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AN APPROACH TO SU_2 GAUGE FIELDS IN MINKOWSKI SPACE - TIME

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ABSTRACT

A reparametrization invariant formulation of SU_2 gauge theory in Minkowski space-time is given in terms of differential forms. A map of space-time into a compact region is used and the SU_2 Maurer-Cartan forms employed to establish a convenient gauge. New solutions to the theory are presented and discussed.

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1. The intensive study of SU_2 gauge theory has uncovered a rich variety of solutions to the free field equations. Solutions now exist in both Euclidean and Minkowski¹⁾ space. The self-dual and anti-self-dual solutions in Euclidean space have received particular attention and considerable progress has been made recently in constructing all such solutions²⁾. A notable feature of the techniques pioneered by Atiyah and Ward³⁾ is the use of complexified Euclidean space. The normal procedure is then to retain real non-singular configurations in this space and identify those with finite action as contributions to tunnelling amplitudes between topologically distinct vacua. Indeed one of the motivations for pursuing Euclidean solutions is that quantization procedures based on functional integrals can usually be performed here with more confidence. It should, however, be remembered that the imaginary time technique used to pass from Minkowski to Euclidean space relies on transferring one complex field degree of freedom onto the time coordinate. In a Minkowski space formulation classically forbidden field configurations can correspond to complex solutions of the classical field equations. In a theory with several degrees of freedom per point of space-time it may not always be possible to use the imaginary time map into Euclidean space. In other words, Minkowski space solutions with several complex degrees of freedom may constitute a class worthy of attention.

The use of complex classical trajectories in computing tunnelling amplitudes is not new and can probably be traced to techniques invented by L. Landau⁴⁾. The methods have been put into practice in the study of molecular transitions⁵⁾. To see how one may incorporate complex classical field configurations into the quantum theory we need only recall that the semi-classical field propagator is proportional to the exponential of i times the classical action. If this action is complex, the propagator can immediately acquire a damped behaviour. It is of interest to note however that the existence of a complex potential for the theory does not necessarily imply a complex action as we shall explicitly demonstrate in the following. If the classical action is real the fluctuation potential that enters into the equation for small oscillations about the complex solution may be complex, producing complex eigenmodes that again produce damping.

In the following we formulate SU_2 gauge theory in Minkowski space using the language of differential forms. This is a particularly useful formalism here since it emphasizes the intrinsic geometrical nature of the theory and being manifestly reparametrization invariant enables one to work in whatever co-ordinates one pleases with relative ease. A physicist's eye view of the rudiments of this language may be found in Ref. 6) which also contains a number of references to the mathematical literature.

Our purpose is to draw attention to a convenient co-ordinate system to study Minkowski space solutions and to offer new solutions in a gauge appropriate to these co-ordinates. We shall restrict ourselves to real co-ordinates both for Minkowski space and the gauge group manifold. However, in view of the comments above we shall exhibit both real and complex new solutions in addition to recovering many of the known Minkowski space solutions. In order to distinguish solutions (that might otherwise be considered as gauge equivalent) we also compute their associated gauge invariant action densities.

2. The basic action for the theory is, apart from possible boundary contributions of the form $\Theta F \wedge F$,

$$S = \frac{1}{g^2} \int \text{Tr} (F \wedge * F) \quad (1)$$

where F is an SU_2 algebra valued two-form expressible in terms of a connection one-form A as

$$F = dA + A \wedge A \quad (2)$$

Our convention for the algebra basis is to use

$$A = A^j \sigma_j \quad (3)$$

$$F = F^j \sigma_j \quad (4)$$

in terms of Pauli matrices σ_j , $j = 1, 2, 3$. It is convenient to take F and A as anti-Hermitian and normalize the self-coupling g of the free field with (2). We call a solution real if the coefficient of $i\sigma_j$ in the associated form is real. The field equations following from (1) may be written

$$D * F \equiv d * F - * F \wedge A + A \wedge * F = 0 \quad (5)$$

where the dual operation is taken with respect to a specified orientation of the base space and a given metric. In terms of real co-ordinates (t', r', θ, ϕ) we

