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APPROACHES TO THE GRAVITATIONAL SPIN $3/2$ AXIAL ANOMALY

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ABSTRACT

The spin $3/2$ axial anomaly is discussed from various points of view. Two consistent theories for a spin $3/2$ field interacting with gravity are considered: supergravity and a real quantized spin $3/2$ field in a classical gravitational background. The Feynmann graph method, the zeta function regularization method, the point splitting method and the topological method all yield the same result for the latter theory, but in supergravity different gauge choices lead to different values.

1. INTRODUCTION

With the advent of supergravity, it has become interesting to study properties of spin $\frac{3}{2}$ fields. One class of problems concerns the anomalies present in theories containing such fields.

Both the axial anomaly^{1,2)} and the trace anomaly³⁾ have been calculated, but the calculation in Ref. 1, which used Feynman graph methods, has yielded a numerical result which differs from the calculation of Ref. 2, which was based on point-splitting or topological methods. In this paper we reconsider the axial anomaly and compare the various approaches towards this calculation. As we will see, there are novel features present owing to the gauge character of the spin $\frac{3}{2}$ field.

As is well known, consistency requires particular couplings for a higher spin field. In the case of spin $\frac{3}{2}$, we consider two possible systems which are consistent: i) supergravity itself^{4,5)} in which one treats both the spin 2 and spin $\frac{3}{2}$ fields on an equal, quantized footing; or ii) a quantized Rarita-Schwinger field in a classical background gravitational field. In both cases there are constraints to be satisfied in order that the system be consistent. In the case where both fields are quantized, all couplings must be exactly as in supergravity, while in the case of a quantized spin $\frac{3}{2}$ field in a non-dynamical background field, consistency requires that the background field satisfies the Einstein equations $R_{\mu\nu} = 0$. In both cases we need a gauge-fixing term in order to eliminate the fermionic gauge invariance of massless spin $\frac{3}{2}$ fields. In this way we can obtain propagators, or, equivalently, the kinetic operators in the Euclidean version are elliptic. As we shall see, the ambiguity mentioned before can be traced back to the choice we have in the way we treat the gravitational field; in particular, using different gauge-fixing procedures may lead to different numerical values of the axial anomaly. This is perhaps not so surprising, because the classical axial current is not invariant under fermionic gauge transformations.

The description of a spin $\frac{3}{2}$ particle by means of the Rarita-Schwinger field introduces certain complications not present in the calculation of the spin $\frac{1}{2}$ anomaly. These complications have to do with redundant degrees of freedom that are carried by such a field. For a massive spin $\frac{3}{2}$ particle there are four redundant degrees of freedom (four spin $\frac{1}{2}$ states which appear in the decomposition of the vector-spinor ψ_μ into irreducible components), while in the massless case there are two more, corresponding to the $\pm\frac{1}{2}$ helicity states of the pure spin $\frac{3}{2}$ irreducible components. In conventional supergravity calculations the usual gauge-fixing procedure introduces two real Faddeev-Popov spin $\frac{1}{2}$ ghosts (one complex ghost) whose role is to cancel four of the redundant degrees of freedom. The remaining two decouple in any calculation where all external lines are on-shell.

In a calculation such as that of the anomaly, where at least one external line is off-shell (that associated with the axial vertex), whether or not the decoupling occurs has to do with the detailed form of the fermionic gauge-fixing term. In particular, for the case of an external gravitational field where the most natural choice is a gravitationally covariant gauge-fixing term, the remaining two degrees of freedom do not decouple and a third real ghost must be introduced to cancel them⁵⁾. [We note that this choice is not admissible in supergravity since it would lead to non-unitarity⁶⁾.]

Once the ghosts have been introduced they will contribute to the chiral current in such a way as to cancel the current carried by the redundant degrees of freedom. It turns out that in the massless case four of these degrees of freedom have the same chiral charge as the spin $\frac{3}{2}$ particle, while two have the opposite charge. Thus in calculating the spin $\frac{3}{2}$ anomaly, the net effect is to subtract from the anomaly calculated with the Rarita-Schwinger field the anomaly of *one* real spin $\frac{1}{2}$ fermion.

We calculate the axial anomaly by using several apparently unrelated methods: a Feynman graph method, a proper-time method, a point-splitting method, and a topological method. These methods all lead to the same result in the most symmetric case of a gravitationally covariant gauge-fixing term for the fermionic gauge invariance (-21 times the result for a Majorana spin $\frac{1}{2}$ field). All the methods can in principle be applied to fully quantized supergravity. However, as we have already mentioned, in that case the most symmetric gauge fixing is not admissible since it would lead to non-unitarity and the number -21 cannot be obtained. We now give a brief discussion of the various methods used.

The Feynman graph method is based on perturbation theory, and simply uses propagators and vertices for gauge and ghost fields to build Feynman diagrams from which we obtain matrix elements. The anomaly is obtained by evaluating the matrix element of the axial current between the vacuum and a two-graviton state. Only one diagram need be considered if we impose gravitational gauge invariance of the matrix element under a gauge change of the external gravitational fields which are on-shell. We discuss the plausibility of the assumption, but a proof using Slavnov-Taylor identities is lacking. This diagram is a spin $\frac{3}{2}$ triangle loop with two external gravitons so that the calculation is very similar to Adler's and Rosenberg's calculations of the spin $\frac{1}{2}$ anomaly in quantum electrodynamics⁷⁾.

The proper-time method⁸⁾ exploits the fact that the short-distance behaviour of Green's functions in a given background metric is that of free fields to leading order, with computable covariant corrections. The axial current can be regularized by zeta-function regularization⁹⁾ or by point-splitting techniques^{10,11)}. In both approaches it is possible to express the axial anomaly in terms of the corrections

to the short-distance behaviour of the free fields mentioned above. In the zeta-function method we can obtain the anomaly from a general algorithm¹²⁾, which is a generalization of DeWitt's proper-time method. It appears similar to an algorithm of 't Hooft derived for counter terms¹³⁾, but whereas 't Hooft's algorithm is only for integrals, our lemma is for densities such as the axial anomaly. In the point-splitting method, we deal directly with the Green's functions whose short-distance behaviour is computable. In both cases we get the same answer: -21 . In both cases the computation is fully covariant, and a covariant gauge-fixing term is chosen. In principle, however, we could extend both methods to non-covariant gauge-fixing terms.

There are various ways of convincing oneself that the anomaly of an axial current is proportional to the index density of the Euclidean kinetic operator restricted to fields of positive chirality^{11,14)}. Here the index is equal to the difference between the number of solutions of the Euclidean spin $\frac{3}{2}$ field equation with positive and negative γ_5 -parity. Another way of phrasing this definition is to say that the index is equal to the multiplicity of zero eigenvalues of positive γ_5 -parity minus the multiplicity of zero eigenvalues of negative γ_5 -parity. The correspondence between the skewness of zero modes and the axial anomaly can be inferred either in the framework of proper-time methods or by various intuitive arguments. This is the starting point for the topological method.

The topological method is based on the Atiyah-Singer theorem^{15,16)}, which states that the index of an elliptic operator, although an analytic quantity by its definition, has in fact a topological meaning. It is computable as a topological invariant ("characteristic class") which in turn can be expressed as a definite polynomial in the curvature quantities. The set-up is now the following: we have an oriented Riemannian manifold on which a Rarita-Schwinger field lives, and an elliptic operator which maps Rarita-Schwinger fields satisfying constraints (for example, $D \cdot \psi = \gamma \cdot \psi = 0$, but the exact form of the constraints is never needed) with positive γ_5 -parity into fields of negative γ_5 -parity. The existence of such an elliptic operator is inferred from the existence of the two consistent spin $\frac{3}{2}$ theories mentioned above.

The actual calculation is even further simplified by reducing it to the spin $\frac{1}{2}$ case, and the number -21 is obtained. In principle, one could also choose two non-covariant constraints on the Rarita-Schwinger fields, which should lead to different numbers, and in this case the index would not only depend on the topology of the manifold but also on non-generic features of the elliptic operator, whereas in the covariant case the index formula is such that all covariant elliptic operators yield the same results.

An inspection of the details of the arguments of the proper-time method and of the topological method reveals a rather close parallelism even at the intermediate steps. From a higher perspective, one might say that the same index is evaluated once analytically and once topologically. There is also a connection with the Feynman method, as the following sequence shows:

$$\begin{array}{ccccccc} \text{Feynman graphs} & & \text{Feynman graphs} & & \text{Zeta function} & & \text{Topological} \\ \text{regularized by} & \sim & \text{with} & \sim & \text{regularization} & \sim & \text{methods} \\ \text{gravitational} & & \text{Pauli-Villars} & & & & \\ \text{covariance} & & \text{regularization} & & & & \end{array}$$

Our conventions are those of Ref. 1.

