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POINCARÉ AND CONFORMAL SUPERGRAVITY MODELS
WITH CLOSED ALGEBRAS

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ABSTRACT

We present applications of a recent formulation of supergravity, in which only three auxiliary fields are needed to close the gauge algebra. Previously constructed models, with and without matter, in asymptotically flat or de Sitter space are shown to arise from minimal couplings only. Because the transformation rules of the gauge fields are model-independent, several new models are obtained by adding separately invariant actions.

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1. INTRODUCTION

Recently, we found the minimal set of auxiliary fields, needed to close the gauge algebra of supergravity¹⁾. The result turned out to be very simple: only a scalar S , a pseudoscalar P and an axial vector field A_μ are needed. Using these auxiliary fields, one can write gauge and matter actions which are separately invariant. An obvious, but invaluable, consequence is that one can now obtain actions for complicated systems by simply adding invariant subactions, without the tiresome order-by-order constructions which were needed previously. The action and transformation rules of the gauge fields are fixed once and for all, while the irreducible matter actions considered in this article are obtained by minimal coupling with respect to all local symmetries present. Thus this simplifies the matter-coupling problem considerably and clarifies the structure of supergravity.

Before proceeding, we wish to stress that there are actually two theories of supergravity: Poincaré supergravity^{2,3)} which is based on the graded Poincaré or de Sitter group, and conformal supergravity^{4,5)}, which is based on the graded conformal group. At first sight, they seem quite different: the first has only one local supersymmetry invariance, while conformal supergravity has two local supersymmetries, called Q - and S - conformal supersymmetry. Surprisingly, we have found that there is a close relationship. For example, as we shall show below, the Poincaré supersymmetry laws for the gauge fields which appear in both supergravities, are a linear combination of the two conformal supersymmetry laws.

We study the Maxwell-Einstein system⁶⁾ with spin content $(2, \frac{3}{2}) + (1, \frac{1}{2})$, the Poincaré supergravity gauge action with cosmological term⁷⁾ and the coupling of Poincaré and conformal supergravity. In these cases we find that all interactions and transformation rules are minimal, i.e. they can be obtained by replacing ordinary derivatives by completely covariant derivatives in the kinetic terms and flat-space laws. Thus it seems that the structure of these matter couplings is now understood. There are a large number of other systems to be investigated, in particular, the scalar multiplet coupling⁸⁾ and the $O(n)$ models⁹⁾. It is to be hoped that our method will be useful in constructing new models, and that similarly simple results will follow.

The use of auxiliary fields enables one to write down one of the two one-loop counter terms of Poincaré supergravity completely, with all fields off shell. The other counter term⁵⁾ we give explicitly in linearized form, again with all fields off shell. We also analyse the particle content of a general supersymmetric action of $\alpha R + \beta R^2 + \gamma R_{\mu\nu}^2$ type.

In flat space our conventions are the positive Pauli metric

$$\delta_{\mu\nu} = \delta_{\mu}^{\nu} = \delta^{\mu\nu} = (+, +, +, +) ; \epsilon^{1234} = \epsilon_{1234} = 1$$

while all five gamma matrices are Hermitian with unit square

$$\gamma_5^2 = \gamma_{\mu}^2 = 1, \mu=1,4 ; \sigma_{\mu\nu} = \frac{1}{4}(\gamma_{\mu}\gamma_{\nu} - \gamma_{\nu}\gamma_{\mu}) ; \{\gamma_{\mu}, \gamma_{\nu}\} = 2\delta_{\mu\nu}$$

The matrices γ_4 and γ_5 are imaginary. We define spin-2 and spin- $\frac{3}{2}$ curvatures by

$$R = e_a^{\nu} e_b^{\mu} (\partial_{\mu} \omega_{\nu}^{ab} + \omega_{\mu}^{ac} \omega_{\nu c}^b) - (\mu \leftrightarrow \nu)$$

$$R^{\mu} = e^{-1} \epsilon^{\mu\nu\sigma} \gamma_5 \gamma_{\nu} D_{\rho} \psi_{\sigma} , D_{\rho} \psi_{\sigma} = \partial_{\rho} \psi_{\sigma} + \frac{1}{2} \omega_{\rho ab} \tau^{ab} \psi_{\sigma}$$

The connection $\omega_{\mu ab}$ contains gauge torsion and never torsion of matter; thus it is minimally supersymmetric covariant. *A priori*, in the Maxwell-Einstein system the terms quartic in the spin- $\frac{1}{2}$ fields could be obtained either from matter torsion⁶⁾ or from eliminating the axial-vector auxiliary field¹⁾, but since only in the latter case the local algebra closes, we believe the formulation without matter torsion to be the correct one.

2. THE GAUGE ALGEBRAS OF POINCARÉ AND CONFORMAL SUPERGRAVITY

The gauge action of Poincaré supergravity, usually referred to as the gauge action of supergravity, contains three local symmetries: general coordinate invariance (G), local Lorentz invariance (L) and local supersymmetry (S). In second-order formalism, without auxiliary fields, there are only two independent fields: the spin-2 vierbein field e_{μ}^a and the spin- $\frac{3}{2}$ gravitino field ψ_{μ} . The transformation rules of these two gauge fields under the three local symmetries are as follows:

$$\delta_G e_{\mu}^a = e_{\mu}^a \partial_{\nu} \xi^{\nu} + \xi^{\alpha} \partial_{\alpha} e_{\mu}^a , \delta_G \psi_{\mu} = \psi_{\alpha} \partial_{\mu} \xi^{\alpha} + \xi^{\alpha} \partial_{\alpha} \psi_{\mu}$$

$$\delta_L e_{\mu}^a = \lambda^a_b e_{\mu}^b , \delta_L \psi_{\mu} = \frac{1}{2} \lambda_{ab} \tau^{ab} \psi_{\mu}$$

$$\delta_S e_{\mu}^a = \kappa \bar{\epsilon} \gamma^a \psi_{\mu} , \delta_S \psi_{\mu} = \frac{2}{\kappa} D_{\mu} \epsilon$$

