}{\partial E_N} + E \frac{\partial}{\partial E} + \alpha' \frac{\partial}{\partial \alpha'} - (p_i + p_j - 1) \right] (G_R)_{ij} = 0 \quad (3.31)$$

Combining (3.31) and (3.26)

$$\left\{ \left[E \frac{\partial}{\partial E} + (\alpha' - \frac{1}{2}) \frac{\partial}{\partial \alpha'} - \beta(g) \frac{\partial}{\partial g} - (p_i + p_j - 1) \right] \delta_{ik} \delta_{jl} \right. \quad (3.32)$$

$$\left. - H_{ik} \delta_{jl} - \delta_{ik} H_{jl} \right\} (G_R)_{kl} = 0$$

We now examine the consequences of (3.32) for the $E \rightarrow 0$ behaviour. For simplicity, we assume that g lies exactly at the value $g_1 = O(\sqrt{\epsilon})$ for which $\beta(g_1, \epsilon) = 0$. For $\epsilon = 0$, the matrix H_{ik} is just equal to zero, and the $(G_R)_{ij}$ have canonical scaling behaviour. For ϵ small, H_{ik} has small elements. In order to solve (3.32), the matrix on the left-hand side must first be diagonalized. Bearing in mind that, because of the decoupling result, this can be carried out in a subspace with $\delta_i + \delta_j$ fixed, we find

$$\left\{ E \frac{\partial}{\partial E} + (\alpha' - \frac{1}{2}) \frac{\partial}{\partial \alpha'} - \lambda^{(n)} \right\} G_R^{(n)} = 0 \quad (3.33)$$

where $\lambda^{(n)}$ are eigenvalues and $G_R^{(n)}$ the corresponding eigenvectors. For small ϵ , the eigenvalues have the form $(p_i + p_j - 1) + O(\epsilon)$. The solution of (3.33) is, as $E \rightarrow 0$

$$G_R^{(n)} \sim (-E)^{\lambda^{(n)}} f_n(\alpha' E^{-2}) \quad (3.34)$$

where, as usual,

$$z = 1 - \sum (\alpha'_i g_i) / \alpha' \quad (3.35)$$

$(G_R)_{ij}$ is a linear combination of terms like (3.34). Comparison with (3.30) yields

$$\begin{aligned} (G_R)_{ij} \sim \sum_n C_{ij}^{(n)} E_N^{p_i + p_j - 1} \left(\frac{E_N}{\alpha'} \right)^{\frac{1}{2}(q_i + q_j - D) + \frac{D}{4}(n_i + n_j)} \\ \times \left(-\frac{E}{E_N} \right)^{\lambda^{(n)} + z \left[\frac{1}{2}(q_i + q_j - D) + \frac{D}{4}(n_i + n_j) \right]} \\ \times \phi^{(n)} \left[\left(-\frac{E}{E_N} \right)^{-2} \frac{\alpha' k^2}{E_N} \right] \end{aligned} \quad (3.36)$$

where the constants $C_{ij}^{(n)}$ depend on the normalization prescription, and the $\phi^{(n)}$ are universal scaling functions. For ϵ small, the above expression simplifies to (keeping only the essential dependence on E and $\vec{k}$)

$$\begin{aligned} (G_R)_{ij} \sim (-E)^{\delta_i + \delta_j - 3 - p_i - p_j + (\alpha_i + \alpha_j + \beta)\epsilon} \\ \times \sum_n C_{ij}^{(n)} (-E)^{\lambda^{(n)}\epsilon} \phi^{(n)}((-E)^{-2} k^2) \end{aligned} \quad (3.37)$$

where the sum is now restricted to those n for which $\lambda^{(n)} = -1 + \mu^{(n)}\epsilon + O(\epsilon^2)$ (i.e., the unperturbed operators have $p_i = p_j = 0$), and

$$\alpha_i = \frac{1}{24} \left[\frac{1}{2} q_i - 5 n_i \right] + O(\epsilon) \quad (3.38)$$

$$\beta = 5/12 + O(\epsilon)$$

where we have used the result $z = 1 + \epsilon/24 + O(\epsilon^2)$.

From (3.37) we see that, for $p_i = p_j = 0$, the scaling behaviour of $(G_R)_{ij}$ differs from its canonical form by only small powers of E . To the extent that the ϵ -expansion gives the correct order of magnitude for the $D = 2$ exponents, we expect the same to be true in the physical number of dimensions.