2. THE FEYNMAN GRAPH METHOD

The calculation of the matrix element of the axial current follows the method of Rosenberg and Adler⁷⁾ for the axial anomaly in quantum electrodynamics. The idea is i) to write down the most general tensor structure for this (*a priori* divergent) matrix element; ii) to define the divergent form factors in terms of the convergent form factors by *imposing* gravitational conservation; iii) to calculate a convergent form factor by means of ordinary Feynman graph techniques; and iv) to express the axial anomaly in terms of this finite form factor.

We begin by defining the axial current. Using the Noether prescription for the action and chiral transformation,

$$\mathcal{L} = -\frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \bar{\Psi}_\mu \gamma_5 \gamma_\nu (\partial_\rho + \frac{1}{2} \omega_{\rho ab} \sigma^{ab}) \Psi_\sigma \quad (2.1)$$

$$\delta \Psi_\mu = -i \gamma_5 \Psi_\mu \quad (2.2)$$

we find the classical axial current

$$j_\mu^5 = -\frac{i}{2} \varepsilon^{\mu\lambda\nu\sigma} \bar{\Psi}_\lambda \gamma_\nu \Psi_\sigma \quad (2.3)$$

If, for example, in supergravity we introduce the gauge-fixing term and ghost action

$$\mathcal{L}' = \frac{1}{4} \bar{\Psi} \cdot \gamma \not{\partial} \gamma \cdot \Psi - \bar{C} \gamma^\mu D_\mu C + \dots \quad (2.4)$$

where C ($\bar{C}$) is the supersymmetry ghost (antighost) field, then we get for the quantum axial current an additional contribution from the Noether prescription,

$$\Delta j_\mu^5 = \frac{i}{4} \bar{\Psi} \gamma_\mu \gamma_5 \gamma \Psi + i \bar{C} \gamma_\mu \gamma_5 C \quad (2.5)$$

Different gauge-fixing terms will lead to different Δj_μ^5 .

The sign of these last two terms is not arbitrary but is fixed by requiring chiral invariance of the terms $\bar{C} \gamma^\mu \partial_\alpha \psi_\mu C^\alpha$ and $\bar{C} \gamma^\mu \partial_\mu C$ in the complete ghost action of supergravity¹⁷⁾ (C^α is the general coordinate ghost). For comparison we give the analogous results for a real (Majorana) spin $1/2$ field:

$$\mathcal{L}_{1/2} = -\frac{e}{2} \bar{\lambda} \gamma^\mu (\partial_\mu + \frac{1}{2} \omega_{\mu ab} \sigma^{ab}) \lambda \quad (2.6)$$

$$\delta \lambda = -i \gamma_5 \lambda \quad (2.7)$$

$$\partial_\mu^5 (1/2) = \frac{i}{2} \bar{\lambda} \gamma_\mu \gamma_5 \lambda \quad (2.8)$$

On the other hand in the case of a spin $3/2$ particle in a background gravitational field the covariant gauge-fixing term and ghost action are

$$\mathcal{L}' = \sqrt{g} \left[\frac{1}{4} \bar{\Psi} \gamma \not{\partial} \gamma \Psi - \bar{C} \not{\partial} C - \frac{1}{2} \bar{C}' \not{\partial} C' \right] \quad (2.4')$$

where C' is the third (Majorana) ghost mentioned in the introduction⁵⁾. We now get for the quantum axial current instead of Eq. (2.5) the additional contribution

$$\Delta j_\mu^5 = \sqrt{g} \left[\frac{i}{4} \bar{\Psi} \gamma_\mu \gamma_5 \gamma \Psi + i \bar{C} \gamma_\mu \gamma_5 C - \frac{i}{2} \bar{C}' \gamma_\mu \gamma_5 C' \right] \quad (2.5')$$

Note the signs of the ghost contributions, reflecting the chiral charge of the ghosts⁵⁾. The net effect is that from a single real ghost.

We consider the most general form of the matrix element of the axial current $J_\mu^5 = j_\mu^5 + \Delta j_\mu^5$ between the vacuum and a state with two gravitons with momenta q^1 and q^2 and polarization indices $(\alpha\beta)$ and $(\gamma\delta)$:

$$M(\mu; q^1, \alpha, \beta; q^2, \gamma, \delta) = \langle g: q^1, \alpha, \beta; g: q^2, \gamma, \delta | J_\mu^5 | 0 \rangle$$

It is given by the following expression, where we have retained only terms which do not vanish on-shell:

$$\begin{aligned}
 M(\mu; q^1, \alpha, \beta; q^2, \gamma, \delta) = & \epsilon(\mu, \alpha, \gamma, q^1) [A \delta_{\beta\delta} + B q_{\beta}^2 q_{\delta}^1] + \\
 & + \epsilon(\mu, \alpha, q^1, q^2) [C q_{\gamma}^1 \delta_{\beta\delta} + D q_{\beta}^2 q_{\gamma}^1 q_{\delta}^1] + \\
 & + (\text{Bose and index symmetrization})
 \end{aligned}
 \tag{2.9}$$

We have used the notation $\epsilon(\mu, \alpha, q_1, q_2) = \epsilon_{\mu\alpha\rho\sigma} q_{\rho}^1 q_{\sigma}^2$, and we have omitted many terms with $q_{\alpha}^1, q_{\beta}^1, q_{\gamma}^2, q_{\delta}^2, \delta_{\alpha\beta}$, and $\delta_{\gamma\delta}$. Bose-symmetrization means adding the same term with $(q_1, \alpha\beta)$ and $(q_2, \gamma\delta)$ interchanged, and dividing the sum by two. Index symmetrization is the same operation with respect to $(\alpha\beta)$ and $(\gamma\delta)$.

As we shall see, the form factor A is at most quadratically divergent, B and C are at most logarithmically divergent, and D is finite. We now impose "gravitational conservation", by which we mean

$$q_{\alpha}^1 M(\mu; q^1, \alpha, \beta; q^2, \gamma, \delta) = 0 \tag{2.10}$$

Note that this implies conservation with respect to $q_{\beta}^1, q_{\gamma}^2, q_{\delta}^2$ as well. From this conservation relation we obtain

$$A = -(q^1 \cdot q^2) B = (q^1 \cdot q^2) C = -\frac{1}{2} (q^1 \cdot q^2)^2 D \tag{2.11}$$

Once again we have omitted terms which vanish on-shell, in particular $(q^1)^2 = (q^2)^2 = 0$, in addition to the previous cases mentioned. Thus our conservation condition simply implies that one replaces in the matrix element one polarization tensor $\epsilon_{\alpha\beta}^1(q_1)$ by $(q_{\alpha}^1 \epsilon_{\beta}^1 + q_{\beta}^1 \epsilon_{\alpha}^1)$.

From (2.11) we see that if D is convergent, so are A, B , and C . We stress that our method of invoking gravitational covariance in order to remove the divergent parts of the matrix element is but one particular choice of a regularization scheme. We will come back at the end of this section to the question, Is one allowed to use this requirement of gravitational conservation?

The divergence of the matrix element of the axial current is given by

$$\begin{aligned}
 \text{Div} = \partial_{\mu} < g : q^1, \alpha, \beta; g : q^2, \gamma, \delta | J^5{}^{\mu} | 0 > = \\
 = -i(q^1 + q^2)_{\mu} M(\mu; q^1, \alpha, \beta; q^2, \gamma, \delta) \epsilon_{\alpha\beta}(q^1) \epsilon_{\gamma\delta}(q^2)
 \end{aligned}
 \tag{2.12}$$

From (2.9) we then get

$$D_{IV} = -i \epsilon_{\alpha\beta}(q^1) \epsilon_{\gamma\delta}(q^2) \epsilon(q^2, \alpha, \gamma, q^1) [A \delta_{\beta\delta} + B q_\beta^2 q_\delta^1] \quad (2.13)$$

where, again, we must Bose symmetrize. Using the conservation equation, we can rewrite this result as

$$D_{IV} = -i \epsilon_{\alpha\beta}(q^1) \epsilon_{\gamma\delta}(q^2) \epsilon(q^2, \alpha, \gamma, q^1) \cdot \left[-\frac{1}{2} (q^1 \cdot q^2)^2 \delta_{\beta\delta} + \frac{1}{2} (q^1 \cdot q^2) q_\beta^2 q_\delta^1 \right] D \quad (2.14)$$

