Ref.TH.2491-CERN

THE CLASSICAL YANG-MILLS FIELD OF A QUARK SOURCE

M. Magg

CERN -- Geneva

ABSTRACT

The Coulomb solution of the classical Yang-Mills equations with external source is unstable for sufficiently strong coupling constant. We show that the Yang-Mills system has no other physically static, spherically symmetric solution. Some consequences for the quark confinement problem in non-Abelian gauge theories are pointed out.

Ref.TH.2491-CERN

18 April 1978

1. INTRODUCTION

In this paper we are studying the classical Yang-Mills field of a static and spherically symmetric external source. The quantum analogue of this field theory is considered to be a model for a (spinless) quark and its gluon field in the limit of a large quark mass. It is widely believed that the quark-confinement mechanism in the quark gluon theory is virtually due to gluon dynamics alone and that one will not miss the point if the quarks are taken to be infinitely heavy¹⁾.

The Coulomb potential is the quasi-Abelian, static solution of the Yang-Mills equations with static external source. For sufficiently strong coupling this solution becomes unstable and hence physically meaningless²⁾. This paper is a step in the direction to find the stable solution(s) of the Yang-Mills equations with strong coupling to an external static source.

Some authors used massless scalar electrodynamics as a toy model to study the behaviour of classical and quantum mechanical Yang-Mills systems^{3),4)}. Mandula has found a static, spherically symmetric solution for the external source problem in massless scalar electrodynamics with an energy less than the corresponding Coulomb solution which shows a complete screening of the external charge. Contrary to what this hint seems to indicate we will show that in the Yang-Mills case there is no further static and spherically symmetric solution besides the Coulombic one.

We did not succeed to prove this uniqueness theorem without the symmetry condition, which is much more restrictive for massless vector fields than for scalar ones.

In the following section we introduce our notation and we define our concept of a physically static and spherically symmetric gauge field. The third section characterizes gauge potentials of physically static gauge fields. The uniqueness statement for static spherically symmetric solutions of the Yang-Mills source is proved in the fourth section. Finally we comment on the consequences of our mathematical result for quantum Yang-Mills theory and possible relations to quark confinement.

2. PHYSICALLY STATIC AND SYMMETRIC GAUGE FIELDS

We are studying solutions of the classical $SU(2)$ Yang-Mills equations coupled to an external source j^u :

$$D_i F^{i0}(\vec{r}, t) = j^0(\vec{r}, t) \quad (1a)$$

$$D^0 F^{0i}(\vec{r}, t) + D_j F^{ji}(\vec{r}, t) = j^i(\vec{r}, t) \quad (1b)$$

The symbol D_μ denotes the usual covariant derivative and $F^{\mu\nu}$ is the field strength tensor. The gauge potentials as well as all other fields are written in matrix notation

$$A^\mu = \sum_{a=1}^3 A^{a\mu} T^a$$

$$T^a = -\frac{i}{2} \sigma^a$$

using the Pauli matrices σ^a . The restriction to gauge groups $SU(2)$ is simply for convenience, all our arguments can be carried over to $SU(3)$ as well.

Here we are interested only in external sources j^μ with vanishing three-vector parts in some rest frame

$$j^\mu(\vec{r}, t) = \delta^{\mu 0} j^0(\vec{r}, t) \quad (2)$$

In the following we always refer to this specific frame, defined by Eq. (2).

We call a solution of Eqs. (1) and (2) (physically) static if all gauge invariant fields which can be formed out of the potentials are time-independent. In the same spirit we call a solution spherically symmetric if all gauge invariant scalar fields which can be constructed out of the potentials are spherically symmetric.

Both these definitions are manifestly gauge invariant and give a fairly appropriate description of a physically static (spherically symmetric) gauge field. In this context, other quite more restrictive concepts for time independence and symmetry can be found in the literature⁵⁾. They have the advantage to be mathematically very convenient, but it is a priori not clear to which extent these definitions exhaust the physical meaning of time independence or spherical symmetry, respectively. Part of this paper can be understood as an extension of the uniqueness theorem derived in Ref. 5) to all physically static and spherically symmetric gauge fields. For the rest of this paper we work in a class of gauges (we call them "privileged" gauges) for which the external source points in a constant direction in internal space

$$j^0(\vec{r}, t) = g(\vec{r}, t) T^3 \quad (3)$$

Within this class of gauges defined by Eq. (3), we are free to use an Abelian subgroup of gauge transformations