Having determined the scaling behaviour of $(G_R)_{ij}$, we must now solve (3.12) for the physical quantities G_{ij}

$$G_{ij} = \sum_{k\ell} (Z^{-1})_{ik} (Z^{-1})_{j\ell} (G_R)_{k\ell} \quad (3.39)$$

Since $Z_{ik} = 0$ for $\delta_k > \delta_i$, it follows that $(Z^{-1})_{ik} = 0$ for $\delta_k > \delta_i$. Thus G_{ij} involves all renormalized Green functions $(G_R)_{k\ell}$ of equal or lower over-all dimension $\delta_k + \delta_\ell$. Thus the dominant contribution as $E \rightarrow 0$, at least for small ϵ , will come from operators with $\delta_k = \delta_\ell = 1$, with leading corrections from $\delta_k + \delta_\ell = 3$, $\delta_k + \delta_\ell = 4$. As we shall show, the critical exponents $\lambda^{(n)}$ corresponding to these operators are calculable in terms of the leading exponent η .

The operators with $\delta_k = \delta_\ell = 1$ are, of course, just ψ and $\bar{\psi}$. We shall argue later that $(Z^{-1})_{ik} \neq 0$ in this case. Thus every Green function G_{ij} is asymptotically proportional to the two-point function.

4. CALCULATION TO $O(\epsilon)$

In this section we shall illustrate the general formalism given above by a calculation to $O(\epsilon)$. As previously mentioned, we are then concerned with the massless theory in $D = 4$. We shall regularize the theory with a cut-off $e^{-\alpha_0' k^2/\Lambda}$ in the propagator. Although this is somewhat more cumbersome than using dimensional regularization^{2,5)}, in fact the latter is inadequate for our problem. For example, most of the Z_{ij} defined in Section 3 have quadratic or higher divergences. Such divergences correspond to poles at $D = 2$, etc. in the dimensional regularization scheme, and thus do not appear in the Laurent expansion about $D = 4$. Thus dimensional regularization would set these Z_{ij} equal to zero. This is fine if the renormalized Green functions are the physical quantities, as in most of high-energy physics. However, in our case, it is the unrenormalized quantities which must be solved for, and we must retain all divergences. It was the use of dimensional regularization which led the authors of Ref. 12 to misleading results.

Another rather more pedantic objection to dimensional regularization is the use of ϵ both as a regulator, and as the parameter which is eventually set equal to 2 in the physical number of dimensions. This can lead to confusion and incorrect results in higher orders.

We first briefly describe the usual renormalization of the theory using a cut-off, to the one-loop level necessary for the $O(\epsilon)$ calculation. We use the Feynman rules of Ref. 2. Then

$$i\Gamma^{(1,1)}(E, \vec{k}) = E - \alpha'_0 k^2 + \frac{1}{2} \frac{r_0^2}{(2\pi)^4} \int \frac{d^4 k'}{E - \alpha'_0 (2k^2 + \frac{1}{2}k'^2)} e^{-\alpha'_0 (2k^2 + \frac{1}{2}k'^2)/\Lambda} \\ + \frac{1}{2} \frac{r_0^2}{(2\pi)^4} \int \frac{d^4 k'}{2\alpha'_0 k'^2} e^{-2\alpha'_0 k'^2/\Lambda} + O(r_0^4) \quad (4.1)$$

Keeping only the leading logarithmic divergences as $\Lambda \rightarrow \infty$ we then have

$$Z^{-1} = 1 - \frac{g_0^2}{(2\pi)^4} \frac{\pi^2}{8} \log(\Lambda/E_N) + O(g_0^4) \quad (4.2)$$

$$Z^{-1}\alpha' = \alpha'_0 \left[1 - \frac{g_0^2}{(2\pi)^4} \frac{\pi^2}{16} \log(\Lambda/E_N) + O(g_0^4) \right] \quad (4.3)$$

where $g_0 = r_0/\alpha'_0$. The three-point function is, at the normalization point

$$\Gamma^{(1,2)}(-E_N, -\frac{E_N}{2}, -\frac{E_N}{2}, k_i^2 = 0) \quad (4.4)$$

$$= \frac{r_0}{(2\pi)^{5/2}} - 2 \frac{r_0^3}{(2\pi)^{13/2}} \int \frac{d^4 k}{(E_N + 2\alpha'_0 k^2)(E_N/2 + 2\alpha'_0 k^2)} e^{-3\alpha'_0 k^2/\Lambda} + O(r_0^5) \\ = \frac{r_0}{(2\pi)^{5/2}} \left[1 - \frac{g_0^2}{(2\pi)^4} \frac{\pi^2}{2} \log(\Lambda/E_N) + O(g_0^4) \right] \quad (4.5)$$