Let us now compute the same matrix element of the operator $P = R_{\mu\nu\alpha\beta} R_{\rho\sigma}^{\mu\nu} \epsilon^{\alpha\beta\rho\sigma}$. After defining the fundamental graviton field by $\kappa h_{\mu\nu} = g_{\mu\nu} - \delta_{\mu\nu}$ and linearizing the Riemann curvature as follows:

$$R_{\mu\nu\alpha\beta}^{(L)} = \frac{\kappa}{2} (h_{\mu\beta, \nu\alpha} + h_{\nu\alpha, \mu\beta} - h_{\mu\alpha, \nu\beta} - h_{\nu\beta, \mu\alpha}) \quad (2.15)$$

we find for this matrix element

$$\begin{aligned} \langle g: q^1, \alpha, \beta; g: q^2, \gamma, \delta | P | 0 \rangle &= \\ &= -2 \epsilon(q^2, \alpha, \gamma, q^1) \left[-\frac{1}{2} (q^1 \cdot q^2)^2 \delta_{\beta\delta} + \frac{1}{2} q^1 \cdot q^2 q_\beta^2 q_\delta^1 \right] \epsilon_{\alpha\beta}(q^1) \epsilon_{\gamma\delta}(q^2) \end{aligned} \quad (2.16)$$

We conclude from (2.14) and (2.16) that

$$D_{IV} = i (q^1 \cdot q^2) D \langle g; g | P | 0 \rangle$$

or, in operator form,

$$\partial^\mu T_\mu^5 = (i q^1 \cdot q^2 D) \epsilon^{\alpha\beta\rho\sigma} R_{\mu\nu\alpha\beta} R_{\rho\sigma}^{\mu\nu} \quad (2.17)$$

We now calculate D in the following way. We consider all one-loop diagrams with the axial current $j_\mu^5 + \Delta j_\mu^5$ at one vertex and two external gravitons. The only parts of the Lagrangian which we need, whether using supergravity or an external gravitational background field, are given by

$$\mathcal{L} = -\frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \bar{\Psi}_\mu \gamma_5 \gamma_\nu (\partial_\rho + \frac{1}{2} \omega_{\rho ab}(e) \sigma^{ab}) \Psi_\sigma \quad (2.18)$$

$$+ \mathcal{L}_{(f,x)} + \mathcal{L}_{(ghost)}$$

It follows that we only need consider spin $3/2$ gauge field loops and spin $1/2$ ghost field loops. Our procedure is now to collect from all contributing diagrams only the terms with the tensor structure of the form factor D given in (2.9).

It is not difficult to verify that among the contributing diagrams of Fig. 1, only the triangle (1a) contributes to D. In the other diagrams it is possible to route momenta so that the loop integrals yield tensors constructed out of q_1 , or q_2 , or $q_1 + q_2$ only. Since the D term involves the antisymmetric expression $\varepsilon(\mu, \alpha, q_1, q_2)$, it is clear that it cannot pick up contributions from these diagrams.

In Fig. 1a we must consider contributions both from the spin $3/2$ field and the spin $1/2$ ghost. These can be computed in a straightforward manner once the propagators and vertices are given. In order to proceed we must specify in detail the form of the gauge-fixing term. We shall consider the following possibilities:

$$\mathcal{L}_1(f,x) = \frac{1}{4} \bar{\Psi} \cdot \bar{\gamma} \bar{\gamma}^\lambda \partial_\lambda \bar{\gamma} \cdot \Psi \quad \mathcal{L}_1(ghost) = -\bar{c} \bar{\gamma}^\nu D_\nu c \quad (2.19)$$

$$\mathcal{L}_2(f,x) = \frac{1}{4} \bar{\Psi} \cdot \bar{\gamma} \bar{\gamma}^\lambda \partial_\lambda \bar{\gamma} \cdot \Psi \quad \mathcal{L}_2(ghost) = -\bar{c} \bar{\gamma}^\nu D_\nu c \quad (2.20)$$

$$\mathcal{L}_3(f,x) = \frac{1}{4} \bar{\Psi} \cdot \bar{\gamma} \bar{\gamma}^\lambda D_\lambda \bar{\gamma} \cdot \Psi \quad \mathcal{L}_3(ghost) = -\frac{1}{2} e \bar{\chi} \bar{\gamma}^\nu D_\nu \chi \quad (2.21)$$

The derivative D is the gravitational covariant derivative, with the torsionless spin connection $\omega_{\mu ab}(e)$ ⁴⁾. Further, $\bar{\gamma}_\nu$ is constant, while $\gamma_\nu = e_\nu^a \gamma_a$ is coordinate dependent. The choices $\mathcal{L}_1$ and $\mathcal{L}_2$ are conventional for quantum supergravity, while the covariant choice $\mathcal{L}_3$ is natural for the case of the Rarita-Schwinger field in a gravitational background, but forbidden in quantum supergravity, since it violates unitarity ⁶⁾. They all lead to the same spin $3/2$ propagator $P_{\mu\sigma} = (2p^2)^{-1} \times \bar{\gamma}_\sigma \bar{\gamma}_\lambda p_\lambda \bar{\gamma}_\mu$, but to different vertices, since $\mathcal{L}_2$ and $\mathcal{L}_3$ contain the gravitational field while $\mathcal{L}_1$ does not. All terms $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ lead to different axial currents but they contribute in the same way to the invariant D. Furthermore, choices $\mathcal{L}_1$, $\mathcal{L}_2$ lead to a conventional complex ghost contribution. Choice $\mathcal{L}_3$ leads to one complex ghost and one real ghost of opposite chirality. The net effect is that of a single real ghost which is what appears in Eq. (2.21).

The choice $\mathcal{L}_1$ leads to the following three-point vertex between the incoming gravitino of momentum $k + q$ and polarization λ , an outgoing gravitino of momentum k and polarization κ , and an outgoing on-shell graviton with polarization tensor $\varepsilon_{\alpha\beta}(q)$:

$$V_{\kappa\lambda}^0(k, q, k+q) = -\frac{1}{4} \left[\varepsilon^{\kappa\beta\rho\lambda} \gamma_5 \gamma_\alpha (2k+q)_\rho \varepsilon_{\alpha\beta}(q) + 2\varepsilon^{\kappa\nu\alpha\lambda} \varepsilon^{\nu\beta\sigma\delta} \gamma_\delta q_\sigma \varepsilon_{\alpha\beta}(q) \right] \quad (2.22)$$

This vertex comes from the $h\psi\psi$ terms in the Rarita-Schwinger action only.

The choice $\mathcal{L}_2$ leads to the vertex $V_{\kappa\lambda}^0 + V_{\kappa\lambda}^{\gamma\cdot\psi}$, where

$$V_{\kappa\lambda}^{\gamma\cdot\psi} = -\frac{1}{2} \left[\delta_{\alpha\kappa} \gamma_\beta (\not{k} + \not{q}) \gamma_\lambda + \delta_{\alpha\lambda} \gamma_\kappa \not{k} \gamma_\beta \right] \varepsilon_{\alpha\beta}(q) \quad (2.23)$$

Thus, $V^{\gamma\cdot\psi}$ is obtained by retaining only the $h\psi\psi$ terms in $\mathcal{L}_2$.

Finally, the choice $\mathcal{L}_3$ leads to $V_{\kappa\lambda}^0 + V_{\kappa\lambda}^{\gamma\cdot\psi} + V_{\kappa\lambda}^{\not{D}}$, where

$$V_{\kappa\lambda}^{\not{D}} = -\frac{1}{2} \not{k}_\beta \gamma_\kappa \gamma_\alpha \gamma_\lambda \varepsilon_{\alpha\beta}(q) \quad (2.24)$$

As far as the ghost vertex is concerned, the choice of $\mathcal{L}_1(\text{ghost})$ leads to a zero three-point vertex for an on-shell graviton. This follows from the symmetry, conservation and tracelessness of the on-shell graviton. The choices $\mathcal{L}_2$ or $\mathcal{L}_3$ give the vertex

$$V = -\gamma_\alpha \not{k}_\beta \varepsilon_{\alpha\beta}(q) \quad (2.25)$$

The only contribution comes from the vierbein in γ^ν in $\mathcal{L}_2, \mathcal{L}_3$. The evaluation of the expression corresponding to Fig. 1a is straightforward and we refer the reader to Ref. 1 for details. After writing down the expression, we must take a trace over 14 gamma matrices and extract the quantity D from the resulting integral. This integral is *a priori* cubically divergent, but since the coefficient of D contains five momentum factors it is clear that D corresponds to a convergent integral which can easily be isolated after Feynman parametrization.

