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AMBIGUITIES AND BREAKDOWN CONFIGURATIONS
OF THE UNITARY GAUGE

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A B S T R A C T

The unitary gauge, used for revealing the particle content of spontaneously broken gauge theories, is shown not to fix the gauge uniquely, even after any unbroken gauge symmetries are taken care of. There are discrete ambiguities, which can largely be removed by giving a more precise gauge condition, and continuous ambiguities, which can occur in configurations that cannot be represented smoothly in the unitary gauge (breakdown configurations). Examples of breakdown configurations are instanton-type fields interpolating between topologically distinct vacua.

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1. - INTRODUCTION

One of the main lessons of the recent developments in non-Abelian gauge theories has been the importance of global topological properties of (classical) field configurations. In pure Yang-Mills theories one has realized that the existence of vacuum configurations in a temporal ($A_{\alpha 0} = 0$) gauge, which cannot be continuously transformed into each other, implies a rich vacuum structure with attendant physical effects ^{1),2)}. On the other hand, one has learned that the Coulomb gauge condition does not uniquely fix the gauge ³⁾, leading to ambiguities in the description of a given configuration in this gauge. The vacuum configuration can be made unique, however, if one imposes the additional requirement that the potentials $A_{\alpha\mu}$ vanish faster than r^{-1} for large spacelike distances r . The price one has to pay is, of course, that all the interesting topological structure disappears and that perfectly regular but topologically non-trivial configurations cannot be represented at all by continuous potentials $A_{\alpha\mu}$ in this gauge ^{4),5)}. Such non-representable configurations will be called breakdown configurations of the gauge.

Non-Abelian gauge theories are also interesting as building blocks for unified theories of weak and electromagnetic (and strong ?) interactions. In these models one introduces additional scalar fields transforming non-trivially under the gauge group G . This allows new ways of fixing the gauge, namely by placing gauge conditions on the scalar fields. The particle content of spontaneously broken gauge theories is read off the Lagrangian in the unitary gauge, which places a condition on the direction of the Higgs field vector in the space spanning the appropriate representation of G ⁶⁾. This paper is devoted to investigating the ambiguities and breakdown configurations of the unitary gauge. Examples of breakdown configurations have, in fact, been known for a long time : these are the magnetic monopole solitons [for a recent review, see Ref. 7]. It is well known that a monopole configuration is necessarily singular in the unitary gauge.

The content of the paper is briefly as follows : Section 2 establishes notation and gives the customary definition of the unitary gauge. In Section 3 we study the occurrence of ambiguities and discuss how a sharper definition of the unitary gauge and smoothness requirements can remove some of the ambiguities. The study of breakdown configurations is taken up in Section 4. Singularities in the field of directions of the

Higgs field imply breakdown configurations, and this is important in understanding how the vacuum structure is described in the unitary gauge. In the Appendix it is shown that the unitary gauge vacuum is unique [as long as the unbroken gauge group is at most a product of $U(1)$ factors], whereas an analysis in a temporal gauge leads to a vacuum structure completely analogous to that of a pure Yang-Mills theory ^{2),8)}. The instanton-like configurations connecting topologically distinct vacua in a temporal gauge are shown in a number of models to be breakdown configurations of the unitary gauge. The paper concludes with a discussion of the relevance of these considerations to the quantum theory.

Finally, it should be pointed out that an analysis of the ambiguities of the unitary gauge description of an $SU(N)$ gauge theory with the Higgs field in the adjoint representation has already appeared ⁹⁾. The present discussion overlaps as little as possible with that paper.

2. - THE UNITARY GAUGE

The field theory we want to investigate is described by the Lagrangian (the notation and conventions follow Weinberg ⁶⁾) :

$$\mathcal{L} = -\frac{1}{4} F_{\alpha\mu\nu} F_{\alpha}^{\mu\nu} + \frac{1}{2} (D_{\mu}\Phi, D^{\mu}\Phi) - \mathcal{P}(\Phi), \quad (2.1)$$

where

$$F_{\alpha}^{\mu\nu} = \partial^{\mu} A_{\alpha}^{\nu} - \partial^{\nu} A_{\alpha}^{\mu} - C_{\alpha\beta\gamma} A_{\beta}^{\mu} A_{\gamma}^{\nu},$$

$$(D_{\mu}\Phi)_{\rho} = \partial_{\mu}\Phi_{\rho} + i(\theta_{\alpha})_{\rho\gamma} \Phi_{\gamma} A_{\alpha\mu}.$$

The gauge group is $\mathcal{G}$, assumed to be compact and a product of semi-simple and Abelian factors. It is generated by the generators T_{α} , $\alpha=1,\dots,N$, normalized so that the structure constants $C_{\alpha\beta\gamma}$ in

$$[T_{\alpha}, T_{\beta}] = i C_{\alpha\beta\gamma} T_{\gamma}$$

are real and completely antisymmetric. The Higgs field φ is written in terms of real components φ_p , $p=1, \dots, n$, and it transforms according to the real n dimensional representation D_B of $\mathcal{G}$, generated by $\{\theta_\alpha\}$, which are Hermitean and antisymmetric

$$\theta_\alpha^\dagger = \theta_\alpha = -\theta_\alpha^T \quad (2.2)$$

This means that scalar products like

$$(\varphi, \varphi) \equiv \varphi_p \varphi_p$$

are $\mathcal{G}$ invariant and that the representation matrices $\{D_B(g)\}$ form a subgroup of $O(n)$.