express the metric of Minkowski space as

$$g = -e^0 \otimes e^0 + \sum_{k=1}^3 e^k \otimes e^k \quad (6)$$

with

$$\begin{aligned} e^0 &= \lambda dt' \\ e^1 &= \lambda dr' \\ e^2 &= \lambda \sin r' d\theta \\ e^3 &= \lambda \sin r' \sin \theta d\phi \\ \lambda &= \frac{1}{2} \sec \frac{1}{2}(t'+r') \sec \frac{1}{2}(t'-r') \end{aligned} \quad (7)$$

and

$$\begin{aligned} -\pi &< t'+r' < \pi \\ -\pi &< t'-r' < \pi \\ 0 &< \phi < 2\pi \\ 0 &< \theta < \pi \\ r' &\geq 0 \end{aligned} \quad (8)$$

The orientation of our tetrad is fixed by taking the volume element

$$\xi = e^1 \wedge e^2 \wedge e^3 \wedge e^0 \quad (9)$$

This metric is conformally related to the familiar one with $x^2 = r^2 - t^2$ by the transformations

$$t' = \tan^{-1} \left\{ \frac{x^2 - 1}{2r} \right\} \quad (10)$$

$$r' = -\tan^{-1} \left\{ \frac{x^2 + 1}{2t} \right\} \quad (11)$$

Such a representation of space-time was originally devised by Penrose in a study of Einstein spaces⁷⁾. Since the topology of SU_2 is S^3 it is convenient to map Minkowski space onto the group with the unitary matrix

$$U = U + i \underline{r} \cdot \underline{\sigma} \quad (12)$$

where

$$\begin{aligned} U &= \cos r' \\ \underline{r} &= \sin r' \underline{n}(\theta, \phi) \end{aligned} \quad (13)$$

and $\underline{n}$ is a unit vector on S^2 .

With this map the left invariant Maurer-Cartan one-forms $\Omega = U^{-1} dU = \Omega_j \sigma^j$ can be computed in Minkowski space and used to establish a convenient gauge in which to search for solutions. They are given as the forms

$$\begin{aligned} \Omega_1 &= i dr' \sin \theta \sin \phi + i d\theta \{ \cos r' \sin r' \sin \phi - \sin^2 r' \cos \phi \} \\ &\quad + i d\phi \{ \sin r' \cos r' \sin \theta \cos \phi + \sin^2 r' \cos \theta \sin \theta \sin \phi \} \end{aligned} \quad (14)$$

$$\begin{aligned} \Omega_2 &= i dr' \sin \theta \cos \phi + i d\theta \{ \sin r' \cos r' \cos \theta \cos \phi + \sin^2 r' \sin \phi \} \\ &\quad + i d\phi \{ \sin^2 r' \cos \theta \sin \theta \cos \phi - \cos r' \sin r' \sin \theta \sin \phi \} \end{aligned} \quad (15)$$

$$\begin{aligned} \Omega_3 &= i dr' \cos \theta - i d\theta \sin r' \cos r' \sin \theta \\ &\quad - i d\phi \sin^2 r' \sin^2 \theta \end{aligned} \quad (16)$$

These forms describe a pure gauge connection for the theory in the sense that $A = \Omega$ yields $F = 0$. The hypersurfaces on which U and Ω are defined are hypersurfaces

of constant t' embedded in Minkowski space. Consequently one expects $*(d\Omega)$ to involve a form normal to the embedded hypersurface. Indeed, one finds after a short calculation

$$* d\Omega = d\rho \wedge \Omega \quad (17)$$

where it is often convenient to absorb two and use $\rho = 2t'$. This relation suggests that we look for solutions involving functions of the single variable ρ . Since $d\Omega = -\Omega \wedge \Omega$ any potential one-form A^j that is expressed as a linear combination of basis forms Ω^j with functions of ρ as coefficients will yield fields expressible in terms of the two-forms $*d\Omega$ and $d\Omega$. Accordingly we write