We shall express our results in the following form: we write

$$\partial^\mu \bar{J}_\mu^5 = -\lambda \frac{i}{768\pi^2} \varepsilon^{\alpha\beta\rho\sigma} R_{\mu\nu\alpha\beta} R_{\rho\sigma}{}^{\mu\nu} \quad (2.26)$$

where $\lambda = -1$ yields the value of the axial anomaly of a spin $1/2$ Majorana fermion in a gravitational background field^{11,18}). In our case, the value of λ depends on the choice of the gauge-fixing term and can be obtained from the following table:

$\lambda = \lambda(\text{ghost}) + \lambda(\text{non-ghost})$																			
$\lambda(\text{ghost})$	=	0	for the first choice																
		2	for the second choice																
		1	for the third choice																
$\lambda(\text{non-ghost})$	=		<table> <tr> <th></th><th>V^0</th><th>$V^{\gamma \cdot \psi}$</th><th>$V^{\not{D}}$</th></tr> <tr> <th>V^0</th><td>4</td><td>2</td><td>0</td></tr> <tr> <th>$V^{\gamma \cdot \psi}$</th><td>2</td><td>6</td><td>2</td></tr> <tr> <th>$V^{\not{D}}$</th><td>0</td><td>2</td><td>2</td></tr> </table>		V^0	$V^{\gamma \cdot \psi}$	$V^{\not{D}}$	V^0	4	2	0	$V^{\gamma \cdot \psi}$	2	6	2	$V^{\not{D}}$	0	2	2
	V^0	$V^{\gamma \cdot \psi}$	$V^{\not{D}}$																
V^0	4	2	0																
$V^{\gamma \cdot \psi}$	2	6	2																
$V^{\not{D}}$	0	2	2																

The table corresponds to the fact that we have done the calculation for $\mathcal{L}_3$, corresponding to the two vertices

$$(V^0 + V^{\gamma \cdot \psi} + V^{\not{D}}) (V^0 + V^{\gamma \cdot \psi} + V^{\not{D}}) \quad (2.27)$$

by evaluating separately the contributions from products of any two V 's. Each product gives one of the numbers in the table. Therefore:

$$\text{For } \mathcal{L}_1: \lambda = 4$$

$$\text{For } \mathcal{L}_2: \lambda = 4 + 2 + 2 + 6 = 14$$

$$\text{For } \mathcal{L}_3: \lambda = 4 + 2 + 2 + 6 + 2 + 2 + 2 = 20 .$$

Thus we obtain the following results by adding up the ghost contribution:

For $\mathcal{L}_1$ the anomaly is -4 times that of a spin $1/2$ Majorana fermion;

For $\mathcal{L}_2$ the anomaly is -16 times that of a spin $1/2$ Majorana fermion;

For $\mathcal{L}_3$ the anomaly is -21 times that of a spin $1/2$ Majorana fermion.

A fourth choice $\mathcal{L}(\text{fix}) = \bar{\psi} \cdot \bar{\gamma} \not{D} \bar{\gamma} \cdot \psi$ gives an anomaly -6 times that of a spin $1/2$ Majorana fermion.

Thus different choices of gauge-fixing terms lead to different values of the anomaly. This is perhaps not astonishing if one realizes that the axial current itself is not a (Rarita-Schwinger) gauge-invariant object. There is *a priori* no reason to expect the anomaly to be gauge-independent. What one should expect is that in a theory of supergravity which contains the axial vector current in a coupling to an axial vector field, S-matrix elements should be gauge-invariant.

In such a theory one could envisage calculating "pion" decay into two gravitons through a spin $\frac{3}{2}$ loop and expect the result to be independent of gauge. However, at the present time the only locally supersymmetric theories containing the axial current coupled to an axial vector field also contain a cosmological term¹⁹⁾ which renders these theories unsuitable as a testing ground since it is not clear how to define the S-matrix in a de-Sitter space.

If we consider the axial anomaly in the context of a spin $\frac{3}{2}$ gravitino in a gravitational background metric the completely covariant choice $\mathcal{L}_3$ seems to be the only sensible one. Even there, however, the answer could be sensitive to the choice of a gauge *parameter* in front of $\mathcal{L}_3$. We have checked that to first order in such a parameter the result does not change, but this question deserves further study. At any rate, for the choice $\mathcal{L}_3$ we obtain the same result with the Rosenberg-Adler method as with the other covariant methods discussed in the following sections.

Finally, we come back to the question whether we may impose gravitational conservation on the matrix element $M(\mu; q_1, \alpha\beta; q_2, \gamma\delta)$ in a calculation which uses non-covariant choices of gauge-fixing terms such as $\mathcal{L}_1$ or $\mathcal{L}_2$. It is known from explicit calculations of the graviton self-energy due to a gravitino loop using the gauge-fixing term $\mathcal{L}_1$, that the result satisfies the conservation equation even off-shell²⁰⁾. This would suggest that using conservation is valid. A second possible source of difficulties is the fact that the axial vector itself is not gravitationally invariant (it transforms as a world four-vector). This is to be contrasted with the situation in quantum electrodynamics where the axial current $\bar{\lambda} \gamma^\mu \gamma^5 \lambda$ is U(1) *invariant*. The same situation is present for the gravitational spin $\frac{1}{2}$ anomaly, where again the axial current is not gravitationally invariant. However, the Rosenberg-Adler method leads here to the correct value¹⁾. Thus it is conceivable that if the conservation condition is not valid, it is only because of the presence of the non-covariant gauge-fixing term *together* with the fact that the axial current is not Rarita-Schwinger gauge-invariant. In principle this could be answered by appealing to suitable Slavnov-Taylor identities. We plan to study this problem in the future. We should stress, however, that in our opinion, even if the conservation condition is not satisfied for choices $\mathcal{L}_1$ or $\mathcal{L}_2$ it is valid for the choice $\mathcal{L}_3$.

A last point to be mentioned is that we have used, in the matrix element M , symmetric polarization tensors $\epsilon_{\alpha\beta}(q_1)$ for the gravitons. Whereas after stripping the polarization tensors from M , M will in general no longer be symmetric (similar to the non-transversality of the three-point function in Yang-Mills theory with on-shell momenta), in Eq. (2.9) we were able to restrict our attention to symmetric M since we always saturated with symmetric tensors.

3. ZETA-FUNCTION AND POINT-SPLITTING METHODS

Working with a Majorana spin $\frac{3}{2}$ field ψ_μ in curved space time, we have the classical Noether current (2.3) for chiral transformations. The corresponding current in the quantized theory, when a fully covariant Feynman-type gauge (background gauge) is used, is given by the sum of Eqs. (2.3) and (2.5'). It is the anomaly of this current that is treated here.

The sum of (2.3) and (2.5') is regrouped according to the definitions:

$$J_{RS}^{5\mu} = -|g|^{1/2} \frac{i}{4} \bar{\psi}_\rho \gamma^\sigma \gamma^\mu \gamma^\rho \gamma_5 \psi_\sigma \quad (3.1)$$

$$J_G^{5\mu} = |g|^{1/2} \left[i \bar{c} \gamma^\mu \gamma_5 c - \frac{i}{2} \bar{c}' \gamma^\mu \gamma_5 c' \right] \quad (3.2)$$

It should be noted that the axial current, though being the quantum Noether current for chiral transformations, is not invariant under BRS transformations of the fermion fields²¹⁾. Consequently, unphysical degrees of freedom cannot be expected to decouple from the current, and one will in general expect a result depending on the choice of gauge.

When we calculate the expectation value of the current, the only new thing to be considered is the term $J_{RS}^{5\mu}$ involving the field ψ_μ , because the term $J_G^{5\mu}$ gives just the usual spin $\frac{1}{2}$ induced current, but with opposite sign, because of the Bose statistics of the ghost fields.

The induced current from $J_{RS}^{5\mu}$ is given by

$$\langle J_{RS}^{5\mu} \rangle(x) = i \frac{i}{2} |g|^{1/2} \text{tr} \Gamma^\mu \gamma_5 G(x, x) \quad (3.3)$$

Here we have introduced the generalized Dirac matrices Γ^μ by

$$(\Gamma^\mu)_{\rho\sigma} = -\frac{1}{2} \gamma_\sigma \gamma^\mu \gamma_\rho \quad (3.4)$$

with the important algebraic property

$$\{\Gamma^\mu, \Gamma^\nu\}_{\rho\sigma} = 2 g^{\mu\nu} g_{\rho\sigma} \quad (3.5)$$

Also, the Green's function $G(x, x')$ is defined as the solution with causal boundary conditions of:

$$\Gamma^\mu D_\mu G(x, x') = |g|^{-1/2} \delta(x, x') \quad (3.6)$$

(we disregard here the possibility of zero modes). It is also convenient^{8,11)} to define a Green's function for the square of $\Gamma_\mu D^\mu$ with the same boundary conditions:

$$-(\Gamma^\mu D_\mu)^2 \mathcal{G}(x, x') = |g|^{-1/2} \delta(x, x') \quad (3.7)$$

The two Green's functions are related by

$$G(x, x') = -\Gamma^\mu D_\mu \mathcal{G}(x, x') \quad (3.8)$$

and (3.3) becomes

$$\langle J_{RS}^{5\mu} \rangle(x) = \frac{1}{2} |g|^{1/2} \text{tr} \Gamma^\mu \gamma_5 \Gamma^\lambda D_\lambda \mathcal{G}(x, x') \Big|_{x'=x} \quad (3.9)$$

The expression on the right-hand side of Eq. (3.9) is of a purely formal character, since $\mathcal{G}(x, x')$ diverges for $x' = x$. In order to overcome this difficulty we can introduce a regularization procedure. In this section two such schemes will be considered: the ζ -function method⁹⁾, and the point separation method^{10,11)}. They are apparently different, but on closer inspection intimately related.