$$g(\vec{r}, t) = \exp(\chi(\vec{r}, t) T^3) \quad (4)$$

There is a well-known consistency condition for the external source

$$D_\mu j^\mu(\vec{r}, t) = 0 \quad (5)$$

For source (2) this equation is equivalent with to the following statements

$$A^0(\vec{r}, t) = A^{30}(\vec{r}, t) T^3 \quad (6)$$

$$g(\vec{r}, t) = g(\vec{r}) \quad (7)$$

The source density ρ is a gauge invariant quantity and therefore it has to be spherically symmetric for a spherically symmetric solution of Eqs. (1) and (2). For technical reasons we assume $\rho(r) \neq 0$ almost everywhere in $0 \leq r < \infty$. Sources with compact support may be obtained by a weak limiting process. The mathematical heart of our argument is the well-known impossibility to comb a hedgehog (every continuous tangent field on the sphere S^2 has a zero)⁶⁾. We will repeatedly make use of this statement. As an immediate consequence one finds that for a spherically symmetric solution every gauge invariant vector field is in fact a spherically symmetric radial field. The proof is simple. The set of gauge invariant vector fields which can be derived from the potential contains the unit radial field

$$\frac{1}{g'(r)} \vec{\nabla} g(r)$$

It can be used to project any given vector field onto the tangent space of the spheres $r = \text{const}$. For a spherically symmetric solution this tangentially projected field has to have constant length on the spheres $r = \text{const}$, hence it vanishes identically. The original field was a spherically symmetric radial field.

3. POTENTIAL OF A STATIC SOLUTION

Our physically motivated definition of time independence does not refer directly to the gauge potentials and therefore seems hard to handle in concrete calculations, which mostly start with some ansatz for the gauge potential. Fortunately, this difficulty can be resolved for solutions of the Yang-Mills equations with external source (2).

Theorem

Given a static solution of the Yang-Mills equations with external source (2), there is a gauge such that the potentials A^μ do not depend on time.

The proof starts with the inspection of the following gauge invariant fields:

$$\text{Tr}\{j^0 F^{\mu\nu}\}, \quad \text{Tr}\{F^{\mu\nu} F^{\kappa\lambda}\}, \quad \text{Tr}\{j^0 [F^{\mu\nu}, F^{\kappa\lambda}]\}$$

Their time independence constrains the field strength tensor to the form

$$\begin{aligned} F^{3\mu\nu}(\vec{r}, t) &= F^{3\mu\nu}(\vec{r}) \\ F_{\pm}^{\mu\nu}(\vec{r}, t) &= \exp(\pm i\Lambda(\vec{r}, t)) \cdot F_{\pm}^{\mu\nu}(\vec{r}) \\ (\Lambda(\vec{r}, 0) &= 0) \end{aligned} \quad (8)$$

Here it was convenient to decompose the field with respect to the matrices

$$T_{\pm} = \frac{1}{\sqrt{2}} (T_1 \pm iT_2)$$

We now choose the gauge $A^0 = 0$.

The electric fields are now time derivatives of the potentials

$$F^{0i}(\vec{r}, t) = \partial_0 A^i(\vec{r}, t) \quad (9)$$

The field $\text{Tr}\{j^0 D^i F^{i0}\}$ (no summation over i) is time independent only if

$$\partial_0 (A_+^i(\vec{r}, t) A_-^i(\vec{r}, t)) = 0 \quad (10)$$

Only the phase of $A_{\pm}^i$ depends on time. Because of Eq. (8) this phase is identical with Λ and, moreover, Λ is linear in time

$$\Lambda(\vec{r}, t) = t \cdot \Lambda(\vec{r}) \quad (11)$$

The potential has now the following form in the $A^0 = 0$ gauge

$$\begin{aligned} \vec{A}^3(\vec{r}, t) &= \vec{A}^3(\vec{r}) + t \dot{\vec{A}}^3(\vec{r}) \\ \vec{A}_\pm(\vec{r}, t) &= \vec{A}_\pm(\vec{r}) \cdot \exp(\pm i t \Lambda(\vec{r})) \end{aligned} \quad (12)$$

A gauge transformation of type (4) (with Λ instead of χ) changes the potentials

$$\begin{aligned} A^{30}(\vec{r}, t) &= -\Lambda(\vec{r}) \\ A_\pm^0(\vec{r}, t) &= 0 \\ \vec{A}^3(\vec{r}, t) &= \vec{A}^3(\vec{r}) + t (\dot{\vec{A}}^3(\vec{r}) - \vec{\nabla} \Lambda(\vec{r})) \\ \vec{A}_\pm(\vec{r}, t) &= \vec{A}_\pm(\vec{r}) \end{aligned} \quad (13)$$

Simultaneously, this gauge rotation cancels the time-dependent phase of the field strength tensor (8).