$$A^j = P^{jk}(\rho) \Omega^k \quad j,k=1,2,3$$

or more compactly

$$A^j = \underline{P}^j \cdot \underline{\Omega} \quad (18)$$

in terms of R^3 vectors. Using (17) and (18) in (2) we find ($j = 1, 2, 3$)

$$F^j = \underline{P}'^j \cdot * d\underline{\Omega} + \underline{h}^j \cdot d\underline{\Omega} \quad (19)$$

where

$$\underline{P}'^j = \frac{d\underline{P}^j}{d\rho} \quad (1, j, k \text{ cyclic } 1, 2, 3)$$

$$\underline{h}^j = \underline{P}^j - \underline{P}^j \times \underline{P}^k$$

Since

$$* F^j = - \underline{P}'^j \cdot d\underline{\Omega} + \underline{h}^j \cdot * d\underline{\Omega} \quad (20)$$

this field has no definite real dual parity in Minkowski space and one should compute $D*F$. Using (5) we find that the field equations are solved if

$$\underline{P}_i'' + \underline{h}_i - (\underline{h}_j \times \underline{P}_k) + (\underline{h}_k \times \underline{P}_j) = 0 \quad (21)$$

(i, j, k cyclic 1, 2, 3)

These constitute nine second order ordinary differential equations for the nine functions $P^{jk}(\rho)$. They could be written as 18 coupled first-order autonomous equations for Runge-Kutta numerical iteration in the parameter ρ .

We can make analytic progress if we now adopt the diagonal form

$$\begin{aligned} \underline{P}^1 &= \underline{L}_1 f^1 \\ \underline{P}^2 &= \underline{L}_2 f^2 \\ \underline{P}^3 &= \underline{L}_3 f^3 \end{aligned} \quad (22)$$

where $\underline{i}_1, \underline{i}_2, \underline{i}_3$ are an orthonormal basis in R^3 . The equations then become

$$-f_i'' = f_i - 3f_j f_k + f_i(f_j^2 + f_k^2) \quad (23)$$

for the three functions f_i . These have the first "energy" integral

$$C = f_1'^2 + f_2'^2 + f_3'^2 + (f_1 - f_2 f_3)^2 + (f_2 - f_3 f_1)^2 + (f_3 - f_1 f_2)^2 \quad (24)$$

which may be used to identify the non-Abelian self-potential of the system.

Before proceeding further with these equations we shall express some gauge invariant forms in this gauge. The action density is

$$T_r(F_\wedge * F) = \sum_{j=1}^3 \{ f_j'^2 - h_j^2 \} \hat{\varepsilon} \quad (25)$$

where

$$\hat{\varepsilon} = 8 \sin^2 r' \sin \theta \, d\varphi_\wedge dr'_\wedge d\theta_\wedge d\phi$$

and the "magnetic" components are defined as

$$h_i = f_i - f_j f_k \quad (i, j, k \text{ cyclic})$$

The boundary contribution mentioned earlier, which is locally the exterior derivative of a three-form is

$$T_r(F_\wedge F) = -2 \sum_{j=1}^3 f_j' h_j \hat{\varepsilon} \quad (26)$$

Finally we mention the energy-momentum three-form

$$t_c = (F_\wedge (*F)_{cb} - *F_\wedge F_{cb}) e^b \quad (27)$$

which is related to the energy momentum tensor t^b_a by

$$t_a = t^b_a * e_b$$

Integrals of these quantities are over the finite domain (8) and consequently give finite numbers for solutions non-singular in this region.

3. A great simplification in (23) occurs if we assume a solution with all f_i equal to some f say. The resulting equation

$$-f'' = f(1-f)(1-2f) \quad (28)$$

is essentially the one found in Refs. 8) and 9) using the $O(4) \times O(2)$ subgroup structure of the $SO(4,2)$ conformal group of space-time. The correspondence is $\rho = 2\tau$, $f = -q/2$ in the notation of Ref. 8) and implies a different normalization of the original fields. In these co-ordinates (28) has a real solution in terms of Jacobi elliptic functions. Depending on the value of C one has the representation