The starting point of the ζ -function method is the observation that $\mathcal{G}(x, x')$, according to its definition (3.7) can be written formally:

$$\mathcal{G}(x, x') = \langle x | \frac{1}{-(\Gamma^\mu D_\mu)^2 - i\epsilon} | x' \rangle \quad (3.10)$$

An object of a similar structure which is regular for $x' \rightarrow x$ is now obtainable by raising the denominator to a sufficiently large power, so we are led to define

$$\zeta(s, x) = \langle x | \frac{1}{(-(\Gamma^\mu D_\mu)^2 - i\epsilon)^{s+1}} | x \rangle \quad (3.11)$$

(the similarity to the usual Riemann ζ -function becomes more apparent when a representation in terms of eigenfunctions of the Dirac operator is used).

The ζ -regulated version of (3.9) is

$$\langle J_{RS}^{5\mu} \rangle_{\text{reg}}(x) = \frac{1}{2} |g|^{1/2} \text{tr} \Gamma^\mu \gamma_5 \Gamma^\lambda D_\lambda \langle x | \frac{1}{(-(\Gamma^\mu D_\mu)^2 - i\epsilon)^{s+1}} | x \rangle \quad (3.12)$$

with the divergence

$$\partial_\mu \langle J_{RS}^{5\mu} \rangle_{\text{reg}}(x) = |g|^{1/2} \text{tr} \gamma_5 \zeta(s, x) \quad (3.13)$$

which then must be continued analytically to $s = 0$ to give the final anomaly equation,

$$\partial_\mu \langle \bar{\psi}_{R5}^{\mu} \rangle(x) = |g|^{1/2} \text{tr } \gamma_5 \zeta(0, x) \quad (3.14)$$

The evaluation of $\zeta(0, x)$ can be carried out by means of the following representation of the ζ -function:

$$\zeta(s, x) = \frac{i^s}{\Gamma(s)} \int_0^\infty t^{s-1} dt \langle x | e^{it((\Gamma^\mu D_\mu)^2 + i\varepsilon)} | x \rangle \quad (3.15)$$

where t is the proper-time variable of Schwinger⁸⁾. In a space with a Euclidean signature, where the Dirac operator is elliptic, a more convenient variable is the corresponding proper temperature, which is the imaginary proper time¹⁶⁾.

For $s \rightarrow 0$ only the singular part of the integral in (3.15) will contribute, and it can be evaluated by means of the asymptotic expansion⁸⁾,

$$\langle x | e^{it(\Gamma^\mu D_\mu)^2} | x \rangle = -\frac{i}{16\pi^2 t^2} \left[a_0(x) + a_1(x)it + a_2(x)(it)^2 + O(t^3) \right] \quad (3.16)$$

Inserting this expansion into (3.15), we get for $s \approx 0$

$$\int_0^\infty t^{s-1} dt \langle x | e^{it(\Gamma^\mu D_\mu)^2 + i\varepsilon} | x \rangle \approx \frac{i}{16\pi^2 s} a_2(x) \quad (3.17)$$

and hence

$$\zeta(0, x) = \frac{i}{16\pi^2} a_2(x) \quad (3.18)$$

The coefficient $a_2(x)$ is evaluated by the coincidence limit method of Synge²²⁾ and De Witt⁸⁾. The derivation follows exactly that given by De Witt⁸⁾ for the scalar and spinor cases and by Brown and Cassidy²³⁾ for the vector case, but the outcome can in fact be considered as a consequence of a general algorithm

$$a_2(x) = \frac{1}{180} R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau} + \frac{1}{12} \gamma_{\mu\nu} \gamma^{\mu\nu} + \frac{1}{2} X^2 \quad (3.19)$$

valid when the background metric fulfils the Einstein equation. This algorithm applies to the expansion analogous to (3.16) for all differential operators of the form

$$\mathcal{D} = -D_\lambda D^\lambda - X \quad (3.20)$$

where X is some polynomial in the Riemann tensor, and D_λ is the covariant derivative for the field in question. Also,

$$\gamma_{\mu\nu} = [\gamma_\mu, \gamma_\nu] \quad (3.21)$$

A detailed derivation of (3.19) will be given elsewhere¹²⁾. A similar algorithm was obtained by Gilkey²⁴⁾ (see also Ref. 25). Note that the algorithm, although similar to that found in perturbation theory by 't Hooft¹³⁾ in the case of bosons, in fact is radically different in the case of fermions. The reason for this is that the square of the full Dirac operator appears in (3.11); this effectively reduces the fermionic case to the bosonic one. Also the algorithm used here, in contrast to that of 't Hooft, gives direct information on local quantities.

Specializing to a Rarita-Schwinger field, we get from (3.21),

$$(\gamma_{\mu\nu})_{\lambda\rho} = R_{\mu\nu\lambda\rho} + \frac{1}{2} \sigma_{\alpha\beta} R^{\alpha\beta}_{\mu\nu} g_{\lambda\rho} ; \sigma_{\alpha\beta} = \frac{1}{4} [\gamma_\alpha, \gamma_\beta] \quad (3.22)$$

Also,

$$\begin{aligned} \{-(\gamma_\mu D^\mu)^2\}_{\lambda\rho} &= -(D_\mu D^\mu)_{\lambda\rho} - \frac{1}{4} ([\gamma_\mu, \gamma_\nu] \gamma^{\mu\nu})_{\lambda\rho} = \\ &= -(D_\mu D^\mu)_{\lambda\rho} - \sigma_{\mu\nu} R^{\mu\nu}_{\lambda\rho} \end{aligned} \quad (3.23)$$

which, according to (3.20), means

$$(X)_{\lambda\rho} = \sigma_{\mu\nu} R^{\mu\nu}_{\lambda\rho} \quad (3.24)$$

Insertion of (3.22) and (3.24) into (3.19) now leads by combination with (3.14) and (3.18) to the result

$$\begin{aligned} \partial_\mu \langle J_{RS}^{5\mu} \rangle(x) &= \frac{i|g|^{1/2}}{16\pi^2} \text{tr} \gamma_5 \left(\frac{1}{48} \sigma_{\alpha\beta} R^{\alpha\beta}_{\mu\nu} g_{\lambda\rho} \sigma_{\sigma\delta} R^{\sigma\delta\mu\nu} g^{\rho\lambda} \right. \\ &\quad \left. + \frac{1}{2} \sigma_{\alpha\beta} R^{\alpha\beta}_{\mu\nu} \sigma_{\sigma\delta} R^{\sigma\delta\nu\mu} \right) = \\ &= -\frac{i}{16\pi^2} \left(\frac{1}{2} - \frac{1}{12} \right) \epsilon_{\alpha\beta\sigma\delta} R^{\alpha\beta}_{\mu\nu} R^{\sigma\delta\mu\nu}(x) \end{aligned} \quad (3.25)$$

In order to get the corresponding result for $J^{5\mu}$ itself we have also to take the term $J_G^{5\mu}$ in Eq. (3.2) into account. This just contributes the standard real spin $\frac{1}{2}$ result (see Refs. 18 and 11), and the effect is to change (3.25) into

$$\begin{aligned} \partial_\mu \langle J_{RS}^{5\mu} \rangle(x) &= -\frac{i}{16\pi^2} \left(\frac{1}{2} - \frac{3}{48} \right) \epsilon_{\alpha\beta\sigma\delta} R^{\alpha\beta}_{\mu\nu} R^{\sigma\delta\mu\nu}(x) \\ &= -\frac{i}{16\pi^2} \frac{21}{48} \epsilon_{\alpha\beta\sigma\delta} R^{\alpha\beta}_{\mu\nu} R^{\sigma\delta\mu\nu}(x) \end{aligned} \quad (3.26)$$

which is the final result.