(1)

where $\sigma^{ab} = \frac{1}{4}(\gamma^a \gamma^b - \gamma^b \gamma^a)$ and $D_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{2} \omega_{\mu ab} \sigma^{ab} \epsilon$. The non-propagating spin connection field $\omega_{\mu ab}$ is expressed in terms of the two gauge fields e_μ^a and ψ_μ by solving its algebraic field equation

$$\omega_{\mu ab} = \omega_{\mu ab}(e) + \frac{\kappa^2}{4} (\bar{\Psi}_\mu \gamma_a \Psi_b - \bar{\Psi}_\mu \gamma_b \Psi_a + \bar{\Psi}_a \gamma_\mu \Psi_b) \quad (2)$$

The spin connection field without torsion $\omega_{\mu ab}(e)$ is easily found by requiring that

$$\partial_\mu e_\nu^a + \omega_\mu^{ab} e_{b\nu} - \Gamma_{\mu\nu}^\alpha e_\alpha^a = 0$$

We find

$$\omega_{\mu ab}(e) = \frac{1}{2} e_a^\nu (\partial_\mu e_{b\nu} - \partial_\nu e_{b\mu} + e_b^\tau e_\mu^\tau \partial_\tau e_{c\nu}) - (a \leftrightarrow b) \quad (3)$$

Thus $\omega_{\mu ab}$ is supersymmetric covariant, i.e. it does not contain $\partial \epsilon$ terms after a local supersymmetry variation.

The commutator of two bosonic symmetries shows that the parameters ξ^α and λ^{ab} are a world-tensor Lorentz-scalar, and a world-scalar Lorentz-tensor, respectively. Acting on e_μ^a or ψ_μ , the uniform result is

$$\begin{aligned} [\delta_G(\xi^\alpha), \delta_G(\gamma^\alpha)] &= \delta_G [\gamma^\alpha \partial_\alpha \xi^\beta - \xi^\alpha \partial_\alpha \gamma^\beta] \\ [\delta_L(\lambda_{ab}), \delta_G(\xi^\alpha)] &= \delta_L [\xi^\alpha \partial_\alpha \lambda_{ab}] \\ [\delta_L(\lambda_{ab}), \delta_L(\omega_{ab})] &= \delta_L [\lambda_a^c \omega_{cb} - \omega_a^c \lambda_{cb}] \end{aligned} \quad (4)$$

Similarly, the commutator of a bosonic and a fermionic symmetry shows that ϵ is a world-scalar Lorentz-spin- $\frac{1}{2}$ parameter

$$\begin{aligned} [\delta_S(\epsilon), \delta_G(\xi^\alpha)] &= \delta_S [\xi^\alpha \partial_\alpha \epsilon] \\ [\delta_S(\epsilon), \delta_L(\omega_{ab})] &= \delta_S [\frac{1}{2} \omega_{ab} \gamma^{ab} \epsilon] \end{aligned} \quad (5)$$

But the commutator of two supersymmetries acts differently on e_μ^a than on ψ_μ . For e_μ^a one finds

$$[\delta_S(\epsilon_1), \delta_S(\epsilon_2)] = \delta_G[t^c e_c^\alpha] + \delta_L[t^c \omega_{cab}] + \delta_S[-\frac{\kappa}{2} t^c \psi_c] \quad (6)$$

$$t^c = 2(\bar{\epsilon}_2 \gamma^c \epsilon_1) \quad (\text{one } e^a_\mu)$$

hence local symmetries with field-dependent parameters. But for ψ_μ one finds, in addition to these terms, extra terms

$$[\delta_S(\epsilon_1), \delta_S(\epsilon_2)] \psi_\mu = \text{as for } e^a_\mu + (\bar{\epsilon}_1 \gamma^\alpha \epsilon_2) \left[\frac{3}{4} \gamma_\alpha g_{\mu\rho} \right. \\ \left. - \frac{1}{2} \gamma_\mu g_{\alpha\rho} - \sigma_{\alpha\mu} \gamma_\rho \right] R^\rho + (\bar{\epsilon}_1 \sigma^{\rho\eta} \epsilon_2) \left[-\frac{1}{2} \sigma_{\rho\eta} \gamma^\alpha \psi_{\alpha\mu} \right. \\ \left. - \gamma_\mu \gamma^\alpha \psi_{\alpha\rho} - \frac{1}{2} \sigma_{\rho\eta} \gamma_\mu \sigma^{ab} \psi_{ab} - e \epsilon_{\mu\rho\eta\tau} \gamma_5 R^\tau \right] \quad (7)$$

with $\gamma^\alpha \psi_{a\mu} = R_\mu - \frac{1}{2} \gamma_\mu \gamma^\alpha R$ and $\sigma^{ab} \psi_{ab} = \frac{1}{2} \gamma^\alpha R$. Thus, the gauge algebra of Poincaré supergravity closes only on shell.