Any other fields, such as spinor fields, have been suppressed in (2.1), as they will not play any rôle for the considerations in this paper.

To define the unitary gauge, we choose a constant vector λ in the manifold M_0 of the absolute minima of the $\mathcal{G}$ invariant Higgs field self-interaction $\mathcal{F}(\varphi)$ to represent the vacuum configuration of the Higgs field. Then we require for all α ⁶⁾

$$(\theta_\alpha \lambda, \varphi(x)) = 0 \quad (2.3)$$

which, by the antisymmetry of θ_α , Eq. (2.2) is equivalent to

$$(\lambda, \theta_\alpha \varphi(x)) = 0 \quad (2.3a)$$

Since the gauge condition (2.3) is linear and homogeneous in $\varphi(x)$, it is really a condition on the direction of $\varphi(x)$ in the n dimensional vector space spanning the representation D_B .

There is an important difference between placing the gauge condition on the potentials A_α^μ or on a matter field $\varphi(x)$. Since a transformation of the potentials

$$A_{\alpha}^{\mu} T_{\alpha} \rightarrow A'_{\alpha}^{\mu} T_{\alpha} = g(x) A_{\alpha}^{\mu} T_{\alpha} g^{-1}(x) + i(\partial^{\mu} g(x)) g^{-1}(x) \quad (2.4)$$

involves the derivatives of the gauge group element $g(x)$, transforming a given configuration (assumed smooth) into a gauge fixed by a condition on the A_{α}^{μ} 's implies solving partial differential equations which automatically guarantee smoothness of the required gauge transformation $g(x)$, and hence of the transformed configuration. On the contrary, the transformation law for the matter field

$$\varphi(x) \rightarrow \varphi'(x) = D_B(g(x)) \varphi(x) \quad (2.5)$$

is purely local, and the problem of transforming a given smooth configuration into the unitary gauge can be solved at each point x separately with no smoothness implied, if several solutions $g(x)$ are possible. In classical physics, and probably also in semi-classical quantum physics, it is reasonable to impose the condition that the transformed configuration is also smooth, and this might already dispose of some ambiguities. A discrete ambiguity : there exists a finite number of gauge transformations $g_1(x), \dots, g_r(x)$ taking $A_{\alpha}^{\mu}(x), \varphi_p(x)$ into the unitary gauge, therefore does not necessarily lead to a breakdown of the unitary gauge description, but may merely imply the existence of r different, equivalent descriptions of the same configuration (like in the case of a discrete global symmetry), which is, of course, an interesting observation in itself. A continuous ambiguity [a continuity of possible solutions $g(x)$] can, however, as we shall see, lead to a genuine breakdown of the unitary gauge : no smooth description is possible.

3. - AMBIGUITIES IN THE UNITARY GAUGE

The problem to be investigated is : given a smooth Higgs field configuration $\varphi(x)$, find smooth gauge transformations $g(x)$ solving the gauge conditions

$$(\theta_\alpha \lambda, D_B(g(x)) \varphi(x)) = 0, \quad \alpha = 1, \dots, N. \quad (3.1)$$

If, for a given x , the solution is not unique, we say that there is an ambiguity. If we have constructed solutions for all x , but cannot connect them to a smooth function $g(x)$, we say that $\varphi(x)$ is a breakdown configuration of the unitary gauge.

In this section we shall address ourselves to the ambiguity problem. We shall be considering the Higgs field at a fixed point x , and since the condition (3.1) is local, we can drop the label x . The discussion is independent of the manifold on which the fields are defined.

A neat formulation of the problem can be given using the continuous, differentiable mapping from $\mathcal{G}$ to the reals introduced by Weinberg⁶⁾ and given by

$$F_\varphi(g) = (\lambda, D_B(g) \varphi). \quad (3.2)$$

It is easy to show that the gauge transformations g taking φ to the unitary gauge are precisely the stationary points of F_φ : since

$$\delta F_\varphi(g) = (\lambda, i\epsilon_\alpha \theta_\alpha D_B(g) \varphi) = -i\epsilon_\alpha (\theta_\alpha \lambda, D_B(g) \varphi) \quad (3.3)$$

for arbitrary infinitesimal parameters ϵ_α , the condition that F_φ be stationary at g is equivalent to (3.1). Thus we have to find all the stationary points of F_φ .

If $\mathcal{G}$ is compact, F_φ must have at least one maximum and one minimum ; hence a gauge transformation taking φ to the unitary gauge always exists, but at the same time we see that there is always at least one discrete ambiguity : both the g for which F_φ is maximal and the g for which F_φ is minimal take φ to the unitary gauge. A simple example is

$G = SO(3)$, D_B the vector representation and $\varphi \neq 0$. The g giving a maximum of F_φ rotates φ parallel to λ , the one giving a minimum rotates φ antiparallel to λ .

When and what discrete ambiguities occur is very much a model dependent question. The reader is referred to the article by Balachandran et al.⁹⁾ for a discussion of discrete ambiguities for the case when φ is in the adjoint representation of $G = SU(N)$.

It should be remarked that a more judicious choice of the unitary gauge condition can eliminate most of the discrete ambiguities : we saw that the gauge conditions (2.3) are equivalent to requiring that φ is such that F_φ , Eq. (3.2), has a stationary point at $g=e$, the unit element. A sharper condition is that this stationary point should be the absolute maximum of F_φ , or that

$$(\lambda, \varphi(x)) \text{ is maximal.} \quad (3.4)$$

(We choose the maximum rather than the minimum in order to have λ itself in the unitary gauge.) In the simple example quoted above, the condition (3.4) eliminates the possibility that φ is antiparallel to λ .