When point-splitting is used, the current (3.9) is regulated by¹¹⁾

$$\langle J_{RS}^{SM} \rangle_{\text{reg}}(x, x') = \frac{1}{2} |g|^{1/2} \text{tr} \Gamma^M(x) \gamma_5 \Gamma^\lambda(x') D_\lambda G(x, x') I(x', x) g(x', x) \quad (3.27)$$

where $I(x', x)$ is the bispinor function⁸⁾ and $g(x', x)$ the bivector function (in matrix form). The divergence of this expression is, by Eq. (3.7) reduced to (see Ref. 3)

$$\partial_\mu \langle J_{RS}^{SM} \rangle_{\text{reg}}(x, x') = |g|^{1/2} \text{tr} \Gamma^M(x) \gamma_5 \Gamma^\lambda(x') D_\lambda G(x, x') D_\mu \{ I(x', x) g(x', x) \} \quad (3.28)$$

whence the anomaly is obtained by matching singular terms in the Green's function with terms in the power expansion of $D_\mu \{ I(x', x) g(x', x) \}$ in such a way that the product is finite and directional-independent.

The short-distance behaviour of $G(x, x')$ can, when Eq. (3.20) is used, be read off from Refs. 11 and 26:

$$G_{\mu\nu}(x, x') \simeq \frac{i}{8\pi^2\sigma} I(x, x') g_{\mu\nu}(x, x') - \frac{i}{16\pi^2} \log \sigma \sigma_{\alpha\beta} R^{\alpha\beta}{}_{\mu\rho} g^\rho{}_\nu(x, x') I(x, x') \quad (3.29)$$

The factor i has its origin in the fact that we are dealing with a space with Minkowski signature; $\sigma = \sigma(x, x')$ is the geodesic interval^{8, 22)}.

From (3.29) we now see that there are two possibilities to get something non-vanishing and finite for $x' \rightarrow x$ in Eq. (3.28): we can either take the product of the leading singularity of $D_\lambda G$ with the part of $D_\mu \{ Ig \}$ which is of third order in $x' - x$ or we can take the product of the next leading singularity of $D_\lambda G$ with the first-order part of $D_\mu \{ Ig \}$. Considering the former possibility first, we have to use (see Ref. 11, App. B)

$$\begin{aligned} I_{;\mu}(x', x) g^{\lambda\lambda'}(x', x) + I(x', x) g^{\lambda\lambda'}_{;\mu}(x', x) \simeq \\ \frac{1}{48} I(x', x) g^{\lambda\lambda'}(x', x) \sigma_{\alpha\beta} R^{\lambda}{}_{\tau\mu\eta} R^{\alpha\beta}{}_{\lambda\rho} \sigma^\tau \sigma^\eta \sigma^\rho \\ - \frac{1}{24} I(x', x) g^{\lambda\lambda'}(x', x) R^{\lambda\lambda'\eta}{}_\rho R_{\lambda\lambda\mu\eta} \sigma^\rho \sigma^\lambda \sigma^\lambda \end{aligned} \quad (3.30)$$

where $\sigma^\lambda = \partial^\lambda \sigma$. Using (3.29) we then find that (3.28) contains a term

$$\begin{aligned}
 & \frac{i}{24} \frac{|g|^{1/2}}{16\pi^2 \sigma^2} \text{tr} \gamma_5 (\gamma^\lambda \gamma^\mu \gamma^\nu \gamma^\rho + \gamma^\rho \gamma^\nu \gamma^\mu \gamma^\lambda) \sigma_\lambda \cdot \\
 & \cdot \left(\frac{1}{2} g_{\tau\tau} \sigma_{\alpha\beta} R_{\tau\mu\eta}^{\alpha\beta} R_{\lambda\rho}^{\alpha\beta} \sigma^\tau \sigma^\mu \sigma^\eta \sigma^\rho - \right. \\
 & \quad \left. - R_{\tau\tau\nu}^{\alpha\beta} R_{\nu\kappa\mu\lambda} \sigma^\rho \sigma^\kappa \sigma^\mu \sigma^\lambda \right) = \\
 & = -\frac{i}{48\pi^2 \sigma^2} \epsilon^{\lambda\mu\tau\tau} R_{\tau\tau\nu}^{\alpha\beta} R_{\tau\kappa\mu\lambda} \sigma_\lambda \sigma^\rho \sigma^\kappa \sigma^\mu \sigma^\nu = \\
 & = \frac{i}{192\pi^2 \sigma^2} \epsilon^{\mu\lambda\tau\tau} R_{\tau\tau\nu\rho} R_{\nu\rho}^{\mu\lambda}
 \end{aligned} \tag{3.31}$$

where in the last step the identity (C.3) of Ref. 11 was used and a directional-dependent term was disposed of.

When we consider the contribution from the next-leading singularity of G, we have to use (see Ref. 11, App. B)

$$\begin{aligned}
 & I_{;\mu}(x', x) g_{\tau'}^{\tau} (x, x') + I(x', x) g_{\tau'}^{\tau}{}_{;\mu}(x', x) \simeq \\
 & \simeq \frac{1}{4} I(x', x) g_{\tau'}^{\tau} (x', x) \sigma_{\alpha\beta} R_{\mu\nu}^{\alpha\beta} \sigma^\nu - \frac{1}{2} I(x', x) g_{\tau'\eta}^{\tau} (x', x) R_{\mu\nu}^{\tau\eta} \sigma^\nu
 \end{aligned} \tag{3.32}$$

and the only non-vanishing term arises when we substitute

$$g^{\lambda\tau'}{}_{;\mu}(x, x') \simeq -\frac{i\sigma_\mu}{16\pi^2 \sigma} \sigma_{\alpha\beta} R^{\alpha\beta\lambda}{}_\rho g^{\rho\tau'}(x, x') I(x, x') \tag{3.33}$$

The outcome is

$$\begin{aligned}
 & \frac{i|g|^{1/2}}{32\pi^2 \sigma} \text{tr} \gamma_5 (\gamma^\lambda \gamma^\mu \gamma^\nu \gamma^\rho + \gamma^\rho \gamma^\nu \gamma^\mu \gamma^\lambda) \sigma_\lambda \sigma_{\alpha\beta} R^{\alpha\beta}{}_{\tau\eta} \cdot \\
 & \cdot \left(\frac{1}{4} g_{\tau\tau}^{\eta} \sigma_{\delta\delta} R^{\tau\delta}{}_{\mu\nu} \sigma^\nu - \frac{1}{2} R_{\tau\mu\nu}^{\eta} \sigma^\nu \right) = \\
 & = -\frac{i|g|^{1/2}}{32\pi^2 \sigma} \text{tr} \gamma_5 \sigma_{\alpha\beta} R^{\alpha\beta\tau\rho} R_{\lambda\rho\mu\nu} \gamma^\mu \gamma^\lambda \sigma^\nu \sigma_\lambda = \\
 & = -\frac{i}{16\pi^2 \sigma} \epsilon^{\alpha\beta\mu\lambda} R_{\alpha\beta}{}^{\tau\rho} R_{\tau\rho\mu\nu} \sigma^\nu \sigma_\lambda = \\
 & = -\frac{i}{32\pi^2} \epsilon^{\alpha\beta\mu\nu} R_{\alpha\beta\tau\rho} R^{\tau\rho}{}_{\mu\nu}
 \end{aligned} \tag{3.34}$$

where in the last step (C.2) of Ref. 11 was used and a directional-dependent term was disposed of. Adding (3.31) and (3.34) we are again led to Eq. (3.25).

Obviously, the part of (3.25) arising from $\frac{1}{2}X^2$, corresponds to (3.34), and the part arising from $\frac{1}{12}Y_{\mu\nu}Y^{\lambda\nu}$ corresponds to (3.31).

Thus, in the case of the completely covariant choice of gauge (2.21), both the ζ -function regularization method and the point-splitting method yield, for the gravitational real spin $3/2$ axial anomaly, a result which is -21 times that for a real spin $1/2$ fermion.

4. THE TOPOLOGICAL METHOD

The starting point of the topological method is the observation that in a very general situation current anomalies are related to the asymmetry of the zero-mode structure of the Euclidean kinetic operator and, hence, to the evaluation of an index. In Ref. 11 this relationship was explained in detail for the spin $1/2$ axial current and for two new currents, the Euler current and the Hirzebruch current. The cases which are studied in this work are formally completely analogous, and we need not repeat the corresponding discussion for them.

The physical situation under consideration can be translated into the following mathematical framework.

We are given a four-dimensional Euclidean oriented space-time manifold M , which, for convenience we assume to be compact without boundary. M will also be a Riemannian manifold, with metric g . On M there are defined certain fields (like spinor fields or Rarita-Schwinger fields, four-vector fields), which assume their values in linear vector spaces, whose dimension equals the number of components of the fields. Every such field naturally leads to the introduction of an affiliated vector bundle²⁷⁾ over M , which, roughly speaking, just means that to every space-time point in M we attach the same linear vector space of all possible values the field can assume there.