Recently it was shown that by adding three fields to the local supersymmetry transformation rules, the resulting algebra closes¹⁾. These three fields are a scalar field S , a pseudoscalar field P and an axial vector field A_μ . The new transformation rules are given by*)

$$\delta e^a_\mu = \kappa \bar{\epsilon} \gamma^a \psi_\mu$$

$$\delta \psi_\mu = \frac{2}{\kappa} D_\mu \epsilon + \frac{1}{3} \gamma_\mu (S - i \gamma_5 P) \epsilon + i (A_\mu - \frac{1}{3} \gamma_\mu A) \gamma_5 \epsilon$$

$$\delta S = \frac{1}{2} \bar{\epsilon} \gamma \cdot R - \frac{\kappa}{2} \bar{\epsilon} \gamma \cdot \psi S - \frac{i\kappa}{2} \bar{\epsilon} \gamma_5 \gamma \cdot \psi P + \frac{i\kappa}{2} \bar{\epsilon} \gamma_5 A \cdot \psi$$

$$\delta P = \frac{-i}{2} \bar{\epsilon} \gamma_5 \gamma \cdot R - \frac{\kappa}{2} \bar{\epsilon} \gamma \cdot \psi P + \frac{i\kappa}{2} \bar{\epsilon} \gamma_5 \gamma \cdot \psi S + \frac{\kappa}{2} \bar{\epsilon} A \cdot \psi$$

$$\delta A_\mu = \frac{3i}{2} \bar{\epsilon} \gamma_5 (R_\mu - \frac{1}{3} \gamma_\mu \gamma \cdot R) + \frac{i\kappa}{2} \bar{\epsilon} \gamma_5 \psi_\mu S + \frac{\kappa}{2} \bar{\epsilon} \psi_\mu P \quad (8)$$

$$- \frac{\kappa}{2} \bar{\epsilon} \gamma \cdot \psi A_\mu + \kappa \bar{\epsilon} A \psi_\mu - \frac{e\kappa}{4} \epsilon_{\mu\nu\rho\sigma} A^\nu \bar{\epsilon} \gamma_5 \gamma^\rho \psi^\sigma$$

*) We have replaced $S \rightarrow iS$ and $P \rightarrow iP$ in Ref. 1.

The new action is obtained by simply adding the squares of S , P , and A_μ to the old action

$$\mathcal{L} = \mathcal{L}(SG) + \frac{e}{3} (-S^2 - P^2 + A_\mu A^\mu) \quad (9)$$

and the gauge action of Poincaré supergravity is given by

$$\mathcal{L}(SG) = \frac{1}{2\kappa^2} e R - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \bar{\Psi}_\mu \gamma_5 \gamma_\nu D_\rho \Psi_\sigma \quad (10)$$

while the symbol R^μ denotes the spin- $3/2$ field equation

$$e R^\mu = \epsilon^{\mu\nu\rho\sigma} \gamma_5 \gamma_\nu D_\rho \Psi_\sigma \quad (11)$$

(a torsion term in R^μ , obtained by partially integrating the derivative D_ρ , vanishes identically after a Fierz rearrangement)¹⁰⁾.

In first-order Poincaré supergravity, the commutator of two local supersymmetries holds:

- for e_μ^a only modulo the ω field equation;
- for ψ_μ only modulo the ω and ψ field equations;
- for $\omega_{\mu ab}$ only modulo the ω , ψ and e field equations.

We have not tried to close this algebra, since second-order formalism seems to be the more natural in the context of our work below.

The gauge action of conformal supergravity⁴⁾ contains the following local symmetries:

- Q-supersymmetry, S-supersymmetry
 - Weyl invariance, chiral invariance
 - general coordinate and local Lorentz invariance .
- (12)

It contains three gauge fields, the vierbein and gravitino fields e_μ^a and ψ_μ , and a chiral gauge field A_μ . Conformally invariant theories have no lower spin gauge components. The local symmetries remove these lower spin gauge components: Q- and S-supersymmetry remove the $\partial \cdot \psi$ and $\gamma \cdot \psi$ spin- $1/2$ components of ψ_μ , chiral invariance the $\partial \cdot A$ spin-0 component of A_μ , while Weyl and the gravitational invariances remove the spin-1 and two spin-0 components of e_μ^a . It follows that invariance under local conformal boosts is not needed to remove any remaining gauge components, and indeed the conformal gauge fields are inert under this symmetry. The transformation rules under Q-supersymmetry read*)

$$\begin{aligned}\delta_Q e_\mu^a &= \bar{\epsilon} \gamma^a \psi_\mu, & \delta_Q \psi_\mu &= 2 \left(D_\mu + \frac{i}{2} A_\mu \gamma_5 \right) \epsilon \\ \delta_Q A_\mu &= \frac{i}{2} \bar{\epsilon} \gamma_5 \gamma^\lambda \left(\hat{D}_\lambda \psi_\mu - \hat{D}_\mu \psi_\lambda + \frac{\epsilon}{2} \gamma_5 \epsilon_{\lambda\mu\rho\sigma} \hat{D}_\rho \psi_\sigma \right) \\ \hat{D}_\lambda \psi_\mu &= \partial_\lambda \psi_\mu + \frac{1}{2} \omega_{\lambda ab} \tau^{ab} \psi_\mu + \frac{i}{2} A_\lambda \gamma_5 \psi_\mu\end{aligned}\tag{13}$$

Under S-supersymmetry, Weyl and chiral rotations one has

$$\delta_S e_\mu^a = 0, \quad \delta_S \psi_\mu = -\gamma_\mu \epsilon, \quad \delta_S A_\mu = -\frac{3i}{2} \bar{\epsilon} \gamma_5 \psi_\mu\tag{14}$$

$$\delta_W e_\mu^a = \Lambda e_\mu^a, \quad \delta_W \psi_\mu = -\frac{1}{2} \Lambda \psi_\mu, \quad \delta_W A_\mu = 0\tag{15}$$

$$\delta_A e_\mu^a = 0, \quad \delta_A \psi_\mu = -\frac{i\alpha}{2} \gamma_5 \psi_\mu, \quad \delta_A A_\mu = \partial_\mu \alpha\tag{16}$$

Explicit evaluation shows that the local commutator algebra closes off shell⁴⁾.