Let us now turn to investigating the possibility of continuous ambiguities. We need the following lemma : if λ and φ are such that F_φ , Eq. (3.2) has a stationary point for $g=e$ (i.e., φ is already in the unitary gauge), then all of G_λ and G_φ , the little groups of λ and φ , respectively, are stationary points of F . This statement is obvious for G_φ ; for G_λ one has :

$$\begin{aligned} \delta F|_{g_\lambda \in G_\lambda} &= i\epsilon_\alpha (\lambda, \theta_\alpha D_B(g_\lambda) \varphi) = -i\epsilon_\alpha (D_B(g_\lambda)^{-1} \theta_\alpha \lambda, \varphi) \\ &= -i\epsilon_\alpha Q_{\alpha\beta}(g_\lambda) (\theta_\beta D_B(g_\lambda^{-1}) \lambda, \varphi) \\ &= i\epsilon_\alpha Q_{\alpha\beta}(g_\lambda) (\lambda, \theta_\beta \varphi) = 0, \end{aligned}$$

by hypothesis. $a(g)$ is the adjoint representation and we used

$$D_B(g)^{-1} \theta_\alpha D_B(g) = Q_{\alpha\beta}(g) \theta_\beta.$$

If $\mathcal{G}_\lambda$ and/or $\mathcal{G}_\varphi$ are non-trivial and continuous, there thus exists a continuous ambiguity. The $\mathcal{G}_\lambda$ ambiguity is well known: $\mathcal{G}_\lambda$ is nothing but the unbroken gauge group S , and the gauge conditions (2.3) must be supplemented by gauge fixing conditions for S . If the connected part of S is at most a product of $U(1)$ factors, this can be done without introducing any new ambiguities, as is well known from electrodynamics. We shall assume that additional conditions have been imposed, removing the S ambiguity.

Let now F_φ be stationary for an arbitrary $g_0 \neq e$. Then

$$\hat{F}_\varphi(g) = F_\varphi(gg_0)$$

will be stationary at $g=e$, and the lemma implies that $\hat{F}_\varphi$ is stationary on $\mathcal{G}_\lambda$ and $\mathcal{G}_{\varphi_0}$, where $\varphi_0 = D_B(g_0)\varphi$. This means that F_φ will be stationary on the right coset $\mathcal{G}_\lambda g_0$, this ambiguity being removed by the S gauge conditions, and on the left coset $g_0 \mathcal{G}_\varphi$. Now if the latter coset is not contained in the first, or

$$g_0 \mathcal{G}_\varphi g_0^{-1} \not\subset \mathcal{G}_\lambda, \quad (3.5)$$

the latter ambiguity will provide a genuine continuous ambiguity. Thus, if the Higgs field φ is such that its little group $\mathcal{G}_\varphi$ is continuous but not conjugate to a subgroup of $\mathcal{G}_\lambda = S$ by an element g_0 for which $F_\varphi(g)$ is stationary, there will be a continuous ambiguity. (Note that φ and λ may well belong to the same stratum¹⁰⁾, i.e., their little group may be conjugate, for an ambiguity to occur.)

The points $\varphi \in \mathbb{R}^n$ satisfying the unitary gauge conditions can be characterized as follows: if we use the original conditions (2.3) we find that for any real μ_1, μ_2 , $\mu_1 \varphi^{(1)} + \mu_2 \varphi^{(2)}$ satisfies the conditions if $\varphi^{(1)}$ and $\varphi^{(2)}$ do. The unitary gauge vectors φ form a hyperplane L in $\mathbb{R}^n$ containing $\varphi=0$ and $\varphi=\lambda$. If instead we adopt the stricter conditions (3.4), μ_1 and μ_2 have to be non-negative, or in this case the unitary gauge vectors φ lie in a closed region $K \subset L$ bounded by a cone with tip at $\varphi=0$ (Fig. 1).

Adopting the stricter gauge conditions (3.4) one can show that those unitary gauge φ which allow a continuous ambiguity lie on the boundary of K . For such φ , the function $F_\varphi(g)$ is constant and equal to its maximal value $F_\varphi(e)$ on $\mathcal{G}_\varphi \not\subset S$. Take now another unitary gauge vector φ_1 which does not have any continuous ambiguity. As a function on $\mathcal{G}_\varphi$ $F_{\varphi_1}(g)$ has its absolute maximum at $g=e$. Consider now

$$F_{\varphi+x\varphi_1}(g) = F_\varphi(g) + x F_{\varphi_1}(g) ,$$

where x is a real number, and let g vary over $\mathcal{G}_\varphi$. For $x > 0$, $F_{\varphi+x\varphi_1}(g)$ has a unique maximum at $g=e$, i.e., $\varphi+x\varphi_1 \in K$, for $x < 0$ a minimum at $g=e$, hence $\varphi+x\varphi_1 \notin K$. Hence any neighbourhood of φ contains vectors both in K and in its complement, that is φ lies on ∂K .

As a simple application of this result, consider the case when L is just a line in $\mathbb{R}^n$, i.e., the unitary gauge conditions fix the direction of the Higgs field. Then K is a half-line starting from $\varphi=0$, which obviously always allows a continuous ambiguity. In the case considered this is the only possible continuous ambiguity, since ∂K is just the point $\varphi=0$. This situation occurs in many simple models such as the Georgi-Glashow or Weinberg-Salam models.