So the dimensionless renormalized coupling g is

$$g = g_0 \left(1 - \frac{g_0^2}{(2\pi)^4} \frac{3\pi^2}{8} \log(\Lambda/E_N) + O(g_0^4) \right) \quad (4.6)$$

$$\beta(g) = E_N \frac{\partial g}{\partial E_N} = \frac{g^3}{(2\pi)^4} \frac{3\pi^2}{8} + O(g^5) \quad (4.7)$$

The above expressions give the correct $O(\epsilon)$ expansion of the exponents if we use $\beta(g, \epsilon) = (-\epsilon/4)g + \beta(g)$.

We now consider the renormalization of the operators $\psi^n, \bar{\psi}^n$ for general n . All vertices $\Gamma_{\psi^n}^{(N)}$ which have $n \geq N$ are divergent in perturbation theory. However, the only divergent diagrams at the one-loop level occur in $\Gamma_{\psi^n}^{(n)}$ and $\Gamma_{\psi^n}^{(n-1)}$; they are shown in Fig. 2. Therefore, by a generalization of the argument following (3.5), they can be made finite by subtracting insertions of the operators ψ^{n-1} , $\psi^{n-2}\partial_y\psi$, and $\psi^{n-2}\nabla^2\psi$. Following (3.6) we therefore write, to this order

$$\begin{aligned} (\Gamma_{\psi^n}^{(N)})_R = Z^{N/2} \left[Z_{n,n} \Gamma_{\psi^n}^{(N)} + Z_{n,\partial_y} \Gamma_{\psi^{n-2}\partial_y\psi}^{(N)} \right. \\ \left. + Z_{n,\nabla^2} \Gamma_{\psi^{n-2}\nabla^2\psi}^{(N)} + Z_{n,n-1} \Gamma_{\psi^{n-1}}^{(N)} + \dots \right] \end{aligned} \quad (4.8)$$

The additional operators which we have introduced will also require subtractions. Since, however, we wish ultimately to solve for $\Gamma_{\psi^n}^{(N)}$ to first order, and the off-diagonal Z_{ij} vanish to zeroth order, we need consider the renormalization of the additional operators only to zeroth order. For the case $N = n$, (4.8) reduces to

$$(\Gamma_{\psi^n}^{(n)})_R = Z^{n/2} Z_{n,n} \Gamma_{\psi^n}^{(n)} \quad (4.9)$$

The other vertices vanish because they have no one-particle irreducible pieces. Since

$$\Gamma_{\psi^n}^{(n)} = n! + O(g_0^2) \quad (4.10)$$

we impose the normalization condition

$$\Gamma_{\psi^n}^{(n)}(E_i = -E_N/n, k_i^2 = 0) = n! \quad (4.11)$$

To the one-loop level (keeping leading terms only as $\Lambda \rightarrow \infty$)

$$\Gamma_{\psi^n}^{(n)}(E_i = -E_N/n, k_i^2 = 0) = n! \left[1 - \frac{g_0^2}{(2\pi)^4} \frac{n(n-1)\pi^2}{4} \log(\Lambda/E_N) + O(g_0^4) \right] \quad (4.12)$$

So

$$Z_{n,n} = 1 + \frac{g_0^2}{(2\pi)^4} \frac{(4n^2 - 5n)\pi^2}{16} \log(\Lambda/E_N) + O(g_0^4) \quad (4.13)$$

For the case $N = n - 1$, all vertices in (4.8) contribute. We impose the normalization conditions

$$\Gamma_{\psi^n}^{(n-1)}(E_i = k_i^2 = 0) = 0 \quad (4.14)$$

$$\left. \frac{\partial}{\partial E_j} \Gamma_{\psi^n}^{(n-1)} \right|_{E_i = -E_n, k_i^2 = 0} = 0 \quad (4.15)$$

$$\left. \frac{\partial}{\partial k_j^2} \Gamma_{\psi^n}^{(n-1)} \right|_{E_i = -E_n, k_i^2 = 0} = 0 \quad (4.16)$$