Another natural vector bundle over M , which we shall need in the sequel, is the tangent bundle TM , which is constructed by attaching to every point the space of all tangent vectors at that point.

The fields fulfil Euclidean equations of motion, and there is a kinetic operator (normally a differential operator) which acts on them.

Writing f_i for the field components, we can express the kinetic operator D locally in the form

$$(Df)_i = \sum_{\alpha, j} a_{ij}^{\alpha}(x) \partial^{\alpha} f_j$$

Here $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is a multi-index, which denotes the order of the partial derivative ∂^{α} . Denoting $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$, we shall normally have $|\alpha| \leq 2$ or even $|\alpha| \leq 1$.

Physically it is clear that the operator D should be *elliptic*, which means that substituting monomials $\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \xi_3^{\alpha_3} \xi_4^{\alpha_4}$ for ∂^α the symbol

$$\sigma_D(x, \xi)_{ij} = \sum_{|\alpha| \leq |\alpha|_{\max}} \alpha_{ij}^\alpha(x) \xi^\alpha$$

should be an invertible matrix for $\xi \neq 0$. This simply means that all the components of the field really propagate (to achieve this, one has to eliminate redundant components of the field, which we assume has been done) and that the corresponding Minkowskian operator has a unique mass shell.

By means of the metric g we can define a unique covariant derivative ∇ , which is torsion-free and conserves scalar products, and the Clifford bundle $C(M)$, obtained by forming the Clifford algebra over the tangent space at every point of M with respect to the metric. This in particular means that we can define γ -matrices at every point of M .

If, in addition, M is a spin manifold, as shall be assumed, spinors can live on M ; which means that one can define a spinor bundle $\Delta^{\frac{1}{2}}(M)$ over M which decomposes into a direct sum

$$\Delta^{\frac{1}{2}}(M) = \Delta_+^{\frac{1}{2}}(M) \oplus \Delta_-^{\frac{1}{2}}(M) \quad (4.1)$$

of spinor bundles with positive and negative chirality (γ_5 -parity). Also a unique covariant derivative $\tilde{\nabla}$ on spinors can be constructed in this case as well as an *elliptic* Dirac operator

$$D_{1/2} = \gamma^\mu \tilde{\nabla}_\mu \quad (4.2)$$

from $\Delta_+^{\frac{1}{2}}(M)$ to $\Delta_-^{\frac{1}{2}}(M)$:

$$D_{1/2} : \Delta_+^{\frac{1}{2}}(M) \rightarrow \Delta_-^{\frac{1}{2}}(M) . \quad (4.3)$$

($D_{1/2}$ anticommutes with γ_5).

In the physical situation under consideration we have Rarita-Schwinger spinors on M of positive and negative chirality. So, we have Rarita-Schwinger bundles

$$\Delta^{\frac{3}{2}}(M) = \Delta_+^{\frac{3}{2}}(M) \oplus \Delta_-^{\frac{3}{2}}(M) \quad (4.4)$$

where $\Delta^{\frac{3}{2}}(M)$ will be defined below [Eq. (4.6)].

The possibility of formulating consistent Rarita-Schwinger equations on M implies the existence of an elliptic Rarita-Schwinger operator

$$D_{3/2} : \Delta_+^{\frac{3}{2}}(M) \rightarrow \Delta_-^{\frac{3}{2}}(M) . \quad (4.5)$$

Massless physical spin $3/2$ particles are described by constrained vector-spinors. One has to remove three massless $1/2$ spinors, two of positive and one of negative γ_5 -parity. The corresponding statement for bundles is

$$\Delta_{\pm}^{1/2}(M) \oplus T_c(M) = \Delta_{\pm}^{3/2}(M) \oplus \Delta_{\pm}^{1/2}(M) \oplus \Delta_{\mp}^{1/2}(M) \oplus \Delta_{\pm}^{1/2}(M) \quad (4.6)$$

For instance, $S = 1/2$ spinors can be eliminated from the Rarita-Schwinger field ψ_{μ} by subsidiary conditions like

$$\gamma^{\mu} \psi_{\mu} = 0 \quad ; \quad \tilde{\nabla}_{\mu} \psi^{\mu} = 0 \quad ; \dots$$

The anomalies of the spin $1/2$ and spin $3/2$ axial vector currents are now determined by the index densities of the elliptic operators $D_{1/2}$ and $D_{3/2}$.

Quite generally, if E and F are complex vector bundles over a compact n -dimensional manifold M , and if

$$D: E \rightarrow F \quad \text{elliptic}$$

one defines index $D \stackrel{\text{Def}}{=} \dim \text{kernel } D - \dim \text{kernel } D^*$, where $\dim \text{kernel } D$ is the (finite) dimension of the space of solutions to the homogeneous equation $Df = 0$, and analogously for the adjoint D^* . For instance, for the Dirac operator $D_{1/2}$, index $D_{1/2}$ is just the number of zero modes of $D_{1/2}$ of positive minus the number of zero modes of negative γ_5 parity.

The index theorem of Atiyah and Singer^{15,16)} gives a formula for index D in terms of topological quantities, which characterize the base manifold M and the bundles over M .

In the case that the operator D has a universal interpretation²⁸⁾ this formula has a particularly simple form. The Dirac operator $D_{1/2}$ always has a universal interpretation, because it can be constructed from the manifold M and its spin structure without further arbitrariness. For the Rarita-Schwinger operator, universal interpretability means that there must be a "natural" covariant way of imposing the subsidiary conditions, which eliminate the redundant degrees of freedom. Assuming this universality the index theorem can be written as

$$\text{index } D = (-1)^{\frac{n(n+1)}{2}} \frac{(\text{ch } E - \text{ch } F) \text{td } T_c(M)}{e(TM)} [M] \quad (4.7)$$

The meaning of the symbols on the right-hand side of Eq. (4.7) is the following:

- a) $T_c(M)$ is the complexified tangent bundle of M .
- b) For any complex vector bundle ξ over M , $\text{ch } \xi$ and $\text{td } \xi$ are given by closed differential forms, non-homogeneous in their degree. In terms of the Chern classes $c_i(\xi)$ their parts of degree ≤ 4 are, for example,

$$\text{td } \xi = 1 + \frac{1}{2} C_1(\xi) + \frac{1}{12} (C_2(\xi) + C_1^2(\xi)) + \dots \quad (4.8a)$$

$$\text{ch } \xi = \dim \xi + C_1(\xi) + \frac{1}{2} (C_1^2(\xi) - 2C_2(\xi)) + \dots \quad (4.8b)$$

and $c_i(\xi)$ are closed differential forms on M of even degree $2i$, which are expressible²⁹⁾ by the curvature form of any connection on ξ . For instance, if $\dim \xi = N$, and one introduces on ξ a $U(N)$ connection A_μ with curvature $F_{\mu\nu}$, then

$$C_1(\xi) = \frac{i}{4\pi} \text{tr} F_{\mu\nu} dx^\mu \wedge dx^\nu \quad (4.8c)$$

$$C_2(\xi) = \frac{1}{32\pi^2} (\text{tr}(F_{\mu\nu} F_{\rho\sigma}) - \text{tr} F_{\mu\nu} \text{tr} F_{\rho\sigma}) dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \quad (4.8d)$$

and it turns out that replacing A_μ by another connection changes c_i only by an exact differential form.

c) $e(TM)$ is the Euler class of the (non-complexified) tangent bundle TM , a closed n -form on M , expressible by the Pfaffian²⁹⁾ of the curvature tensor of M . Integrated over M it gives the Euler characteristic $\chi(M)$ of the manifold M . For $\dim M = 2$,

$$e(TM) = -\frac{1}{8\pi} \varepsilon_{ij} R_{ij} dx^k \wedge dx^l$$

and one obtains the theorem of Gauss and Bonnet. For higher dimension, generalizations of the Gauss-Bonnet theorem emerge.

d) The products in Eq. (4.7) are exterior products of differential forms, commutative because only forms of even degree are involved, and the formal division by $e(TM)$ can really be performed.

e) To compute index D , we now have to isolate the part of degree n on the r.h.s. of Eq. (4.7) and to integrate that differential n -form over the compact manifold M .

A very remarkable feature, evident from Eq. (4.7), is that index D does not depend on the special form of the operator D but only on the bundles E and F and, of course, on the fact that there exists a universally interpretable elliptic operator $D: E \rightarrow F$.

