Ref.TH.2496-CERN

INVERSE SCATTERING-LIKE PROBLEM FOR SUPERSYMMETRIC MODELS

L. Girardello

Istituto di Fisica, Università degli Studi, Milano
I.N.F.N. Sezione di Milano, Italy

and

S. Sciuto ^{*)}

CERN - Geneva

A B S T R A C T

We determine a linear problem associated with the equations of motion of supersymmetric sine-Gordon and alike theories, in a way reminiscent of the inverse scattering method. As a consequence, a rigorous proof of the existence of infinitely many currents for such models is derived.

*) On leave from Istituto di Fisica Teorica dell'Università, Torino.

Ref.TH.2496-CERN

1 May 1978

Recent results ¹⁾⁻³⁾ seem to suggest that two-dimensional field theories which correspond, at the classical level, to integrable Hamiltonian systems, may retain complete integrability when fermionic fields are added in a supersymmetric way. A first signal for that is the existence of higher order bosonic conservation laws in the supersymmetric sine-Gordon model ³⁾.

Now the standard approach for a thorough investigation relies on the inverse scattering method ⁴⁾. This consists of the association of a linear scattering problem with the non-linear equations of motion of the theory. If the scattering data can then be used to construct a complete set of new canonical variables of action-angle type, in terms of which the constant of motion can be expressed, integrability is assured.

The first non-trivial step lies in the determination of the linear problem to be associated with the non-linear system, as there is no systematic way of doing such an association.

Having in mind the supersymmetric sine-Gordon theory (SSG), we indicate here a solution of this question by writing down two systems of linear equations whose compatibility condition is the SSG equation of motion. As a consequence, the proposed ansatz ³⁾ for infinitely many bosonic conservation laws is rigorously confirmed. We follow here the notation of Ref. 3).

Let V_1, V_2 be two bosonic superfields and V_3 a fermionic superfield, all of them dimensionless. Consider the two linear systems of differential equations :

$$D_2 \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} \lambda \theta_1 & D_2 \phi & 0 \\ -D_2 \phi & \lambda \theta_1 & -\sqrt{2}\lambda \\ 0 & \sqrt{2}\lambda & \lambda \theta_1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \quad (1)$$

$$D_1 \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \frac{im}{\sqrt{2}\lambda} \begin{pmatrix} 0 & 0 & -\sin \phi \\ 0 & 0 & -\cos \phi \\ -\sin \phi & -\cos \phi & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}, \quad (2)$$

where

$$D_1 = -\frac{\partial}{\partial \theta_2} + i \theta_2 \partial_+ , \quad D_2 = \frac{\partial}{\partial \theta_1} + i \theta_1 \partial_- ,$$

$$x^\pm = \frac{1}{2}(x^1 \pm x^0) , \quad \partial_\pm = \frac{\partial}{\partial x^\pm} , \quad D_1 D_1 = -i \partial_+ , \quad D_2 D_2 = i \partial_- ,$$

and λ is a complex parameter (c number) with dimension of the mass m .

The integrability condition for Eqs. (1) and (2) is precisely the equation of motion of the SGG theory

$$D_1 D_2 \phi = i m \sin \phi ; \quad (3)$$

hence Eqs. (1) and (2) constitute a linear problem associated with the non-linear equation (3). System (1) leads to

$$\partial_- \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} -i\lambda & \partial_- \phi & -i\sqrt{2}\lambda D_2 \phi \\ -\partial_- \phi & i\lambda & 0 \\ i\sqrt{2}\lambda D_2 \phi & 0 & i\lambda \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \quad (4)$$

which can be interpreted as an isospectral ($\partial_- \lambda = 0$) eigenvalue problem.

We stress that Eqs. (1) and (2) are far from being a trivial generalization of the corresponding equations for the pure bosonic ⁵⁾ case, to which they reduce when fermions are absent.

We do not analyze here the scattering problem : rather, as a first application, we use such equations to derive rigorously an infinite set of conservation laws. Define two new bosonic superfields U and Y_B and a fermionic superfield Y_F as follows

$$U = \ln V_1 + i\lambda x^- , \quad Y_B = \frac{V_2}{V_1} , \quad Y_F = \frac{V_3}{V_1} . \quad (5)$$