Note that we can subtract $\Gamma_{\psi^n}^{(n-1)}$ at zero energy since it is quadratically divergent. To the one-loop level,

$$\Gamma_{\psi^n}^{(n-1)}(E_i, k_i) = \frac{n!}{2} \frac{i\tau_0}{(2\pi)^{3/2}} \sum_{j=1}^{n-1} \int \frac{d^4 k' e^{-\alpha'_0(2k'^2 + \frac{1}{2}k_j'^2)/\Lambda}}{E_j - 2\alpha'_0 k'^2 - \frac{1}{2}\alpha'_0 k_j'^2} + O(\tau_0^3) \quad (4.17)$$

so

$$\Gamma_{\psi^n}^{(n-1)}(0, 0) = -\frac{i\tau_0}{(2\pi)^{3/2}} \frac{n! (n-1) \pi^2}{8} \frac{\Lambda}{\alpha'_0{}^2} + O(\tau_0^3) \quad (4.18)$$

$$\left. \frac{\partial}{\partial E_j} \Gamma_{\psi^n}^{(n-1)} \right|_{\text{NORM. PT.}} = -\frac{i\tau_0}{(2\pi)^{3/2}} \frac{n! \pi^2}{8 \alpha'_0{}^2} \log(\Lambda/E_n) + O(\tau_0^3) \quad (4.19)$$

$$\left. \frac{\partial}{\partial k_j^2} \Gamma_{\psi^n}^{(n-1)} \right|_{\text{NORM. PT.}} = \frac{i\tau_0}{(2\pi)^{3/2}} \frac{n! \pi^2}{16 \alpha'_0} \log(\Lambda/E_n) + O(\tau_0^3) \quad (4.20)$$

At the zero loop level,

$$\Gamma_{\psi^{n-2} \partial_y \psi}^{(n-1)}(E_i, \vec{k}_i) = -(n-2)! \sum_{j=1}^{n-1} E_i + O(\tau_0^2) \quad (4.21)$$

$$\Gamma_{\psi^{n-2} \nabla^2 \psi}^{(n-1)}(E_i, \vec{k}_i) = -(n-2)! \sum_{j=1}^{n-1} k_j^2 + O(r_0^2) \quad (4.22)$$

$$\Gamma_{\psi^{n-1}}^{(n-1)}(E_i, \vec{k}_i) = (n-1)! + O(r_0^2) \quad (4.23)$$

Solving for the Z_{ij} ,

$$Z_{n, n-1} = \frac{i r_0}{(2\pi)^{3/2}} \frac{n(n-1)\pi^2}{8} \frac{\Lambda}{\alpha'_0} + O(r_0^3) \quad (4.24)$$

$$Z_{n, \partial_j} = -\frac{i r_0}{(2\pi)^{3/2}} \frac{n(n-1)\pi^2}{8 \alpha'_0} \log(\Lambda/E_N) + O(r_0^3) \quad (4.25)$$

$$Z_{n, \nabla^2} = \frac{i r_0}{(2\pi)^{3/2}} \frac{n(n-1)}{16 \alpha'_0} \pi^2 \log(\Lambda/E_N) + O(r_0^3) \quad (4.26)$$

To this order, we see that the matrix Z has the form $Z_{ik} = \delta_{ik} + \eta_{ik}$, where the diagonal elements of η_{ik} are $O(r_0^2)$ and the off-diagonal elements $O(r_0)$. The matrix that appears in (3.15) therefore has the form

$$\left(\sum_{\tilde{}} E_N \frac{\partial}{\partial E_N} Z^{-1} \right)_{ijkl} = -\delta_{jl} E_N \frac{\partial}{\partial E_N} \eta_{ik} - \delta_{ik} E_N \frac{\partial}{\partial E_N} \eta_{jl} + \text{higher order terms} \quad (4.27)$$

Since $(\partial/\partial E_N) Z_{n, n-1} = 0$, this verifies the decoupling theorem in this case. (Note that this would have been far less obvious if we had subtracted $\Gamma_{\psi^n}^{(n-1)}$ at $E_N \neq 0$). We now consider (3.32), restricted to a subspace with $\delta_i = n$, $\delta_j = m$, appropriate to the discussion of the Green function $\langle 0 | \psi^n(y, \vec{b}) \bar{\psi}(0, 0)^m | 0 \rangle_R$. The matrix H_{ik} has a diagonal element

$$H_{n, n} = \frac{g^2}{(2\pi)^4} \frac{4n^2 - 5n}{16} \pi^2 + O(g^4) \quad (4.28)$$