For two local Q-supersymmetry transformations one finds

$$\begin{aligned}[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] &= \delta_G[t^c e_c{}^\lambda] + \delta_L[t^c \omega_{cab}] + \delta_Q[-\frac{1}{2} t^c \psi_c] \\ &+ \delta_A\left[\frac{4}{3} t^c A_c\right] + \delta_S\left[-t^c \not{D}_c + \gamma^\mu \epsilon_2 (\bar{\not{D}}_\mu \epsilon_1) - \gamma^\mu \epsilon_1 (\bar{\not{D}}_\mu \epsilon_2)\right]\end{aligned}\tag{17}$$

*) The normalization of A_μ differs from Ref. 4: $A_\mu(\text{Ref. 4}) = -\frac{2}{3} A_\mu(\text{here})$.

where again $t^c = 2\bar{\epsilon}_2 \gamma^c \epsilon_1$, and ϕ_μ is the original S-gauge field which has been expressed by means of three geometrical constraints in terms of the remaining gauge fields as follows:

$$\phi_\mu = \frac{1}{3} \gamma^\nu \left[\hat{D}_\nu \psi_\mu - \hat{D}_\mu \psi_\nu + \frac{e}{2} \gamma_5 \epsilon_{\mu\nu\rho\sigma} \hat{D}^\sigma \psi^\rho \right] \quad (18)$$

3. RELATIONS BETWEEN CONFORMAL AND POINCARÉ SUPERGRAVITY

The transformations of the gauge fields e_μ^a , ψ_μ and A_μ under local Poincaré supersymmetry transformations can be written in a suggestive fashion as follows:

$$\begin{aligned} \delta e_\mu^a &= \kappa \bar{\epsilon} \gamma^a \psi_\mu \\ \delta \psi_\mu &= \frac{2}{\kappa} \left(\hat{D}_\mu + \frac{i\kappa}{2} A_\mu \gamma_5 \right) \epsilon + \frac{1}{3} \gamma_\mu (S - i\gamma_5 P - iA\gamma_5) \epsilon \\ \delta A_\mu &= \frac{3i}{2} \bar{\epsilon} \gamma_5 \left(\hat{R}_\mu - \frac{1}{3} \gamma_\mu \gamma \cdot \hat{R} \right) + \frac{i\kappa}{2} \bar{\epsilon} (S - i\gamma_5 P - iA\gamma_5) \gamma_5 \psi_\mu \end{aligned} \quad (19)$$

The symbol $\hat{R}^\mu$ denotes the chirally covariant spin- $3/2$ field equation

$$e \hat{R}^\mu = \epsilon^{\mu\nu\rho\sigma} \gamma_5 \gamma_\nu \left(\hat{D}_\rho + \frac{i\kappa}{2} A_\rho \right) \psi_\sigma \quad (20)$$

In this way of writing, A_μ plays the role of a chiral gauge field. This is reminiscent of the conformal supergravity action, which indeed has a chiral gauge field, in addition to the fields e_μ^a and ψ_μ . Comparing the above expressions with the transformations of e_μ^a , ψ_μ , and A_μ under Q- and S-supersymmetry transformations of conformal supergravity, one sees that the Poincaré local symmetry rules of e_μ^a , ψ_μ and A_μ are a linear combination of the Q- and S-supersymmetry rules of conformal supergravity with S-supersymmetry parameters $-1/3 (S - i\gamma_5 P - iA\gamma_5) \epsilon$. This observation opens up the perspective of coupling combinations of superconformal matter systems, Poincaré supergravity gauge action and conformal supergravity gauge action, as we now discuss.

As a first application we will write down an R^2 -type locally supersymmetric action. Namely, the sum of the Poincaré and conformal supergravity gauge actions is invariant under Poincaré supersymmetry. The Lagrangian reads schematically

$$\mathcal{L} = \mathcal{L}(\text{CSG}; e, \psi, A) + \mathcal{L}(\text{PSG}; e, \psi, S, P, A) \quad (21)$$

and is of course not conformal invariant. One can eliminate S and P without losing invariance, since their field equations $S = P = 0$ give a consistent truncation, but the gauge algebra then no longer closes. At the linearized level one can display Q- and S- conformal supersymmetry in a manifest way:

$$\mathcal{L}(\text{CSG}, \text{lin}) = e \left[R_{\mu\nu}^2 - \frac{1}{3} R^2 - \bar{\Psi}_\mu (\not{\partial} \delta_{\mu\nu} - \partial_\mu \not{\partial}_\nu) (R^\nu - \frac{1}{3} \gamma^\nu \gamma \cdot R) - \frac{1}{3} F_{\mu\nu}^2 \right] \quad (22)$$

For the non-linear terms in action and transformation laws we refer to Ref. 4. This R^2 -type supersymmetric action can be interpreted as one of the two one-loop counter terms with all fields off shell. Note that one cannot eliminate A_μ algebraically from the total action.