How does a continuous ambiguity assert itself in the canonical formalism in the unitary gauge? Let us transform to a new basis $\bar{T}_\alpha$ in the Lie algebra of $\mathcal{G}$ ($\bar{\Theta}_\alpha$ in the representation D_B), so that S is generated by $\bar{T}_1, \dots, \bar{T}_M$, the rest of the generators $\bar{T}_{M+1}, \dots, \bar{T}_N$ corresponding to the broken symmetry. If φ is in the unitary gauge and allows a continuous ambiguity, i.e., $\mathcal{G}_\varphi \not\subset S$, there must exist at least one non-vanishing linear combination of $\bar{\Theta}_{M+1}, \dots, \bar{\Theta}_N$ which gives zero when acting on φ :

$$\sum_{\alpha=M+1}^N c_\alpha \bar{\Theta}_\alpha \varphi = 0 . \quad (3.6)$$

In constructing the Hamiltonian in the unitary gauge ⁶⁾ one solves the constraint equations (Gauss' law):

$$\sum_{\rho=M+1}^N (\bar{\Theta}_\alpha \varphi, \bar{\Theta}_\rho \varphi) A_{\rho 0} = \partial_i \pi_\alpha^i - \pi_i^\beta A_\beta^i C_{\alpha\beta\gamma} + i(\partial_0 \varphi, \bar{\Theta}_\alpha \varphi)$$

for the variables $A_{\alpha 0}$ $\alpha = M+1, \dots, N$. This requires that the matrix

$$M_{\alpha\beta} = (\bar{\theta}_\alpha \varphi, \bar{\theta}_\beta \varphi), \quad \alpha, \beta = M+1, \dots, N \quad (3.7)$$

is regular, or $\det M \neq 0$. The formal expression for the unitary gauge Hamiltonian becomes singular when $\det M = 0$, which happens if and only if the vectors $\bar{\theta}_\alpha \varphi$, $\alpha = M+1, \dots, N$ are linearly dependent, i.e., if (3.6) holds. Thus, exactly like in the Coulomb gauge ³⁾, the ambiguity of the gauge signals itself as a singularity in the Hamiltonian.

Let us conclude this section by illustrating the question of ambiguities in a model where the geometry is still easily visualizable. This model also reveals how discrete and continuous ambiguities can interplay.

The model has $\mathcal{G} = SO(3)$ ("isospin") with the Higgs field in the tensor representation. The Higgs field can be represented by a symmetric, traceless 3×3 matrix Φ , which we write with the aid of two isovectors ξ, η :

$$\Phi = \xi \xi^T + \eta \eta^T - \frac{2}{3} \xi \cdot \eta \quad (3.8)$$

Since (ξ, η) and $(x\xi, \frac{1}{x}\eta)$ give the same Φ , we can require ξ and η to be of equal length. One still has to keep in mind that (ξ, η) , $(-\xi, -\eta)$ and (η, ξ) all give the same Φ .

Alternatively, we describe Φ in a basis given by symmetric, traceless matrices λ_i , $i = 1, \dots, 5$:

$$\Phi = \sum_{i=1}^5 \varphi_i \lambda_i \quad (3.9)$$

with

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

normalized so that

$$\text{Tr } \lambda_i \lambda_j = 2 \delta_{ij} .$$

Under a gauge transformation $g \in \text{SO}(3)$ Φ transforms as

$$\Phi' = g \Phi g^{-1} = g \Phi g^T \quad (3.10)$$

i.e., the vectors ξ and η are rotated by the same rotation. The action of the generators on the vector $\varphi = (\varphi_1, \dots, \varphi_5)$ can easily be calculated from Eqs. (3.9) and (3.10).

A renormalizable Higgs potential $\mathcal{V}(\Phi)$ can be found ¹¹⁾ such that the vacuum can be chosen as

$$\Lambda = \frac{1}{\sqrt{3}} \begin{pmatrix} -\mu & 0 & 0 \\ 0 & -\mu & 0 \\ 0 & 0 & 2\mu \end{pmatrix} = \mu \lambda_3 \quad (3.11)$$

i.e., $\lambda = (0, 0, \mu, 0, 0)$, or expressed in terms of the vectors ξ, η ,

$$\xi_{vac} = \eta_{vac} = a \underline{e}_3, \quad a^2 = \frac{\sqrt{3}}{2} \mu .$$

The $\text{SO}(3)$ gauge symmetry is broken down to $\text{NSO}(2)$, the normalizer of the $\text{SO}(2)$ subgroup of rotations around the three-axis, i.e., the rotations around $\underline{e}_3$ along with all rotations by π around any non-zero vector perpendicular to $\underline{e}_3$ and all products of these elements ¹²⁾. The unitary gauge conditions (2.3) imply $\varphi_4 = \varphi_5 = 0$, the stricter conditions (3.4) that φ lies in the solid cone

$$K : \quad \varphi_3 \geq + \frac{1}{\sqrt{3}} \sqrt{\varphi_1^2 + \varphi_2^2} . \quad (3.12)$$

A standard analysis of the Lagrangian in the unitary gauge implies that the particle content of the perturbation theory around the vacuum is : charged massive vector bosons described by $A_{1 \pm i 2\mu}$, a photon $A_{3\mu}$, a neutral Higgs boson φ_3 and a pair of doubly charged scalar particles described by $\varphi_1 \pm i\varphi_2$.