Since $H_{\partial_y, n} = H_{\nabla^2, n} = 0$ to this order, this gives us the eigenvalue. (There are, of course, other eigenvalues in this subspace, but to this order they do not couple to $\langle 0 | \psi^n \bar{\psi}^m | 0 \rangle_R$. Putting $g = g_1(\epsilon)$, (3.36) gives

$$\langle 0 | \psi^n \bar{\psi}^m | 0 \rangle_R \sim (-E)^{-3+n+m} \left[(2n^2 - 5n) + (2m^2 - 5m) + 5 \right] (\epsilon/2) + O(\epsilon^2) + O(\epsilon) \quad (4.29)$$

For $n = m = 1$, we recover the known result for the two-point function.

If we now solve for $\langle 0 | \psi^n \bar{\psi}^m | 0 \rangle_\Lambda$, to this order it will involve only $\langle 0 | \psi^n \bar{\psi}^m | 0 \rangle_R$, and Green functions of operators with one less dimension. It will not, in general, involve the two-point function. However, it is clear that the two point function will enter if we work to a sufficiently high order at which $\Gamma_{\psi n}^{(1)}$ is non-zero. The relevant $(Z^{-1})_{ij}$ will thus be $O(\epsilon^{n-1})$. However, since the Z_{ij} are not universal there is no reason to expect this contribution to be small in the general theory, even near $\epsilon = 0$.

5. EXACT RESULTS

In the previous sections, we showed how the dominant contribution to any Green function $\langle 0 | 0_i(y, b) \bar{0}_j(0, 0) | 0 \rangle_\Lambda$ as $y \rightarrow \infty$ is proportional to the two-point function. The principal non-leading terms will then be proportional to the renormalized Green functions of operators with $(\delta_i, \delta_j) = (2, 1), (1, 2)$ and $(2, 2)$. The eigenvalues of the renormalization group equation in these subspaces can be related to the leading exponent η . The operators in question are $\partial_y \psi$, $\nabla^2 \psi$ and ψ^2 , together with their conjugates. The first two are multiplicatively renormalizable since they are trivially related to ψ . There is therefore only one new eigenvalue, which is determined by the renormalization of ψ^2 , since the other operators in the subspace do not couple to it. In analogy with (4.9), we have

$$\left(\Gamma_{\psi^2}^{(2)} \right)_R = Z_{2,2} \Gamma_{\psi^2}^{(2)} \quad (5.1)$$

where we impose

$$\Gamma_{\psi^2}^{(2)}(E_i = -E_n/2, k_i^2 = 0) = 2 \quad (5.2)$$

On the other hand

$$\Gamma^{(1,2)} = \frac{1}{2} \gamma_0 \Gamma_{\psi^2}^{(2)} \quad (5.3)$$

as can be seen by comparing the diagrammatic expansions of each side, or by taking matrix elements of the equations of motion. Now

$$\Gamma^{(1,2)}(E_i = -E_N/2, k_i = 0) = \tau = g \alpha' D_4 E_N^{\epsilon/4} \quad (5.4)$$

so

$$Z_{2,2} = \frac{\tau_0 Z^{\frac{1}{2}}}{g \alpha' D_4 E_N^{\epsilon/4}} \quad (5.5)$$

$$Z_{2,2} E_N \frac{\partial}{\partial E_N} Z_{2,2}^{-1} = -\frac{1}{2} \gamma(g) + \beta(g)/g + \frac{D}{4} \gamma(\alpha; g)/\alpha' + \frac{\epsilon}{4} \quad (5.6)$$

Specializing to $g = g_1$ we have

$$H_{22} = -\frac{1}{2} \gamma - \frac{D}{4} (Z - 1) + \frac{\epsilon}{4} \quad (5.7)$$

which agrees with (4.29) to $O(\epsilon)$. Similarly, using the fact that $Z_{1,1} = Z^{-\frac{1}{2}}$, we find

$$H_{11} = \frac{1}{2} \gamma \quad (5.8)$$

These eigenvalues, when substituted in (3.36), yield

$$(G_{12})_R = (G_{21})_R \sim (-E)^0 \phi_{(1,2)} [(-E)^{-2} k^2] \quad (5.9)$$

$$(G_{22})_R \sim (-E)^{1-\gamma} \phi_{(2,2)} [(-E)^{-2} k^2] \quad (5.10)$$

These results, together with those for the two-point function G_{11} and its related derivatives, give all corrections to scaling through order $E^{1+O(\epsilon)}$, which originate in the multi-Pomeron couplings. Note that $(G_{12})_R$ is non-singular in E at $k = 0$, and thus does not contribute to the total cross-section. The results (5.9), (5.10) agree with those of Ref. (5), where, however, it was also deduced that they hold for the unrenormalized Green functions.