Equations (1) then become :

$$D_2 U = D_2 \phi \cdot Y_B \quad (1'a)$$

$$\sqrt{2\lambda} Y_F = -D_2 \phi - D_2 Y_B - D_2 \phi \cdot Y_B \cdot Y_B \quad (1'b)$$

$$\sqrt{2\lambda} Y_B = D_2 Y_F + D_2 \phi \cdot Y_B \cdot Y_F, \quad (1'c)$$

and the first of Eqs. (2) takes the form :

$$D_1 U = -\frac{im}{\sqrt{2\lambda}} \sin \phi \cdot Y_F. \quad (2'a)$$

The above stated compatibility condition (3) now for Eqs. (1'a) and (2'a) can be read as a supersymmetric covariant conservation law :

$$D_1 J_2(x, \theta; \lambda) = D_2 J_1(x, \theta; \lambda), \quad (6)$$

where

$$\begin{aligned} J_2(x, \theta; \lambda) &= D_2 \phi \cdot Y_B \\ J_1(x, \theta; \lambda) &= \frac{im}{\sqrt{2\lambda}} \sin \phi \cdot Y_F. \end{aligned} \quad (7)$$

Elimination of Y_B through (1'c) gives

$$\sqrt{2\lambda} Y_F = -D_2 \phi - \frac{i \partial_- \phi}{\sqrt{2\lambda}} + \frac{\partial_- \phi \cdot D_2 Y_F \cdot Y_F}{2i\lambda} \quad (8)$$

and relation (6) becomes

$$D_1 (D_2 \phi \cdot D_2 Y_F) = D_2 (im \sin \phi Y_F). \quad (9)$$

We assume now, as in the pure bosonic case, an expansion of Y_F in power series of λ^{-1} :

$$Y_F = \frac{1}{i\sqrt{2\lambda}} \sum_{n=0}^{\infty} \frac{Y^{(n+\frac{1}{2})}}{(2i\lambda)^n} ; \quad (10)$$

by inserting it in (9) and equating the terms of the same power in λ^{-1} we obtain infinitely many conservation laws

$$D_1 J_2^{(n+\frac{1}{2})} = D_2 J_1^{(n+\frac{1}{2})} , \quad n=1,2,3,\dots , \quad (11)$$

where

$$J_2^{(n+\frac{1}{2})} = -i D_2 \phi \cdot D_2 Y^{(n-\frac{1}{2})} , \quad J_1^{(n+\frac{1}{2})} = m \sin \phi \cdot Y^{(n-\frac{1}{2})} .$$

The expression of $J_i^{n+\frac{1}{2}}$ in terms of the superfield Φ , solution to Eq. (3), is attained by substituting (10) in (8). The procedure gives the recursive formula

$$Y^{(n+\frac{1}{2})} = \partial_- Y^{(n-\frac{1}{2})} + \partial_- \phi \cdot \sum_{\ell=1}^{n-1} D_2 Y^{(\ell-\frac{1}{2})} \cdot Y^{(n-\ell-\frac{1}{2})} ,$$

$n=1,2,3,\dots$

$$Y^{(\frac{1}{2})} = -i D_2 \phi .$$

This coincides with the ansatz of Ref. 3).

The general case of supersymmetric models with equations of motion of the type

$$D_1 D_2 \phi = i m V(\phi)$$

with

$$\frac{d^2 V}{d\phi^2} = k V , \quad \text{e.g.} \quad \begin{aligned} V &= \exp \phi \\ V &= \sinh \phi , \end{aligned}$$

for which, at the pure bosonic level, there exist Bäcklund transformations ⁶⁾, can be considered, with trivial modifications of Eqs. (1) and (3), and analogous results are immediately obtained.

ACKNOWLEDGEMENTS

L.G. wishes to thank the Theoretical Physics Division for the kind hospitality extended to him during part of this work.

REFERENCES

- 1) P. Di Vecchia and S. Ferrara - Nuclear Phys. B130, 93 (1977).
- 2) R. Shankar and E. Witten - Harvard Preprint 77/A076 (1977).
- 3) S. Ferrara, L. Girardello and S. Sciuto - CERN Preprint TH. 2474 (1978),
to appear in Physics Letters B.
- 4) C.S. Gardner, J. Green, M. Kruskal and R. Miura - Phys.Rev.Letters 19,
1095 (1967) ;
P.L. Lax - Commun.Pure and Applied Maths. 21, 467 (1968).
- 5) M. Ablowitz, D.J. Kaup, A.C. Newell and H. Segur - Phys.Rev.Letters 30,
1262 (1973), and Studies Applied Maths. 53, 249 (1974) ;
L.A. Takhtadzhyan and L.D. Faddeev - Theor.Math.Phys. 21, 1046 (1974).
- 6) L. Bianchi - Opere Complete, Edizioni Cremonese, Roma (1953).
C. Lowner - J.Anal.Math. 2, 219 (1952) ;
D.W. McLaughlin and A.C. Scott - J.Math.Phys. 14, 1817 (1973).