One might wonder what the second counter term looks like. Its gravitational sector starts with R^2 . At the linearized level⁵⁾, there is the following one-to-one correspondence between the quantities of the scalar multiplet A, B, λ , F, G and the curvatures and auxiliary fields R, R^μ, S, P, A_μ

$$\begin{array}{ccccc} A & B & \lambda & F & G \\ S & P & \gamma \cdot R & R & \partial \cdot A \end{array} \quad (23)$$

By requiring invariance under global supersymmetry transformations at the linearized level, one finds the explicit linearized R^2 action

$$\mathcal{L} = e \left[R^2 + \bar{R} \cdot \gamma \not{\partial} \gamma \cdot R - 4(\partial_\mu S)^2 - 4(\partial_\mu P)^2 + 4(\partial \cdot A)^2 \right] \quad (24)$$

It is likely that this expression can be extended to its full non-linear glory by means of any of the well-known methods, since an explicit calculation of the graviton self-energy due to a spin- $\frac{3}{2}$ loop¹¹⁾ yielded non-zero coefficients for $R_{\mu\nu}^2 - \frac{1}{3} R^2$ and R^2 . One can again rewrite the gravitino part as

$$\bar{R} \cdot \gamma \not{\partial} \gamma \cdot R = 2 \bar{\Psi}^{\mu} (\square \delta_{\mu\nu} - \partial_{\mu} \partial_{\nu}) \gamma^{\nu} \gamma \cdot R \quad (25)$$

Suppose that the full R^2 action is known. Then the effective Poincaré action plus both R^2 and $R_{\mu\nu}^2$ actions have the following form at the linearized level

$$\begin{aligned} \mathcal{L}(\text{eff}) &= e \left[\alpha \mathcal{L}_{R^2} + \beta \mathcal{L}_C + \gamma \mathcal{L}_R \right] \\ \mathcal{L}_R &= -\frac{1}{2k^2} R - \frac{1}{2} \bar{\Psi}_{\mu} R^{\mu} + \frac{1}{3} (-S^2 - P^2 + A_{\mu}^2) \\ \mathcal{L}_{R^2} &= \frac{1}{k^2} R^2 + 2 \bar{\Psi}^{\mu} (\square \delta_{\mu\nu} - \partial_{\mu} \partial_{\nu}) \gamma^{\nu} \gamma \cdot R + 4 (S \square S + P \square P - A^{\mu} \partial_{\mu} \partial \cdot A) \\ \mathcal{L}_C &= \frac{1}{k^2} (R_{\mu\nu}^2 - \frac{1}{3} R^2) - \bar{\Psi}^{\mu} (\square \delta_{\mu\nu} - \partial_{\mu} \partial_{\nu}) (R^{\nu} - \frac{1}{3} \gamma^{\nu} \gamma \cdot R) - \frac{1}{3} F_{\mu\nu}(A)^2 \end{aligned} \quad (26)$$

We now investigate the particle content of $\mathcal{L}(\text{eff})$.

For the purely gravitational sector it is known that these are: one massless and one massive spin-2 excitation, and one massive spin-0 excitation¹²⁾. If all masses are real, only the massive spin-2 excitation is a ghost. We now turn to the other fields.

The spin- $3/2$ equation reads

$$4\alpha (\square \delta_{\mu\nu} - \partial_{\mu} \partial_{\nu}) \gamma^{\nu} \gamma \cdot R - 2\beta (\square \delta_{\mu\nu} - \partial_{\mu} \partial_{\nu}) (R^{\nu} - \frac{1}{3} \gamma^{\nu} \gamma \cdot R) - \gamma R^{\mu} = 0 \quad (27)$$

Contracting with γ^{μ} and using $\partial \cdot R = 0$ yields

$$(\square - m^2) \gamma \cdot R = 0, \quad m^2 = \frac{\gamma}{12\alpha} \quad (28)$$

Consider next a linear combination of R^{μ} and $\gamma \cdot R$ such that it satisfies the massive spin- $3/2$ constraints

$$\hat{R}_\mu = R_\mu + \epsilon (\square \delta_{\mu\nu} - \partial_\mu \partial_\nu) \gamma^\nu \gamma \cdot R \quad (29)$$

The constraint $\partial \cdot \hat{R} = 0$ is manifestly satisfied; the constraint $\gamma \cdot \hat{R} = 0$ requires

$$\epsilon = \frac{-1}{3m^2} \quad (30)$$

A little computation shows that $\hat{R}_\mu$ satisfies the Klein-Gordon equation and is thus a massive spin- $3/2$ field

$$(\square - M^2) \hat{R}_\mu = 0, \quad \partial \cdot \hat{R} = \gamma \cdot \hat{R} = 0, \quad M^2 = \frac{-\gamma}{2\beta} \quad (31)$$

In order not to produce tachyons, when one starts from the Einstein action where $\gamma = 1$, one must have that α is positive and β is negative.

The spin-1 equation reads

$$-8\alpha \partial_\mu \partial \cdot A - \frac{4\beta}{3} \partial_\nu F_{\mu\nu} + \frac{2}{3} \gamma A_\mu = 0 \quad (32)$$

Contracting with ∂_μ one obtains the Proca equation for a massive spin-1 field

$$(\square - m^2) \partial \cdot A = 0, \quad m^2 = \frac{\gamma}{12\alpha} \quad (33)$$

The remainder $\hat{A}_\mu = A_\mu + \eta \partial_\mu \partial \cdot A$ satisfies the Lorentz condition $\partial \cdot \hat{A} = 0$ if η is chosen properly

$$\hat{A}_\mu = A_\mu - \frac{12\alpha}{\gamma} \partial_\mu \partial \cdot A \quad (34)$$

From the original equation for A_μ and the equation for $\partial \cdot A$ it follows that $\hat{A}_\mu$ is a massive spin-1 (Proca) field

$$(\square - M^2) \hat{A}_\mu = 0, \quad \partial \cdot \hat{A} = 0, \quad M^2 = \frac{-\gamma}{2\beta} \quad (35)$$

Finally we turn to the fields S and P. Their field equations are given by

$$(\square - m^2) S = (\square - m^2) P = 0, \quad m^2 = \frac{\gamma}{12\alpha} \quad (36)$$

We see thus that a supersymmetric action of the form

$$\frac{1}{2} \beta R_{\mu\nu}^2 + \frac{1}{2} \left(\alpha - \frac{\beta}{3} \right) R^2 - \frac{\gamma}{2\kappa^2} R + \dots \quad (37)$$

contains the following:

- i) One massive $(2, \frac{3}{2}, \frac{3}{2}, 1)$ multiplet with mass $M^2 = -\gamma/2\beta$, given by $e_{a\mu}, \hat{R}_\mu, \hat{A}_\mu$. If there are no tachyons ($\alpha > 0, \beta < 0$), all excitations are ghosts.
- ii) One massive $(\frac{1}{2}, 0^+, 0^-)$ multiplet with mass $\gamma/(12\alpha)$, given by S, P, and one of the two solutions of $(\square - m^2)\gamma \cdot R = 0$. These fields are physical if there are no tachyons.
- iii) Another massive $(\frac{1}{2}, 0^+, 0^-)$ multiplet with the same mass, given by the other spin- $\frac{1}{2}$ solution of $(\square - m^2)\gamma \cdot R = 0$, $\partial \cdot A$ and the scalar mode in $R + R^2$. These particles are physical if all masses are real. For example, in a suitable gauge $R_\mu \sim \not{\partial} \psi_\mu$ and decomposing $[(\not{\partial} + m)(\not{\partial} - m)\not{\partial}]^{-1}$ one finds that both residues of the massive spin- $\frac{1}{2}$ excitations are positive.
- iv) The usual massless spin $(2, \frac{3}{2})$ multiplet.

4. THE MAXWELL-EINSTEIN SYSTEM

The Maxwell-Einstein system was obtained by coupling the spin- $(1, \frac{1}{2})$ vector multiplet to the spin- $(2, \frac{3}{2})$ gauge action of Poincaré supergravity and was the first example of a matter coupling⁶⁾. Action and transformation rules were obtained by adding successively at each order in κ extra terms to action and transformation rules such that at each order in κ invariance was obtained. It was also shown that this process must stop after a finite number of steps if no spin-0 fields are present⁶⁾. The action obtained reads

$$\begin{aligned} \mathcal{L} = \mathcal{L}(\text{S.G.}) - \frac{e}{4} F_{\mu\nu}^2 - \frac{e}{2} \bar{\lambda} \gamma^\mu D_\mu \lambda + \frac{3}{64} e \kappa^2 (\bar{\lambda} \gamma_5 \gamma_\rho \lambda)^2 \\ + \frac{e \kappa}{2} \bar{\Psi}_\mu \sigma^{\alpha\beta} \gamma^\mu \lambda F_{\alpha\beta}(V) + \frac{e \kappa^2}{4} (\bar{\Psi}_\mu \sigma^{\rho\sigma} \gamma^\mu \lambda) (\bar{\Psi}_\rho \gamma_\sigma \lambda) \end{aligned} \quad (38)$$

We repeat that D_μ always denotes the gravitational covariant derivative with gauge torsion but not matter torsion. The transformation rules were found to be

$$\begin{aligned} \delta e^a_\mu &= \kappa \bar{\epsilon} \gamma^a \psi_\mu, & \delta V_\mu &= -\bar{\epsilon} \gamma_\mu \lambda \\ \delta \psi_\mu &= \frac{2}{\kappa} D_\mu \epsilon + \frac{\kappa}{4} (\bar{\lambda} \gamma_5 \gamma^\rho \lambda) (\gamma_\rho \gamma_\mu - \sigma_{\rho\mu}) \gamma_5 \epsilon \\ \delta \lambda &= [F_{\mu\nu}(V) + \kappa \bar{\Psi}_\mu \gamma_\nu \lambda] \sigma^{\mu\nu} \epsilon \end{aligned} \quad (39)$$

We will now introduce auxiliary fields such that the commutator algebra closes; this will simplify these results and clarify their structure.

The first point to be observed is that the Poincaré supergravity gauge action with auxiliary fields $\mathcal{L}(\text{SG}, e, \psi, S, P, A)$ is separately invariant under the gauge transformation of Section 2, and thus the gauge action need no longer be considered. This is because we expect the gauge algebra to be model-independent and, in particular, the gauge transformation rules to be independent of the matter fields. Their separation of the action into separately invariant parts is one of the great advantages of introducing auxiliary fields. The second point to be stressed is that globally supersymmetric matter systems have in general their own auxiliary fields needed to close the global algebra in flat space. Obviously, these matter auxiliary fields must be carried over to the locally supersymmetric extension of the theory. In the case of the vector multiplet there is one pseudoscalar auxiliary field D , which enters into the action simply as

$$\mathcal{L} = \frac{e}{2} D^2 \quad (40)$$

and into the flat-space transformation rules as

$$\delta \lambda = F_{\alpha\beta}(V) \sigma^{\alpha\beta} \epsilon + i D \gamma_5 \epsilon; \quad \delta V_\mu = -\bar{\epsilon} \gamma_\mu \lambda; \quad \delta D = i \bar{\epsilon} \gamma_5 \not{\partial} \lambda \quad (41)$$