In the theory with only a triple-Pomeron coupling present, it is possible to calculate G_{12} and G_{22} in terms of G_{11} , independent of the renormalization group analysis.

The quantum equations of motion are

$$\frac{\partial \psi}{\partial y} = -\Delta_0 \psi + \alpha'_0 \nabla^2 \psi + i r_0 (\frac{1}{2} \psi^2 + \bar{\psi} \psi) \quad (5.11)$$

$$-\frac{\partial \bar{\psi}}{\partial y} = -\Delta_0 \bar{\psi} + \alpha'_0 \nabla^2 \bar{\psi} + i r_0 (\frac{1}{2} \bar{\psi}^2 + \bar{\psi} \psi) \quad (5.12)$$

Forming the product of (5.11) on the right with $\bar{\psi}(0,0)$, and taking the vacuum expectation value, we find

$$G_{21}(E, \vec{k}) = \frac{2i}{r_0} [(E - \Delta_0 - \alpha'_0 k^2) G_{11}(E, \vec{k}) - 1] \quad (5.13)$$

and, similarly,

$$G_{22}(E, \vec{k}) = \left(\frac{2i}{r_0}\right)^2 (E - \Delta_0 - \alpha'_0 k^2)^2 G_{11}(E, \vec{k}) \quad (5.14)$$

These results may also be readily obtained from diagrammatic considerations. They agree with our general results, except that the piece proportional to $(G_{22})_R$ appears to be absent in G_{22} .

When other, irrelevant, couplings are included, there is no reason to believe it to remain absent. In any case $(G_{22})_R$ must appear in the unrenormalized Green functions of higher dimensional operators.

Another confirmation of our general result comes from the theory in zero transverse dimensions⁶⁾. In that case, the critical Pomeron is attained only in the limit $\Delta_0 \rightarrow -\infty$. For Δ_0 large and negative, the level structure of the Hamiltonian is very simple. There is a first excited state of energy $e^{-2\Delta_0^2/r_0^2}$, while all other states have energy $O(|\Delta_0|)$. Thus, in the limit $\Delta_0 \rightarrow -\infty$,

$$\langle 0 | \psi(y), \bar{\psi}(0) | 0 \rangle \sim \langle 0 | \psi(1) \rangle e^{-E_1 y} \langle 1 | \bar{\psi} | 0 \rangle \quad (5.25)$$

or

$$G_{nm} \sim \left(\frac{a_n \bar{a}_m}{a_1 \bar{a}_1} \right) G_{11} + O(e^{-|\Delta_0| y}) \quad (5.26)$$

where

$$a_n = \langle 0 | \psi^n | 1 \rangle \quad \bar{a}_m = \langle 1 | \bar{\psi}^m | 0 \rangle$$

The coefficients are non-zero and can in fact be determined recursively¹³⁾.

6. CONCLUSIONS

We have shown how every unrenormalized Green function $\langle 0 | 0_i(y, \vec{b}) \bar{0}_j(0, 0) | 0 \rangle_A$ can be represented as a sum over all renormalized Green functions $\langle 0 | 0_k(y, \vec{b}) \bar{0}_l(0, 0) | 0 \rangle_R$ with $\delta_k \leq \delta_i$, $\delta_l \leq \delta_j$. Each renormalized Green function is a sum of scaling terms, whose exponents become degenerate and equal to the canonical value in the limit $D \rightarrow 4$. We have calculated these exponents for the cases $\delta_k, \delta_l = 1, 2$ and have given the $O(\varepsilon)$ expansion of the exponent in each degenerate set which controls the behaviour of $\langle 0 | \psi(y, \vec{b})^n \bar{\psi}(0, 0)^m | 0 \rangle_R$ as $\varepsilon \rightarrow 0$.