For the $\pm$ components of the magnetic field this implies

$$\vec{A}_\pm(\vec{r}) \times (\dot{\vec{A}}^3(\vec{r}) - \vec{\nabla} \Lambda(\vec{r})) = 0 \quad (14)$$

The constraint equation (1a) on the other hand requires

$$\vec{A}_\pm(\vec{r}) \cdot (\dot{\vec{A}}^3(\vec{r}) - \vec{\nabla} \Lambda(\vec{r})) = 0 \quad (15)$$

Therefore either $\dot{\vec{A}}^3 - \vec{\nabla} \Lambda(\vec{r})$ or $\vec{A}_\pm(\vec{r})$ has to vanish. In the first case, Eq. (13) presents a time-independent form for the potential. In the second case, the field is quasi-Abelian and the potentials can be written in the Coulomb form.

4. UNIQUENESS THEOREM

As we mentioned before the Yang-Mills equations with static source (2) have at least one static solution which is simply the quasi-Abelian Coulomb potential.

$$\begin{aligned} A^0(\vec{r}) &= A^3(\vec{r}) T^3 \\ \vec{A}(\vec{r}) &= 0 \\ \Delta A^3(\vec{r}) &= -\rho(\vec{r}) \end{aligned} \quad (16)$$

One knows for the case without external source that Yang-Mills and Maxwell equations have exactly the same static finite energy solutions namely just the trivial one⁷⁾. The following theorem extends this statement to the case with external source.

Theorem

The Coulomb solution (16) is the only static and spherically symmetric solution of the Yang-Mills equations with external source (2).

The preceding theorem allows us to choose the potentials time independent

$$\begin{aligned} A^0(\vec{r}, t) &= -\Lambda(\vec{r}) T^3 \\ \vec{A}(\vec{r}, t) &= \vec{A}^a(\vec{r}) T^a \end{aligned} \quad (17)$$

Spherical symmetry of the vector fields $\text{Tr} \{j_0 F^{0i}\}$, $\text{Tr} \{j_0 F^{jk}\} \epsilon^{ijk}$, $\text{Tr} \{j_0 [F^{0j}, F^{0k}]\} \epsilon^{ijk}$ restricts the potentials quite a lot:

$$\Lambda(\vec{r}) = \Lambda(r) \quad (18)$$

$$\vec{\nabla} \times \vec{A}^3(\vec{r}) = \frac{\vec{r}}{r} g(r) \quad (19)$$

$$\vec{A}^1(\vec{r}) \times \vec{A}^2(\vec{r}) = \frac{\vec{r}}{r} f(r) \quad (20)$$

Assume now $f \neq 0$ in Eq. (20). This means both fields $\vec{A}^1$ and $\vec{A}^2$ are tangential on the spheres $r = \text{const.}$ The inequality

$$|f(r)| \leq |\vec{A}^1(\vec{r})| \cdot |\vec{A}^2(\vec{r})| \quad (21)$$

is then a contradiction to the assumption $f \neq 0$ because the right-hand side of (21) vanishes for some direction, according to the hedgehog theorem. The vector fields $\vec{A}^1$ and $\vec{A}^2$ are therefore parallel to each other. After applying a gauge transformation (4), we can therefore take $\vec{A}^2 = 0$ in Eq. (17). Equation (22) can hold only for vanishing $\vec{\nabla} \times \vec{A}^3$. The potential (17) is then quite simple

$$\begin{aligned} A^0(\vec{r}) &= -\Lambda(r) T^3 \\ \vec{A}(\vec{r}) &= \vec{\nabla} \chi(\vec{r}) T^3 + \vec{A}^1(\vec{r}) T^1 \end{aligned} \quad (22)$$

and the field equations are equivalent to the following system of differential equations

$$\Delta \Lambda(r) - \Lambda(r) (\vec{A}^1(\vec{r}))^2 = g(r) \quad (23)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}^1(\vec{r})) = \Lambda^2(r) \vec{A}^1(\vec{r}) \quad (24)$$

$$(\vec{A}^1(\vec{r}))^2 \cdot \vec{\nabla} \chi(\vec{r}) = 0 \quad (25)$$

Equation (25) means either $\vec{A}_1 = 0$ or $\vec{\nabla} \chi = 0$. $\vec{A}_1 = 0$ is nothing but the Coulomb solution, therefore we are left with the case $\vec{\nabla} \chi = 0$. The field strength tensor is now given by

$$\begin{aligned} F^{0i}(\vec{r}) &= \partial^i \Lambda(r) T^3 - \Lambda(r) A^{1i}(\vec{r}) T^2 \\ F^{ij}(\vec{r}) &= (\partial^i A^{1j}(\vec{r}) - \partial^j A^{1i}(\vec{r})) T^1 \end{aligned} \quad (26)$$

and the assumed rotation invariance of the fields

$$\text{Tr} \{ F^{0i} F^{0i} \}, \quad \text{Tr} \{ F^{ij} F^{ij} \}, \quad \text{Tr} \{ F^{ij} [j_i^0, F^{0i}] \}$$

can hold only with rotation invariant fields

$$|\vec{A}'(\vec{r})|, \quad |\vec{\nabla} \times \vec{A}'(\vec{r})|, \quad \vec{A}'(\vec{r}) \times (\vec{\nabla} \times \vec{A}'(\vec{r}))$$