In terms of the vectors (ξ, η) the unitary gauge configurations allowed by (2.3) are (Fig. 2) : either

$$\xi = (\xi_1, \xi_2, \xi_3) , \quad \eta = \pm (\xi_1, \xi_2, -\xi_3) ,$$

i.e., η is the reflection of ξ in the (12) plane or in the three-axis,
or

$$\xi_3 = \eta_3 = 0 ,$$

i.e., ξ and η lie in the (12) plane.

Take now a generic Φ described by two non-parallel vectors ξ, η of equal length (Fig. 3). It is evident that there is a threefold discrete ambiguity in bringing this Φ to the unitary gauge : first we can either rotate the plane Π spanned by ξ and η to a position perpendicular or parallel to the (12) plane. In the first case there are two possibilities : the line ℓ bisecting the angle between ξ and η is either taken to lie in the (12) plane (Fig. 2a) or parallel to the three-axis (Fig. 2b). Rotating Π to be parallel to the (12) plane again gives the configuration in Fig. 2c. Any remaining ambiguity is a transformation belonging to δ , and is thus removed by supplementary conditions.

Imposing the sharper gauge condition (3.4) eliminates the discrete ambiguity and picks out the configuration in Fig. 2b as the only acceptable unitary gauge configuration.

The little group of the generic Φ is discrete (generated by rotations by π around ℓ and the normal to Π) and will not produce any continuous ambiguity. The situation is different when ξ and η are lying on the same line. Then G_Φ is an $NSO(2)$ group, the connected part of which is the rotations around ξ . With the gauge conditions (2.3) there is a twofold discrete ambiguity : ξ can be rotated either to the three-axis or into the (12) plane. The latter possibility gives a stationary point g_0 of (3.2) for which $G_\Lambda \neq g_0 G_\Phi g_0^{-1}$ and a continuous ambiguity arises. Indeed the determinant of the matrix M , Eq. (3.7) takes the form

$$\det M = \varphi_1^2 + \varphi_2^2 - 3\varphi_3^2 = (\xi_1 \eta_2 - \xi_2 \eta_1)^2 ,$$

the latter equality holding for ξ, η lying in the (12) plane, and this vanishes when $\xi = \pm\eta$. Note that this ambiguity can be avoided by using the discrete ambiguity to choose the first alternative.

This confusing interplay between discrete and continuous ambiguities can be bypassed by using the gauge condition (3.4) instead. Then, if $\xi = \eta$, they have to be rotated to the three-axis, and no continuous ambiguity is present. If $\xi = -\eta$, however, they have to be rotated into the (12) plane and a continuous ambiguity is at hand. In the φ space the ambiguous configurations are lying on the mantle of the cone K [Eq. (3.12)].

Finally, one should mention the possibility $\Phi = 0$ which gives a continuous ambiguity. The model considered does possess static monopole solutions ¹¹⁾ in four-dimensional space-time, in which Φ has a quadratic zero at the centre of the monopole. These solutions are in fact breakdown configurations of the unitary gauge. We now turn our attention to such configurations.

4. - BREAKDOWN CONFIGURATIONS AND VACUUM STRUCTURE

The discussion in the previous section could be carried out point by point and depended only on the group structure of the model and not on the manifold on which the fields were defined. We are now going to impose the condition that the initial, given field configuration $A_{\alpha\mu}(x)$, $\varphi(x)$ is smooth and to enquire into the possibility of transforming this configuration smoothly into the unitary gauge. It is clear that the underlying manifold, or, since we are considering only flat space-time, the number of variables d on which the configuration effectively depends, will play an important rôle. If D is the dimension of space-time, $d = D$ for a general configuration and $d = D-1$ for a general static configuration.

As pointed out in Section 2, smoothness requirements on the transformed field can eliminate ambiguities. To take an extreme example, suppose we again have the model $\mathcal{G} = SO(3)$, D_B being the vector representation, and let the unitary gauge be specified by requiring the Higgs field to point in the three-direction. Let the given Higgs configuration be

$$\underline{Q}(x) = f(x) \underline{e}_1. \quad (4.1)$$

Obviously, a constant gauge transformation g_0 rotating $\underline{e}_1$ to $\underline{e}_3$ takes the configuration into the unitary gauge. Let now $f(x)$ have an isolated zero at x_0 . Then there is an ambiguity at x_0 : at this point any rotation could be applied. It would be ridiculous, however, to pretend that this ambiguity is of any physical importance. Requiring the gauge transformation to be smooth eliminates the ambiguity by fixing $g(x_0) = g_0$, the only continuous extension of g to x_0 .

In general continuous ambiguities can, however, give rise to breakdown configurations, as the following argument shows.

Let x_0 be a point where φ is such that Eq. (3.5) holds, and let L be a world line $x^\mu = x^\mu(s)$, $x^\mu(0) = x_0^\mu$ through this point such that in a neighbourhood of x_0 , L does not meet other points where (3.5) holds. We assume that there are no discrete ambiguities. Then φ can be transformed to the unitary gauge by a unique gauge transformation $g(s)$ along L for $s \neq 0$. The point is now that although $\varphi(x(s))$ is continuous along L , $g(s)$ can have different limits as $s \rightarrow 0$ from above or below; in general we can only say that

$$\lim_{s \rightarrow 0+} g(s) = \lim_{s \rightarrow 0-} g(s) h_\varphi, \quad (4.2)$$

where $h_\varphi \in G_\varphi$. The condition (3.5) says that we cannot rewrite (4.2) as

$$\lim_{s \rightarrow 0+} g(s) = h_\lambda \lim_{s \rightarrow 0-} g(s),$$

with $h_\lambda \in G_\lambda = S$, which would allow the discontinuity in $g(s)$ at $s=0$ to be removed by an S gauge transformation. Thus, a continuous ambiguity arising from the condition (3.5) may induce discontinuities in $g(x)$, and hence singularities in the unitary gauge potentials $A_{\alpha\mu}(x)$.