This complicated structure implies that it will be very difficult to predict relative magnitudes of quantities in the asymptotic region and their relation to low energy phenomenology, as explained in Section 2 is now also more complicated. The 'bare' couplings to the external particles, N_i , will, in effect, be renormalized. For example, the total cross-section for the scattering of A and B is given by the inverse Laplace transform of

$$\sigma(E) = \sum_{i,j} N_i^A N_j^B G_{ij}(E, k=0) = \sum_{i,j} (N_i^A)_R (N_j^B)_R (G_{ij})_R \quad (6.1)$$

where

$$(N_i^{(x)})_R = \sum_j (Z^{-1})_{ij} N_i^{(x)} \quad (x = A, B) \quad (6.2)$$

The Z_{ij} , being non-universal, depend in detail on all parameters of the theory, and are calculable only in perturbation theory. Note however that the dominant contribution in Eq. (6.1) comes from $(G_{11})_R \sim E^{\gamma-1}$ and thus the leading piece in the cross-section factorizes in the limit $s \rightarrow \infty$ as in the previous results²⁻⁵⁾.

Similar techniques are applicable to the consideration of the effect of the irrelevant operators in $\mathcal{L}'$. Expanding the amplitude in a power series in a typical irrelevant operator $\tilde{O}$ will lead to arbitrary numbers of insertions of this operator in the Green functions we have already considered. Thus, in defining suitable renormalized Green functions we are led to the renormalization of the vertex $\Gamma_{\tilde{O}}^{(n,m)}$, which is the one-particle irreducible truncated piece of the Fourier transform of

$$\langle 0 | \psi(y_1, \vec{t}_1) \dots \psi(y_n, \vec{t}_n) \tilde{O}(0,0) \bar{\psi}(y'_1, \vec{t}'_1) \dots \bar{\psi}(y'_m, \vec{t}'_m) | 0 \rangle \quad (6.3)$$

$\Gamma_{\tilde{O}}^{(n,m)}$ is renormalized by subtracting off vertices corresponding to operators of equal and lower dimension, and then making a multiplicative renormalization. For example, a quartic coupling $\bar{\psi}^2 \psi^2$ will require subtractions proportional to $\bar{\psi} \psi$ and $(\bar{\psi}^2 \psi + \bar{\psi} \psi^2)$, as well as operators of equal dimension. The first subtraction implies that inclusion of $\bar{\psi}^2 \psi^2$ will shift Δ_0 away from its critical value, as it should. Assuming this is compensated for, the second subtraction will change the value of g . This is known¹³⁾ to affect the approach to scaling term in σ_{tot} of order $(\ln s)^{-\gamma-\lambda}$. The residual renormalized operator will give a contribution to σ_{tot} with an exponent close to the canonical value for small ε , that is $(\ln s)^{-1+O(\varepsilon)}$.

Multiple insertions of $\tilde{O}$ will be rendered finite by the same renormalization constants evaluated for the single insertion, except for short distance effects, which cannot now be ignored. At this point, life becomes complicated.

Multiple insertions of $\tilde{O}$ are of direct physical interest in calculating s-channel discontinuities of amplitudes. Thus, $\langle 0 | \tilde{O}_3(0, b_3) O_2(y, b_2) O_1(y, b_1) | 0 \rangle$ appears in the calculation of the inclusive cross-section in the triple Regge region (Fig. 3a), and $\langle 0 | \tilde{O}_4(0, b_4) \tilde{O}_3(0, b_3) O_2(y, b_2) O_1(y, b_1) | 0 \rangle$ appears in the double triple Regge region (Fig. 3b). The problem of the short distance affects $b_1 \rightarrow b_2$ (and $b_3 \rightarrow b_4$) does not appear here since O_1 and O_2 (and O_3 and O_4) are constructed from commuting fields ψ_+ and ψ_- which are associated with the Pomeron above and below the M^2 cut¹⁵⁾.