To determine the Rarita-Schwinger anomaly, we have to calculate the right-hand side of Eq. (4.7) for $E = \Delta_+^{3/2}$ and $F = \Delta_-^{3/2}$. Rather than performing this computation from the very beginning, we reduce it to the well-known result for spin $\frac{1}{2}$:

$$\text{index } D_{1/2} = \frac{(\text{ch } \Delta_+^{1/2} - \text{ch } \Delta_-^{1/2}) \text{td } T(M)}{e(TM)} [M] = \hat{A} [M] \quad (4.9)$$

Here $\hat{A}$ is the Hirzebruch's $\hat{A}$ -genus³⁰⁾, whose terms up to order 4 are

$$\hat{A} = 1 - \frac{1}{24} p_1 + \dots, \quad (4.10)$$

where the Pontryagin classes p_i of M are closed $4i$ -forms, defined by

$$p_i = (-1)^i c_{2i}(T_c(M)) \quad (4.11)$$

For instance, in terms of the curvature tensor R_{ijkl}

$$p_1 = \frac{1}{32\pi^2} R^i{}_{jkl} R^j{}_{irs} dx^k \wedge dx^l \wedge dx^r \wedge dx^s$$

Now the Chern character is additive and multiplicative:

$$ch(\xi \oplus \eta) = ch \xi + ch \eta \quad (4.12a)$$

$$ch(\xi \otimes \eta) = ch \xi \, ch \eta \quad (4.12b)$$

Thus, applying ch on Eq. (4.6) leads to

$$ch \Delta_+^{3/2} - ch \Delta_-^{3/2} = (ch(T_c(M)) - 1) (ch \Delta_+^{1/2} - ch \Delta_-^{1/2}) \quad (4.13)$$

Furthermore, by Eqs. (4.8b) and (4.11)

$$ch(T_c(M)) - 1 = 3 + p_1 \quad (4.14)$$

and finally

$$\begin{aligned} \text{index } \mathcal{D}_{3/2} &:= \hat{A}_{3/2}[M] = (3 + p_1 + \dots) (1 - \frac{1}{24} p_1 + \dots) [M] \\ &= (3 + \frac{21}{24} p_1) [M] = -21 \hat{A}[M] \end{aligned} \quad (4.15)$$

So, the anomaly for the Rarita-Schwinger field as obtained from this topological method is -21 times larger than the corresponding spin $\frac{1}{2}$ anomaly.

In the same spirit we can also calculate the γ_5 anomaly for higher spinor fields, assuming that there is a consistent theory with a universally interpretable kinetic operator. Details will be presented in a forthcoming communication.

Another, possibly important, generalization is the introduction of an additional gauge degree of freedom. Higher supergravity theories like $O(2)$ supergravity³¹⁾ are plausible candidates for consistent theories with higher spin and an additional gauge field. Under the same assumptions as before, which are perhaps not implementable without change, we can easily evaluate the γ_5 anomaly also for these cases.

If the spinor fields transform under the "colour" gauge group G like a representation ρ of G on a complex vector space V , the bundles $\Delta_{\pm}^{s/2}$ have to be replaced by bundles $\Delta_{\pm}^{s/2} \otimes \tilde{V}$ where $\tilde{V}$ is a vector bundle over M obtained by attaching the representation space V to every space-time point in M .

Instead of $\mathcal{D}_{s/2}$ we have elliptic operators $\mathcal{D}_{s/2}^V$, which also involve the gauge potential of the G -gauge field on M .

Because of the multiplicative property of the Chern character [Eq. (4.12b)], the index density simply has to be multiplied by

$$\text{ch } \tilde{V} = \dim V + c_1(\tilde{V}) + \frac{1}{2} (c_1^2(\tilde{V}) - 2c_2(\tilde{V})) + \dots \quad (4.16)$$

From the structure of the index formula it is evident that for $\dim M = 4$, the index and hence the anomaly will get an additive contribution, which is due to the gauge degree of freedom. For higher dimensions of M also mixed contributions emerge.

In our cases we get

$$\begin{aligned} \text{index } D_{S/2}^V &= (\dim V + c_1(\tilde{V}) + \frac{1}{2} (c_1^2(\tilde{V}) - 2c_2(\tilde{V})) + \dots) \hat{A}_{S/2}[M] \\ &= \dim V \cdot \hat{A}_{S/2}[M] + \frac{1}{2} a_{S/2} \cdot (c_1^2(\tilde{V}) - 2c_2(\tilde{V})) [M] \end{aligned} \quad (4.17)$$

with

$$a_{1/2} = 1, \quad a_{3/2} = 3 \quad (4.18)$$

as can be seen from Eqs. (4.10), (4.13), and (4.15).

For $S = \frac{1}{2}$, the right-hand side of Eq. (4.17) is known in the mathematical literature as the generalized $\hat{A}$ -genus³⁰⁾.

So, first of all the anomaly is multiplied by $\dim V$ as expected, and secondly it acquires an additional gauge field contribution which, for $S = \frac{1}{2}$ and $G = U(1)$ or $SU(2)$, is just the Adler anomaly⁷⁾ or Bardeen's³²⁾ non-Abelian Yang-Mills anomaly.

Also the coefficient of this part of the anomaly depends on the spin. We can summarize the results obtained under the above-stated assumptions in the following table, where we give the coefficients for the gravity and Yang-Mills part of the axial anomaly for $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$, normalizing to $S = \frac{1}{2}$.

	Gravitation part	Yang-Mills part
$S = \frac{1}{2}$	1	1
$S = \frac{3}{2}$	-21	2

5. CONCLUSIONS

We have shown that various approaches lead to the same value of the axial anomaly due to a spin $\frac{3}{2}$ loop in an external classical gravitational background field, provided a gravitationally covariant gauge-fixing term is chosen. The

actual value for a real spin $\frac{3}{2}$ field is -21 times the value of the real spin $\frac{1}{2}$ axial anomaly. Within the context of supergravity, the gravitational field cannot be treated as a background field, but is treated on an equal footing with the quantized spin $\frac{3}{2}$ field. The completely covariant gauge-fixing term $\psi \cdot \gamma \not{D} \gamma \cdot \psi$ seems to violate unitarity, while we have found upon explicit evaluation that various non-covariant gauge-fixing terms lead to values different from -21. This non-uniqueness we have traced back to a difference between the axial current in quantum electrodynamics and in supergravity: in supergravity the axial current is not invariant under the symmetries of the theory (in particular under local supersymmetry) while in QED it is. A second difference between supergravity and QED is that the supersymmetry gauge-fixing term is not gravitationally invariant. Whether or not this invalidates in supergravity the Rosenberg-Adler method which imposes gravitational conservation of the matrix element, deserves further study by means of Slavnov-Taylor identities; in any case, no problems arise for the covariant gauge-fixing terms.

Comparing the various methods, one can say that the Feynman graph method is the most flexible since covariant and non-covariant gauge-fixing terms can be treated in the same way, but it is algebraically laborious. At the other extreme the topological method is the fastest way to the anomaly in the covariant case. It exhibits most clearly the relation between the axial anomaly and the global topology of the manifold and the bundles, by expressing the anomaly in terms of the zero modes of our elliptic operator, which is essentially only dependent on the global topological framework. On the other hand, since the method is based on universal features only, it allows less insight into the detailed mechanism of the decoupling of the redundant degrees of freedom of the spin $\frac{3}{2}$ field axial anomaly under conditions stated in Section 4.

The zeta-function method can be viewed in Euclidean space-time as an analytic counterpart of the topological method. The point-splitting method as used above can be considered a variant of the zeta-function method which emphasizes the short-distance behaviour of the theory. Both methods are intermediate between the Feynman graph method and the topological method and can be used perturbatively in the background metric to make contact with the Feynman graph method.

Finally, we ought to emphasize that in this work we have treated the *massless* spin $\frac{3}{2}$ case. Massive spin $\frac{3}{2}$ particles can also be described by constrained vector-spinors. However, there is now no fermionic gauge invariance and the subsidiary conditions which remove the redundant components are not to be interpreted as gauge fixing terms. Nevertheless one can treat the massive case quite analogously. The result for the anomaly is -20. This can be seen for instance in the topological method by observing that the subsidiary conditions $\gamma_\mu \psi^\mu = 0$ and

$D_\mu \psi^\mu = 0$ correspond to the elimination of two spinors of opposite chirality. The only necessary modification of the computation is dropping the last summand on the right-hand side of Eq. (4.6). Effectively, one can therefore take the vector-spinor without any ghosts because the contribution of the redundant degrees of freedom cancel for the axial anomaly.

NOTE ADDED

While this paper was being prepared for publication we received preprints by Christensen and Duff and by Critchley who also calculated the axial anomaly.

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Figure caption

Fig. 1 : One-loop diagrams which contribute to the matrix element of the axial current.

Fig. 1