Can we give conditions that establish when a breakdown configuration is at hand? Let us first establish some terminology: two Higgs field configurations φ, φ' on $M \subseteq \mathbb{R}^d$ are said to be **gauge equivalent** if

there is a smooth mapping $g : M \rightarrow \mathcal{G}$ transforming one into the other [Eq. (2.5)]. If a given Higgs field configuration is gauge equivalent to a configuration satisfying the unitary gauge condition, we have a regular configuration ; if no gauge equivalent unitary gauge configuration can be found, we have a breakdown configuration.

The tool for investigating breakdown configurations that first crosses one's mind is homotopy theory. Two gauge equivalent Higgs fields are homotopic if (as we shall assume from now on) $\mathcal{G}$ is connected ; the converse is, however, not true. Hence homotopy arguments will only furnish sufficient conditions for the occurrence of breakdown configurations.

To get non-trivial conditions from homotopy we cannot consider the Higgs field maps $\varphi : M \rightarrow \mathbb{R}^n$ (as all such maps are homotopic). But since the unitary gauge conditions are conditions on the direction of the Higgs field we can instead consider the field of directions

$$\hat{\varphi}_p(x) = \frac{\varphi_p(x)}{(\varphi(x), \varphi(x))^{1/2}}, \quad (4.3)$$

which defines a map $M \rightarrow N \rightarrow S^{n-1}$ where N is the set of zeros of the Higgs field on M and S^{n-1} is the $n-1$ dimensional sphere. We denote by $\pi^{n-1}(M-N)$ the homotopy classes of these maps [do not confuse this with $\pi_n(M)$, the homotopy group of mappings $S^n \rightarrow M$]. Let C_u be the union of all classes in $\pi^{n-1}(M-N)$ to which unitary gauge configurations can belong. C_u certainly includes the homotopy class C_0 of the constant mapping $M \rightarrow \lambda$, since the vacuum vector λ itself is a unitary gauge configuration, but in general other classes also enter in C_u . A sufficient condition for a Higgs field configuration to be a breakdown configuration is then that it belongs to a homotopy class of $\pi^{n-1}(M-N)$ not in C_u .

This condition, fairly general as it is, becomes practically useful if we can find spheres $M \subset \mathbb{R}^d$ such that the Higgs field under consideration has no zero on them. This is because the homotopy groups $\pi_n(S^m) = \pi^m(S^n)$ of mappings between spheres $S^n \rightarrow S^m$ are known and the homotopy classes can be characterized. Let us, for example, treat the following models : the Abelian Higgs model in $d=2$ [$\mathcal{G} = U(1)$, $n=2$], the Georgi-Glashow model in $d=3,4$ [$\mathcal{G} = SO(3)$, D_B the vector representation, $n=3$] and the Weinberg-Salam model in $d=4$ [$\mathcal{G} = SU(2) \times U(1)$, D_B is the complex spinor representation ; $n=4$]. We can classify the

homotopy classes of the mappings $\hat{\varphi} : M \rightarrow S^{n-1}$ where $M = S^{d-1}$ by an integer k $[\pi_n(S^n) = \pi_3(S^2) = \mathbb{Z}]$. If $d = n$, k can be expressed in terms of the Higgs field as the Kronecker index of the map $S^{d-1} \rightarrow S^{n-1}$ (13) :

$$k = \frac{1}{(d-1)! A_{d-1}} \int_M \frac{1}{(\varphi, \varphi)^{d/2}} \epsilon_{i_1 \dots i_d} \varphi_{i_1} d\varphi_{i_2} \wedge \dots \wedge d\varphi_{i_d} \quad (4.4)$$

where A_{d-1} is the area of the $d-1$ dimensional unit sphere. [For the Georgi-Glashow model in $d = 4$, k is given by a more complicated expression, the so-called Hopf invariant ; see Ref. 14).] In all the models under consideration, the unitary gauge condition (3.4) fixes a unique direction ; hence all unitary gauge configurations are homotopic to the constant mapping, or have $k = 0$. Thus we have arrived at the result : for the models cited above, Higgs field configurations with $k \neq 0$ are breakdown configurations.

How can we connect this result with the appearance of continuous ambiguities, which for the cited models only occur for $\varphi = 0$? The answer, when $d = n$, is given by Kronecker's existence theorem (13), which states that if $k \neq 0$, when calculated over M , then the Higgs field must have at least one zero inside M . Moreover, if φ only has isolated zeros, the sum of their indices, calculated over small spheres enclosing only one zero, equals k . An arbitrary configuration can be arbitrarily closely approximated by a configuration with only isolated zeros without changing k .

A zero at x_0 of the Higgs field $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ with non-zero index k implies a singularity in the field of directions (4.3) (examples are given in Fig. 4 for $d = 2$). One easily convinces oneself that there are no continuous gauge transformations rotating the Higgs field into a fixed direction in any neighbourhood containing x_0 . Hence the gauge transformation taking φ to the unitary gauge will necessarily be singular at x_0 , and will induce singularities in the transformed gauge potentials $A_{\alpha\mu}(x)$ at x_0 .