$$f = \frac{1}{2} + b \operatorname{dn}(a(\rho - \rho_0) | m) \quad (29)$$

$$m = 2 - \frac{1}{2a^2}, \quad a^2 = b^2$$

$$f = \frac{1}{2} + b \operatorname{cn}(a(\rho - \rho_0) | m) \quad (30)$$

$$m = b^2/a^2, \quad 2b^2 - a^2 = \frac{1}{2}$$

$$f = \frac{1}{2} + b \operatorname{sn}(a(\rho - \rho_0) | m) \quad (31)$$

$$m = -b^2/a^2, \quad b^2 - a^2 = \frac{1}{2}$$

where ρ_0 is an integration constant. For zero "energy" ($C = 0$) however, there is a complex solution

$$f = \frac{1}{2} \pm \frac{i}{2} \tan \frac{1}{2} (\rho - \rho_0) \quad (32)$$

which bears a striking resemblance to the Euclidean Belavin et al.¹⁾ solution. In the normalization of this paper the instanton becomes

$$f = \frac{1}{2} + \frac{1}{2} \tanh \frac{1}{2} (\rho - \rho_0) \quad (33)$$

where, in terms of the Euclidean radius R , we have defined $\rho = \ln R^2$. The relation between these solutions has been discussed by Rebbi¹⁰⁾ in the hypertoroidal

formalism. One notes that the real part of (32) coincides with the solutions (29), (30) and (31) with $b = 0$ and reproduces essentially the meron solution of Ref. 11) with $C = 3/16$. If we persist with the complex solution (32) then one must clearly ameliorate the unphysical singularities at $t' = \pm\pi/2$ (or $t'^2 - r'^2 = 1$). Since the gauge functions Ω^j depend only on Minkowski angle co-ordinates (their origin is a singularity) this singularity is not a gauge artifice. It is not clear that $\rho \rightarrow i\rho$ is sufficient to identify the solutions (32) and (33). A complex solution more general than (32) is

$$f = \frac{1}{2} \pm i b \operatorname{sc}(a(\varphi - \varphi_0) | m) \quad (34)$$

where

$$a^2 + b^2 = \frac{1}{2}, \quad m = 1 - \frac{b^2}{a^2}, \quad C = \frac{3}{16} - 3a^2 b^2$$

The simple solution (32) corresponds to $a = b$ and $m = 0$. In either case the action density is real

$$\operatorname{Tr}(F_\lambda * F) = -3 \left\{ a^2 b^2 (1 + 2 d c^2(a(\varphi - \varphi_0) | m) n c^2(a(\varphi - \varphi_0) | m)) + \frac{1}{16} \right\} \hat{E} \quad (35)$$

and for suitable restrictions ($2\pi a \leq K(m)$) non-singular. Presumably these solutions are closely related to some of the solutions described in Ref. 12).

4. Returning to the coupled system (23) it is of interest to first enquire about the possibility of zero "energy" ($C = 0$) solutions analagous to (32) or (34). Complex solutions do exist and are given by

$$f_1 = e^{i(\varphi - \varphi_0)} \operatorname{ns}(e^{i(\varphi - \varphi_0)} | m) \quad (36)$$

$$f_2 = e^{i(\varphi - \varphi_0)} \operatorname{dn}(e^{i(\varphi - \varphi_0)} | m) \quad (37)$$

$$f_3 = e^{i(\varphi - \varphi_0)} \operatorname{cs}(e^{i(\varphi - \varphi_0)} | m) \quad (38)$$

for any m . Increasing the value of C we expect real solutions to (23) analogous to (29), (30) and (31) and having the above set as their complex counterparts.

An accurate analytical representation looks somewhat unlikely however. There are of course Abelian solutions obtained by setting any pair of f_j 's zero. One then recovers Abelian waves satisfying

$$f_j'' = -f_j, \quad (39)$$

propagating with zero action and completely decoupled from the other two components. We can now imagine turning on the other two components adiabatically. If there are stable weak field solutions we can treat the quadratic and cubic terms in (23) as perturbing the harmonic wells. Clearly this corresponds to finding weak field approximations with small C . The same technique can be tried on the equation (28) where we have exact solutions.