The same argument, which led to parallelism of $\vec{A}^1$ and $\vec{A}^2$ [after Eq. (20)], proves now the equation

$$\vec{\nabla} \times \vec{A}'(\vec{r}) = \lambda(r) \vec{A}'(\vec{r}) \quad (27)$$

Note that the function λ can depend only on r . Equation (27) is compatible with the equation of motion (27) only if

$$\lambda'(r) \frac{\vec{r}}{r} \times \vec{A}'(\vec{r}) = 0 \quad (28)$$

This means either λ is constant or $\vec{A}^1$ is a rotation invariant field. In the second case $\vec{\nabla} \times \vec{A} = 0$ and therefore $\vec{A}$ itself vanishes and we are again back at the Coulomb potential. $\lambda = \text{const}$ implies $\Lambda = \lambda = \text{const}$ and Eq. (24) is just the wave equation for light with frequency Λ . Equation (23) admits only those solutions of Eq. (24) which have constant length on the spheres $r = \text{const}$. In electrodynamics it is well known that such solutions do not exist. The reason is again the hedgehog theorem. In the limit $r \rightarrow \infty$ every solution of the vector wave equation becomes a tangential field which has to vanish at some point of the sphere $r = \text{const}$. Therefore it vanishes identically.

5. SPECULATIONS AND CONCLUSIONS

We have shown that the Coulomb potential is the only solution of the Yang-Mills equations with external source which is physically static and spherically symmetric. This solution is known to become unstable if the coupling strength exceeds a certain critical value and then becomes physically unacceptable. Therefore, we are left with three possibilities for the external source problem in the Yang-Mills system:

- i) a stable classical solution does not exist;
- ii) the stable solutions have genuine time dependence;
- iii) the stable solutions are physically asymmetric in space.

In cases (ii) and (iii) we have spontaneous symmetry breakdown for time translations or space rotations, respectively. In this context, stability obviously includes the possibility of zero energy modes leading from one solution to the time translated (space rotated) one. In case (i) one loses the starting point for

a semi-classical expansion of the quantum Yang-Mills field of an external (quark) source and one can speculate on the existence of this sector of the quantum field theory. The breakdown of time translation invariance seems to be a strange possibility especially if one thinks of non-periodic time dependence. Case (iii) is somewhat exciting. The spontaneous breakdown of rotation symmetry may be quite helpful in order to get a string-like field configuration in the corresponding quantum theory. Of course, this is only a hint from classical field theory and the string itself has to be formed by quantum effects.

This behaviour of the classical Yang-Mills system is quite different from what is expected by comparison with massless scalar electrodynamics, where Mandula has found a further physically static spherically symmetric solution with an energy lower than that of the Coulomb field. The symmetry condition is much more restrictive for the massless vector fields in the Yang-Mills case than for the charged scalar fields in scalar electrodynamics.

The completely screened quark source which was found in scalar electrodynamics may provide an argument for charge confinement in this theory but not necessarily for confinement of quark-antiquark and three-quark systems. Looking at classical solutions we think the chances for quark confinement in non-Abelian gauge theories are considerably better than for the strong coupling phase of scalar electrodynamics.

ACKNOWLEDGEMENTS

I would like to thank J. Jersák and W. Nahm for carefully reading the manuscript.

REFERENCES

- 1) G. 't Hooft, On the Phase Transition Toward Permanent Quark Confinement, University of Utrecht preprint, December 1977.
- 2) J.E. Mandula, Phys. Letters B67 (1977) 175.
M. Magg, to be published in Phys. Letters B.
- 3) J.E. Mandula, Phys. Letters B66 (1977) 495.
- 4) J.E. Mandula and L. McLerran, Phys. Letters B73 (1978) 193.
- 5) M. Ikeda and Y. Miyachi, Progr. Theor. Phys. 27 (1962) 474.
H.G. Loos, Nuclear Phys. 72 (1965) 677.
- 6) P. Alexandroff and H. Hopf, Topologie, p. 481, Springer Verlag Berlin (1935).
- 7) S. Deser, Phys. Letters B64 (1976) 463.
S. Coleman, New Phenomena in Subnuclear Physics, Part A., Ed. A. Zichichi (1977), Plenum Press New York.