The archetypal examples of breakdown configurations of this kind are the Nielsen-Olesen vortex line in $d = 2$ (15) and the 't Hooft-Polyakov monopole for $d = 3$ (16). Indeed (4.4) is one of the many ways of characterizing the winding number or the magnetic charge, respectively (17).

The latter example shows that the gauge transformation leading to the unitary gauge can be discontinuous also away from the singularity in the field of directions : it is well known that the "hedgehog" configuration $\varphi_a \propto x_a$, $a=1,2,3$ requires the gauge transformation to be singular not only at the centre of the monopole, $\vec{x}=0$, where the singularity in the field of directions and the zero of the Higgs field are located, but also along a line leading from 0 to infinity, the famous Dirac string.

A sufficient condition for the occurrence of strings (when $d=n$) can be given as follows : let $\varphi(x)$ be a configuration with a zero at x_0 with $k \neq 0$, and let $\hat{\varphi}(x)$ be the corresponding field of directions $k \neq x_0$. Let S^{d-1} be a sphere surrounding x_0 ; we ask whether it is possible to find a smooth $g(x)$ on S^{d-1} such that

$$D_B(g(x)) \hat{\varphi}(x) = \hat{\varphi}_0, \quad (4.5)$$

where $\hat{\varphi}_0$ is some fixed direction, corresponding to a unitary gauge configuration. If such a $g(x)$ can be found, $g(x)$ will be singular only at x_0 , if not, $g(x)$ must have a singularity on every sphere S^{d-1} surrounding x_0 , i.e., there must be a string leading out from x_0 . Suppose now a $g(x)$ satisfying (4.5) has been found and that the $(d-1)$ th homotopy group of G is trivial :

$$\pi_{d-1}(G) = 0. \quad (4.6)$$

Then the mapping $g(x)$ on S^{d-1} can be continuously deformed into the unit mapping ; i.e., there exists a homotopy between $\hat{\varphi}$ and $\hat{\varphi}_0$. Since $\hat{\varphi}_0$ has Kronecker index 0, this contradicts the hypothesis $k \neq 0$. Thus (4.6) is a sufficient condition for the occurrence of strings, applicable, e.g., to the 't Hooft-Polyakov monopole $[\pi_2(SO(3)) = \mathbb{Z}]$. The Weinberg-Salam model, the Georgi-Glashow model, both in $d=4$, and the Abelian Higgs model in $d=2$ do not need strings. I do not know a condition which would be both necessary and sufficient for strings to occur in a given model. In general one has to make allowance for such singularities as well.

A nice example of the necessity of breakdown configurations is provided by the study of the vacuum structure in spontaneously broken gauge theories. In a temporal gauge ($A_{00} = 0$), a vacuum configuration is given by

$$A_{\alpha i} T_{\alpha} = i g(\vec{x}) \partial_i g^{-1}(\vec{x})$$

$$\varphi(\vec{x}) = D_B(g(\vec{x})) \lambda, \quad \lambda \in M_0.$$

(4.7)

The familiar argument ¹⁾ for pure Yang-Mills theory in four space-time dimensions (or for the Abelian Higgs model in $D=2$), implying the existence of non-equivalent vacua labelled by a topological quantum number n_T [if $\pi_{D-1}(G)=Z$, as for most cases of interest] can be taken over word for word into the present case of spontaneous symmetry breaking.

In pure Yang-Mills theory (and in the Abelian Higgs model in $D=2$), instanton and multi-instanton solutions to the Euclidean equations of motion interpolate between vacua of different n_T at $x_4=+\infty$ and $x_4=-\infty$, causing tunnelling transitions between them, leading to the picture of the vacuum as a Bloch wave. For the spontaneously broken theory, it has been shown by many authors ¹⁸⁾ that instanton solutions do not exist in $D=4$. Nevertheless, interpolating, instanton-like configurations with finite Euclidean action can readily be constructed, and these are presumably responsible for tunnelling between the vacua. In short, an analysis in a temporal gauge gives exactly the same vacuum structure as in the pure Yang-Mills theory ^{2),8)}.

The situation is different in the unitary gauge. It is easily shown (see the Appendix) that in this gauge the vacuum is unique up to discrete ambiguities [which can be eliminated by adopting the sharpened gauge condition (3.4)], as long as the unbroken gauge group S is at most a product of $U(1)$ factors. In analogy with what was found in the Coulomb gauge ⁵⁾, one expects that the instanton-like configurations cannot be continuously transformed into the unitary gauge, i.e., they would be breakdown configurations of this gauge.

That this is indeed so can be shown in general for the models we took as examples above. Let the smooth instanton-like configuration in the temporal gauge interpolate between a vacuum n_+ at $x_4=+\infty$ and a vacuum n_- at $x_4=-\infty$. It is then characterized by a non-zero winding number or Pontryagin index

$$n_p = n_+ - n_- , \quad (4.8)$$

which can be calculated as a surface integral over the boundary of the region of Euclidean space bounded by the planes $x_4 = T$, $x_4 = -T$ and the surface $x_1 x_1 = R^2$, for sufficiently large R and T .

Now we need only recall that Woo¹⁴⁾ has shown that for the models in question, the Pontryagin index for a finite action configuration equals the Kronecker index (or the Hopf invariant) k . Non-zero k , we argued, implies a breakdown of the unitary gauge. Thus instanton-like configurations are breakdown configurations. This can readily be checked by explicit construction of such configurations.