A very important remark is that the Maxwell-Einstein system is locally scale invariant (Weyl invariant). It is thus also globally conformal invariant, and it seems plausible that it can be made locally superconformally invariant and be coupled to the superconformal gauge action. (This we will indeed do below.) Since the superconformal gauge action only contains the axial-vector auxiliary field A_μ , it follows that one only needs the auxiliary field A_μ and not S or P in the action and transformation rules for the coupling of the vector multiplet to the conformal supergravity gauge action. We will now argue that also the coupling of the vector multiplet to the Poincaré supergravity gauge action will only involve A_μ . Since the Poincaré transformation rules are a linear combination of the conformal Q - and S - supersymmetry rules, one must find these conformal Q - and S - rules for the matter multiplet. But since the superconformal gauge fields are inert under local conformal boosts K_μ , the matter fields will be locally K -inert as well, and from $[K_\mu, Q_a] = (\gamma_\mu)_a^b S_b$ it follows that the matter fields λ , A_μ , and D are also S -inert. [Note that this argument only applies to the vector multiplet, since it transforms according to the group; but it does not apply to the gauge fields, since their transformation rules have been modified by constraints⁴⁾.] Since the auxiliary fields S and P only entered into the coefficients of the S -supersymmetry laws, the transformation rules of λ , A_μ and D under local Poincaré supersymmetry will be independent of S and P . We conclude that only A_μ appears in the coupling of the vector multiplet to Poincaré supergravity. This field A_μ must be coupled in a locally chiral invariant way, since the superconformal gauge action is chirally gauge invariant. (We note that S and P still appear in the Poincaré supergravity gauge action and rules, but can be eliminated consistently, if desired.)

Finally, it should be stressed that field equations transform into field equations according to¹⁾

$$\delta I_j = -I_k \left\{^\alpha R^k_{\alpha,j} \right. \quad (42)$$

If $D = 0$, the λ field equation rotates under a local supersymmetry transformation into the V_μ field equation without a derivative on ϵ (since $\delta V_\mu = -\bar{\epsilon} \gamma_\mu \lambda$; note that since $\delta \lambda = F \cdot \sigma \epsilon + i D \gamma_5 \epsilon$, the opposite is not true). Thus, the λ -field equation must be supersymmetrically covariant, i.e. it does not contain $\partial \epsilon$ if it is subjected to a local supersymmetry variation. This observation even dispenses one from explicitly constructing the Noether current of the matter action, since one can at once replace $\partial_\mu \lambda$ by its covariantization. [In the varied action $\partial \epsilon$ terms of course do appear, since $\delta \mathcal{L} = \partial_\mu \kappa^\mu$ with $\kappa^\mu \epsilon$ -dependent].

With these four preceding observations one can at once write down the action and transformation rules without any algebra

$$\begin{aligned} \mathcal{L} = \mathcal{L}(PSG; e, \psi, S, P, A) - \frac{e}{4} F_{\mu\nu}^2 + \frac{e}{2} D^2 - \frac{e}{2} \bar{\lambda} \gamma^\mu \tilde{D}_\mu \lambda \\ + \frac{e\kappa}{4} (\bar{\psi} \sigma^{\alpha\beta} \gamma^\mu \lambda) F_{\alpha\beta}(v) \\ \delta \lambda = \hat{F}_{\mu\nu} \sigma^{\mu\nu} \epsilon + i D \gamma_5 \epsilon ; \quad \delta A_\mu = -\bar{\epsilon} \gamma_\mu \lambda \\ \delta D = i \bar{\epsilon} \gamma_5 \gamma^\mu \tilde{D}_\mu \lambda - \frac{\kappa}{2} \bar{\epsilon} \gamma_5 \psi D \equiv i \bar{\epsilon} \gamma_5 \gamma^\mu \bar{D}_\mu \lambda \end{aligned} \quad (43)$$

Here $\hat{F}_{\mu\nu}$ is the covariant field strength

$$\hat{F}_{\mu\nu} = (\partial_\mu V_\nu - \partial_\nu V_\mu + \frac{\kappa}{2} \bar{\psi} \gamma_{\mu\nu} \psi) - (\mu \leftrightarrow \nu) \quad (44)$$

while $\bar{D}_\mu \lambda$ is the completely covariant derivative^{*)} i.e. gravitationally, supersymmetric and chirally covariant

$$\bar{D}_\mu \lambda = D_\mu \lambda + \frac{i\kappa}{2} A_\mu \gamma_5 \lambda - \frac{\kappa}{2} \hat{F}_{\alpha\beta} \sigma^{\alpha\beta} \psi_\mu - \frac{i\kappa}{2} D \gamma_5 \psi_\mu \quad (45)$$

The connection terms are of course the Noether couplings arising from the universal coupling

$$\bar{\psi} \gamma_{ss}^\mu + A_\mu \gamma_5^\mu + e_{\alpha\mu} \tau^{\alpha\mu} \quad (46)$$

which have been further covariantized. The last term in the action is a consequence of the requirement that the λ field equation be covariant but not the action. Terms with one λ appear in the field equation with a weight $1/2$ with respect to terms quadratic in λ , and thus one must give the terms linear in λ an extra factor two in the action. A final simplification occurs by replacing eD^2 by $(D')^2$ in the action and transformation rules. Then the D' law becomes

$$\delta D' = i e^{1/2} \bar{\epsilon} \gamma_5 \gamma^\mu \tilde{D}_\mu \lambda \quad (47)$$

and the whole theory is truly minimal^{**) .}

*) Note that Eq. (42) requires that the D -term in the derivative on λ be absent in the action.

**) We thank Dr. M. Roček for this observation.

It is thus clear that the complete coupling is simply a minimal coupling with respect to gravity, supersymmetry and chirality; or, equivalently, that replacing ordinary derivatives by completely covariant derivatives yields at once the total action and transformation rules.

Also clear is that replacing in the total action the Poincaré supergravity gauge action $\mathcal{L}(\text{PSG}, \bar{e}, \psi, S, P, A)$ by the conformal supergravity gauge action $\mathcal{L}(\text{CSG}, e, \psi, A)$ yields again an invariant action, such that the sum is again invariant.

We observe that these above results remain unchanged for the Yang-Mills case⁶⁾. One can also add separately invariant actions, such that the sum is again invariant. For example, one can add a term $\alpha S + \beta P$ plus corresponding $\psi\psi$ terms to the Poincaré or conformal supergravity action, plus the vector multiplet. The sum of all these terms is again invariant since the auxiliary field equations do not interfere.