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REFERENCES

- 1) V.N. Gribov, JETP 26, 414 (1968).
- 2) H.D.I. Abarbanel and J.B. Bronzan, Phys. Letters B48, 345 (1974); Phys. Rev. D 9, 2397 (1974).
- 3) A.A. Migdal, A.M. Polyakov and K.A. Ter-Martirosyan, Phys. Letters B48, 239 (1974); JETP 40, 420 (1974).
- 4) H.D.I. Abarbanel, J.G. Bronzan, R.L. Sugar and A.R. White, Phys. Reports 21C, 119 (1975).
M. Moshe, Recent developments in RFT, to be published in Phys. Reports.
- 5) H.D.I. Abarbanel, J.B. Bronzan, J. Bartels and D. Sidhu, Phys. Rev. D 12, 2459, 2799 (1975).
- 6) J.B. Bronzan, J.A. Shapiro and R.L. Sugar, Phys. Rev. D 14, 618 (1976).
- 7) H.D.I. Abarbanel, J.B. Bronzan, R.L. Sugar and A. Schwimmer, Phys. Rev. D 14, 632 (1976). The treatment of the case $\alpha_0 > \alpha_{0c}$ in this paper is incorrect; it does give however, the correct description of the case $\alpha_0 \leq \alpha_{0c}$.
- 8) E. Brézin, J.C. Le Guillou and J. Zinn-Justin, Field theoretical approach to critical phenomena in Phase Transition and Critical Phenomena, Vol. 6 (eds. C. Domb and M.S. Green), Academic Press Inc., New York, 1976.
- 9) W.R. Frazer, H. Hoffman, J.R. Fulco and R. Sugar, Phys. Rev. D 14, 2387 (1976).
- 10) See, for example, G. Calucci and R. Jengo, Nuclear Phys. B84, 413 (1975).
- 11) N.N. Bogoliubov and D.V. Shirkov, Introduction to the theory of quantized fields (Interscience, New York, 1959).
- 12) C. Pajares and R. Pascual, Phys. Rev. D 16, 1941 (1977).
- 13) J.L. Cardy, Nuclear Phys. B115, 141 (1976).
- 14) W.R. Frazer and M. Moshe, Phys. Letters 57B, 475 (1975); Phys. Rev. D 12, 2370 (1975).
- 15) On Reggeon field theory for cut and uncut Pomerons, see P. Suranyi, Phys. Rev. D 12, 2124 (1975); J.L. Cardy and P. Suranyi, Phys. Rev. D 13, 1064 (1976); and M. Ciafaloni, G. Marchesini and G. Veneziano, Nuclear Phys. B98, 472, 493 (1976). Application of cut Reggeon field theory to discontinuities of the type of Fig. 3 can be found in M. Moshe and F.E. Paige, Phys. Rev. D 16, 869 (1977).

Table 1

Dimensions of some low order operators

$O_i(\vec{b}, y)$	$\delta_i = [O_i]$ at $D = 4$
ψ	1
ψ^2	2
$\partial_y \psi$	
$\nabla^2 \psi$	
ψ^3	3
$\vec{\nabla} \psi \cdot \vec{\nabla} \psi$	
$\psi \partial_y \psi$	
$\partial_y^2 \psi$	
$\psi \nabla^2 \psi$	
$\partial_y \nabla^2 \psi$	

Figure captions

- Fig. 1 : Expansion of $\Gamma_{0i}^{(n)}$ defined by Eq. 3.3 .
- Fig. 2 : Expansion of $\Gamma_{\psi n}^{(n)}$ and $\Gamma_{\psi n}^{(n-1)}$ whose renormalization scheme is described in Section 4.
- Fig. 3 : Multiple insertions of 0_i in the triple Regge region (a) and in the double triple Regge region (b).

$$\Gamma_{\psi}^{(n)} = \delta_{n1}$$

$$\Gamma_{\psi^2}^{(1)} = \text{cloud diagram} + \text{cloud with internal line} + \text{cloud with internal loop} + \text{cloud with two internal lines} + \dots$$

$$\Gamma_{\psi^2}^{(2)} = 2! + \text{triangle diagram} + \text{triangle diagram with internal line} + \dots$$

$$\Gamma_{\psi^2}^{(3)} = \text{triangle diagram} + \text{triangle diagram with internal line} + \text{triangle diagram with internal loop} + \dots$$

FIG.1

$$\Gamma_{\psi^n}^{(n)} = \text{truncated part of } \left\{ \left(\text{triangle diagram} \right) n + \left(\text{triangle diagram with internal line} \right) n + \left[n(n-1)-1 \right] \text{ similar diagrams} + \dots \right\}$$

$$\Gamma_{\psi^n}^{(n-1)} = \text{truncated part of } \left\{ \left(\text{triangle diagram} \right) (n-1) + \left[\frac{n(n-1)}{2} - 1 \right] \text{ similar diagrams} + \dots \right\}$$

FIG. 2

(a)

(b)

FIG.3