5. - DISCUSSION

The phenomena we have discussed - ambiguities and breakdown configurations of the unitary gauge - are yet another example of how insistence on a trivial global topology leads to singularities in the description of topologically non-trivial situations. We have seen that some of the discrete ambiguities can be removed by appealing to smoothness criteria and/or by adopting a sharper unitary gauge condition (3.4), whereas the breakdown configurations seem to be of more physical relevance.

The considerations in this paper have been resolutely classical - the fields were treated throughout as c numbers. What - if any - relevance do the considerations here have for quantum field theory ?

The occurrence of an ambiguity depends on the Higgs field taking on a particular value. But due to quantum fluctuations, the value of a field at a point is not a meaningful concept in quantum field theory, as it is not physically measurable. Therefore, the ambiguities do not seem to be relevant, unless special circumstances are at hand. One such circumstance could be that the ambiguity occurs classically not just at a point, but in a larger region of space-time.

A more important circumstance is the occurrence of a breakdown configuration, which signals a non-trivial topology of the Higgs field. The breakdown configuration is certainly stable against smooth fluctuations and one can expect that there will be effects in the quantum theory related to breakdown configurations in the classical theory.

In this connection, I want to stress the following point : a singularity in the field of directions of the Higgs field is essentially a local phenomenon. Assuming that we have such a singularity at a point x_0 , knowledge of the Higgs field in an infinitesimal neighbourhood of x_0 is sufficient to calculate the Kronecker index of the singularity and thus to establish its nature. "Local" operators creating bare vortices or monopoles will certainly have, as an important part, operators creating point singularities of the field of directions of the Higgs field. On the other hand, suppose we want to set up a quantum field theory in the classical background field of a monopole, as is done in writing down the quantum theory in the one-monopole sector. It is then usually said that the ultra-violet behaviour of such a theory is the same as in the vacuum sector, since at short distances one does not "see" the large scale topological structure of the monopole. This is evidently true everywhere except at the centre of the monopole where the singularity responsible for the breakdown of the unitary gauge sits. In any neighbourhood, no matter how small, containing the centre, the unitary gauge perturbation theory (or any perturbation expansion equivalent to it) is expected to break down, and new short-distance effects may emerge.

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APPENDIX : UNIQUENESS OF THE UNITARY GAUGE VACUUM

We consider a theory where the connected part of the unbroken group S is at most a product of Abelian factors :

$$S = U(1) \times \dots \times U(1) \times \text{discrete} .$$

The conditions specifying a classical vacuum configuration are

$$F_{\alpha}^{\mu\nu} = 0 \quad (A.1)$$

$$(D_{\mu}\varphi)_{\mu} = 0 \quad (A.2)$$

$$\varphi \in M_0 . \quad (A.3)$$

In the unitary gauge, the vacuum state of the Higgs field is represented by a constant vector $\lambda \in M_0$. Discrete ambiguities allow in general gauge equivalent constant vectors λ' which also satisfy the unitary gauge condition :

$$(\lambda, \theta_{\alpha}\lambda') = 0 ; \quad \lambda' = D_B(g)\lambda \in M_0 . \quad (A.4)$$

The vacuum λ can be fixed uniquely by using the improved unitary gauge condition (3.4) ; by Schwarz' inequality the scalar product (λ, x) with $\|x\| = \|\lambda\|$ takes on its maximal value for $x = \lambda$.

We want to show that, once a fixed λ is taken as the solution of (A.3), the vacuum is unique.

Since λ is constant, (A.2) reads

$$(\theta_{\alpha}\lambda) A_{\alpha\mu} = 0 . \quad (A.5)$$

If (A.5) is to be satisfied with non-zero $A_{\alpha\mu}$, the vectors $\theta_\alpha \lambda$ must be linearly dependent. By hypothesis, however, the only linearly dependent combinations of the vectors $\theta_\alpha \lambda$ are the generators of the $U(1)$ groups appearing in $\mathcal{S}$:

$$C_\alpha^{(n)} \theta_\alpha \lambda = 0 .$$

therefore (A.5) implies that the gauge potentials not belonging to these subgroups must vanish. The remaining condition (A.1) is then just the condition for the vanishing of the $U(1)$ field tensors

$$F_{\mu\nu}^{(n)} = \partial_\mu A_\nu^{(n)} - \partial_\nu A_\mu^{(n)} ,$$

where $A_\mu^{(n)} = \sum_\alpha C_\alpha^{(n)} A_{\alpha\mu}$. Thus the vacuum is unique up to $U(1)$ gauge transformations, and this remaining non-uniqueness is removed by suitable $\mathcal{S}$ gauge fixing conditions, e.g., the Coulomb gauge conditions

$$\partial_i A^{(n)i} = 0 , \quad A_i^{(n)} \rightarrow 0 , \quad |\vec{x}| \rightarrow \infty .$$

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FIGURE CAPTIONS

Figure_1 Unitary gauge configurations in φ space.

L : original gauge conditions ;

K : stricter gauge conditions.

Figure_2 a)-c) Unitary gauge configurations of $SO(3)$ tensor model.

Figure_3 A generic configuration.

Figure_4 Singularities in a planar field of directions with
different values of the Kronecker index [adapted from
Ref. 13].

FIG. 1

FIG: 2a

2b

2c

FIG. 3

$k = +3$

$k = -2$

FIG. 4