The interplay between the conformal and Poincaré supergravity should have important consequences for other matter couplings as well. For instance, one can immediately couple the scalar multiplet in improved form to Poincaré supergravity. Recently, the improved (superconformal) scalar multiplet was coupled to conformal supergravity¹³⁾. In that theory the scalar multiplet is unphysical and by fixing the local axial, dilatational and S-supersymmetry gauge, one regains Poincaré supergravity. Using our techniques, one can instead couple a physical scalar multiplet to Poincaré supergravity. The corresponding action is the sum of the Poincaré action with auxiliary fields¹⁾ and the action for the improved (superconformal) multiplet. The Poincaré supersymmetry transformation laws, which leave this action invariant, are the superconformal Q laws plus a field-dependent uniform S transformation with parameter $\epsilon_S = \frac{1}{3}(i\gamma_5 P + iA \gamma_5 - S)\epsilon$. This reproduces the Poincaré transformation rules for the gauge fields e_μ^a , ψ_μ , S , P , A_μ while only the matter field χ gets modified by a term $\delta\chi = (A - iB\gamma_5)\epsilon_S$. The transformation rules for S supersymmetry can be obtained by using Jacobi identities and $[K_\lambda, Q] = \gamma_\lambda S$, while $[Q, \chi]_+ \sim \not{A} + iB\gamma_5$ and $\partial_\mu A \sim [P_\mu, A]$. Since all fields are K-inert, the desired rules follow immediately.

5. PURE SUPERGRAVITY WITH COSMOLOGICAL CONSTANT

We next consider the gauge action of de Sitter supergravity. Action and transformation rules are given by

$$\mathcal{L} = \mathcal{L}(SG, e, \psi) + \frac{eg}{\kappa} \bar{\psi}_\mu \sigma^{\mu\nu} \psi_\nu + \frac{3eg^2}{\kappa^4} \quad (48)$$

$$\delta e^a_\mu = \kappa \bar{\epsilon} \gamma^a \psi_\mu ; \quad \delta \psi_\mu = \frac{2}{\kappa} \left(D_\mu \epsilon + \frac{g}{2\kappa} \gamma_\mu \epsilon \right) \quad (49)$$

Hence it is the sum of the Poincaré supergravity gauge action plus a cosmological and a ψ -mass term; the latter is actually needed in a de Sitter space in order that ψ_μ remains massless. We repeat that $\omega_{\mu ab} = \omega_{\mu ab}(e) + K_{\mu ab}(e, \psi)$.

As noted in the last section we can put aside the action $\mathcal{L}(\text{SG})$, since it is separately invariant. From the term $\partial \psi_\mu \sim \gamma_\mu \epsilon$ above, we see that only the term $\partial \psi_\mu \sim \gamma_\mu \epsilon S$ survives in the full transformation law of ψ_μ after elimination of the auxiliary fields. Hence, only S appears in the matter action. The most general expression which contains a term linear in S , is given by

$$\mathcal{L}_S = a e \frac{S}{\kappa^2} + b \frac{e}{\kappa} \bar{\psi}_\mu \sigma^{\mu\nu} \psi_\nu + c \frac{e}{\kappa} \bar{\psi} \cdot \psi \quad (50)$$

Recalling the transformation laws of Section 2

$$\delta S = \frac{1}{2} \bar{\epsilon} \gamma \cdot R - \frac{\kappa}{2} \bar{\epsilon} \gamma \cdot \psi S + \dots \quad (51)$$

$$\delta \psi_\mu = \frac{2}{\kappa} D_\mu \epsilon + \frac{1}{3} \gamma_\mu \epsilon S + \dots \quad (52)$$

one finds from the $\epsilon \psi S$ terms and the $\bar{\epsilon} R$ terms in the action, using that $\sigma^{\mu\nu} D_\mu \psi_\nu = \frac{1}{4} \gamma \cdot R$, the result $c = 0$ and $a = 2b$, so that

$$\mathcal{L}_S = e b \left[\frac{1}{\kappa} \bar{\psi}_\mu \sigma^{\mu\nu} \psi_\nu + \frac{2}{\kappa^2} S \right] \quad (53)$$

Eliminating S from the S -dependent terms in the total action

$$\delta_S \left(-\frac{e}{3} S^2 + 2b \frac{e}{\kappa^2} S \right) = 0 \quad \rightarrow \quad S = \frac{3b}{\kappa^2} \quad (54)$$

and substituting this result back into action and transformation rules, one obtains with $P = A_\mu = 0$

$$\mathcal{L} = \frac{3eb^2}{\kappa^4} + \frac{eb}{\kappa} \bar{\psi}_\mu \sigma^{\mu\nu} \psi_\nu, \quad \delta \psi_\mu = \frac{2}{\kappa} D_\mu \epsilon + \frac{b}{\kappa^2} \gamma_\mu \epsilon \quad (55)$$

Clearly, with $b = g$ one obtains the desired equations.

In this example, the auxiliary field S was essential. In principle, one could also start from P ; such a term would break parity. However, no new features arise since the chiral rotation $\psi_\mu \rightarrow \exp(-i\pi/4 \gamma_5) \psi_\mu$ and $S \rightarrow P$ transforms this case back to the previous case. In fact, any $(\alpha S + \beta P)$ term leads to a cosmological term with positive coefficient. More interesting is the case of adding a term linear in D to the vector multiplet action, and extending this to a locally supersymmetric invariant. This might have important consequences for the elimination of unwanted cosmological terms.

Note added: After completion of this manuscript we received a preprint by K. Stelle and P.C. West, ICTP preprint 77-78/6, who have found the same results as in Ref. 1, except the matter coupling and relation to conformal supergravity.

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