Writing in this case the solution of

$$f'' + f = \mathcal{F}(f) \quad (40)$$

as

$$f = R(\varphi) \cos(\varphi - \theta(\varphi))$$

where

$$\mathcal{F} = 3f^2 - 2f^3$$

we find that if R and θ are slowly varying functions (compared with $\omega = 1$) they must satisfy the first order equations

$$\frac{dR}{d\varphi} = -\frac{1}{2\pi} \int_0^{2\pi} \mathcal{F}(R \cos \psi) \sin \psi d\psi \quad (41)$$

$$\frac{d\theta}{d\varphi} = \frac{1}{2\pi R} \int_0^{2\pi} \mathcal{F}(R \cos \psi) \cos \psi d\psi \quad (42)$$

Consequently one finds the weak field solution

$$f(s) = R_0 \cos \left\{ \left(1 - \frac{3}{4} R_0^2 \right) s + \theta_0 \right\} \quad (43)$$

where R_0 and θ_0 are constants. This is a wave whose period $2\pi(1 + 3/4 R_0^2)$ depends on the amplitude R_0 . Comparing with the exact solution (29) we see that this corresponds to a small m approximation for oscillations centred about the potential minimum at $f = 0$.

If we repeat this technique for the coupled waves satisfying (23) we can derive the weak field solutions

$$f_1 = R_1 \sin \left\{ \left(1 + \frac{R_2^2 + R_3^2}{8\pi} \right) s + \theta_1 \right\} \quad (44)$$

$$f_2 = R_2 \sin \left\{ \left(1 + \frac{R_3^2 + R_1^2}{8\pi} \right) s + \theta_2 \right\} \quad (45)$$

$$f_3 = R_3 \sin \left\{ \left(1 + \frac{R_1^2 + R_2^2}{8\pi} \right) s + \theta_3 \right\} \quad (46)$$

where R_j and θ_j are constants. A detailed analysis of the stability criteria for these waves is rather delicate and deserves further attention.

There is one other field configuration that may be obtained. At the turning points of the potential there are constant solutions ($f_j = c_j$) which must be the roots of the simultaneous equations:

$$c_1 - 3c_2c_3 + c_1(c_2^2 + c_3^2) = 0 \quad (47)$$

$$c_2 - 3c_3c_1 + c_2(c_3^2 + c_1^2) = 0 \quad (48)$$

$$c_3 - 3c_1c_2 + c_3(c_1^2 + c_2^2) = 0 \quad (49)$$

For such solutions we have

$$\text{Tr} (F_{\wedge} * F) = C \hat{\epsilon} \quad (50)$$

$$\text{Tr} (F_{\wedge} F) = 0 \quad (51)$$

The vacuum solution $(c_1, c_2, c_3) = (1, 1, 1)$ clearly has $C = 0$. The other roots are (up to cyclic equivalences) $(-i, \pm 3, \mp i)$, $(-i, \mp i, \pm 3)$, $(i, \mp i, \mp 3)$, $(i, \pm 3, \pm i)$ and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. The last solution has $C = 3/16$ and corresponds to a gauge coupled meron. Although the other roots are complex they all have real finite action with $C = 8$ and consequently should make a significant contribution to any calculation involving classical solutions¹³⁾.

CONCLUSION

We have formulated the SU_2 Yang-Mills theory in the language of differential forms in real Minkowski space. Since the action is invariant under conformal changes of the metric and Minkowski space is conformally related to a compact region of an Einstein static space we have mapped the theory with such a transformation. Using a gauge appropriate to the topology of $R \times S^3$ we have adopted the SU_2 Maurer-Cartan forms to define a pure gauge connection for the fields F . Working with a parameter normal to the hypersurfaces defined by these forms we have established a set of coupled equations for the components of the vector potential. We have drawn attention to both real and complex solutions since we feel that the latter should not be ignored a priori in Minkowski space.

The methods of this paper are capable of considerable generalization, particularly to configurations depending on more than one variable. We shall report on these generalizations elsewhere.

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