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THRESHOLD EFFECTS IN GAUGE THEORIES

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A B S T R A C T

The β functions are calculated for values of the momenta of the external legs in the region of the masses of the particles in the internal loops. The effects of these β functions in an $SU(2) \times U(1)$ model of weak and electromagnetic interactions and in a grand unified model of strong, weak, and electromagnetic interactions are discussed. The most significant effect is the reduction of the mass of the superheavy gauge bosons in unified theories by a factor of 6, which decreases the estimated proton lifetime by three orders of magnitude.

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1. - INTRODUCTION

In any quantum field theory, the effective couplings between the particles in the theory depend on the momenta of the external legs, due to higher order corrections to the simple, constant, lowest order couplings. If all the external legs are taken at the same Euclidean momentum $q^2 = -Q^2$, the effective couplings, $\bar{g}$, are governed by the equation

$$\frac{\partial \bar{g}(Q^2)}{\partial \ln Q^2} = \beta(\bar{g}(Q^2)) \quad (1.1)$$

The function β can be calculated in perturbation theory for values of the effective couplings which are not too large. If the sign of the first term in this perturbation expansion is negative the effective coupling constant decreases as Q^2 increases and the theory is said to be asymptotically free ¹⁾. If $\sqrt{Q^2}$ is very much larger than any of the masses in the theory then the effective coupling can only depend on Q^2 through powers of $\ln(Q^2/\mu^2)$, where μ is the renormalization point and so the perturbation series for $\bar{g}$ begins

$$\bar{g}(Q^2) = g_\mu + \beta_0 g_\mu^3 \ln\left(\frac{Q^2}{\mu^2}\right) + \dots \quad (1.2)$$

Since $\bar{g}$ must be independent of the subtraction point (it corresponds to a physical measurable quantity) the Q^2 dependence of $\bar{g}$ may be obtained from the μ^2 dependence of the renormalized coupling constant g_μ

$$\frac{\partial g_\mu}{\partial \ln \mu^2} = \beta(g_\mu) \quad (1.3)$$

On the other hand if we have a one-loop correction to the effective coupling and these are particles of mass M within that loop such that $M^2 \gg Q^2$, we know that we can make a Taylor expansion about $Q^2 = 0$ and we get a correction of the form

$$g_\mu^3 \frac{Q^2}{M^2}$$

so that for sufficiently small Q^2/M^2 we may neglect the effect of this loop on the β function.

An approximation which ²⁾ is often made is to assume that a one-loop correction becomes "operative" if the momenta of the external legs would be sufficient, if they were Minkowskian, to excite the particles in the loop, i.e., if $Q^2 \geq (M_1 + M_2)^2$ where M_1 and M_2 are the largest masses of a pair of particles coupling to an external leg. An "operative" loop is included in the calculation of β_0 in Eq. (1.2) and an "inoperative" one is neglected. The Q^2 dependence of β_0 is then given by

$$\beta_0 = \Theta(Q^2 - (M_1^a + M_2^a)^2) \beta_0^a + \Theta(Q^2 - (M_1^b + M_2^b)^2) \beta_0^b + \dots \quad (1.4)$$

However, it is not clear that this is a good approximation since we know that one can feel the effect of a one-loop correction below the threshold and that since Eq. (1.2) is only valid for $\sqrt{Q^2}$ very much larger than the masses of the particles, the effect of a loop does not reach its full value until values of Q^2 which are considerably larger than the threshold. This effect may have significant consequences for the effective coupling, not only in the threshold region but also away from it since the change in effective coupling as one passes through the threshold is not correctly obtained from Eq. (1.4).

For an unbroken gauge theory with massive fermions, the effects of passing through the thresholds of the fermion masses has been calculated by Georgi and Politzer ³⁾. In this paper we consider spontaneously broken gauge theories which, although technically are no longer asymptotically free, behave like asymptotically free gauge theories up to extremely large energies. We have calculated the effects on the β function as one passes through the thresholds of the vector boson masses. Because of the algebraic complexity it has been necessary to use Schoonschip, a computer program which performs algebraic manipulations which was written by M. Veltman in 1967 ⁴⁾.

Away from the threshold region the definition of β is unambiguous (up to possible renormalization prescription dependence which begins only in third order) since the only terms which contain $\ln(Q^2/\mu^2)$ are terms which have been renormalized and are therefore proportional to the bare couplings. In the threshold region, however, other terms may have a Q^2 dependence leading to a certain arbitrariness in the definition of β . We therefore begin by giving a general definition of β . We consider a

fermion of momentum q_μ and polarization ϵ_μ , such that $\partial \cdot \epsilon = 0$, and $p^2 = q^2 = (p+q)^2 = -Q^2$. Squaring the amplitude and summing over the spins of the incoming and outgoing fermions we have

$$\sum_{\text{fermion spins}} |M|^2 = \frac{\epsilon_\mu \epsilon_\nu}{8} \left\{ (g_V^2 + g_A^2) \left[\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) Q^2 F_1(Q^2) - F_2(Q^2) \left(p_\mu - \frac{p \cdot q}{q^2} q_\mu \right) \left(p_\nu - \frac{p \cdot q}{q^2} q_\nu \right) \right] + i g_V g_A \epsilon_{\mu\nu\rho\sigma} p^\rho q^\sigma F_3(Q^2) \right\} \quad (1.5)$$

The term $i \epsilon_{\mu\nu\rho\sigma} p^\rho q^\sigma F_3(Q^2)$ clearly vanishes when multiplied by $\epsilon_\mu \epsilon_\nu$, but it is in principle present since the coupling of the gauge bosons to fermions may in general be a mixture of vector and axial vector couplings with (renormalized) coupling constants g_V and g_A , respectively. In the tree diagram approximation we have

$$F_1(Q^2) = -\frac{1}{4} \quad (1.6a)$$

$$F_2(Q^2) = 1 \quad (1.6b)$$

If we make the definitions

$$H_1(Q^2) = (F_2(Q^2) - 4 F_1(Q^2)) / 2 \quad (1.7a)$$

$$H_2(Q^2) = (F_2(Q^2) + 4 F_1(Q^2)) / 2 \quad (1.7b)$$

then in the tree diagram $H_1(Q^2) = 1$ and $H_2(Q^2) = 0$. Since the β function outside the threshold region is derived from the part of the coupling which is proportional to the tree diagram coupling, a reasonable general definition of β is

$$\beta(Q^2) = Q^2 \frac{\partial}{\partial Q^2} H_1(Q^2) \quad (1.8)$$

$H_1(Q^2)$ can be projected out by replacing $\epsilon_\mu \epsilon_\nu$ by the projection operator

$$P_{\mu\nu} = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) + \frac{2}{Q^2} \left(p_\mu - \frac{p \cdot q}{q^2} q_\mu \right) \left(p_\nu - \frac{p \cdot q}{q^2} q_\nu \right) \quad (1.9)$$

Note that had we chosen a different set of values for p^2 , q^2 , and $(p+q)^2$ (albeit deep Euclidean for all three) we would obtain a different value for β in the threshold region. Similarly β would be different if we had considered higher order corrections to the triple gauge boson vertex rather than the fermion-antifermion gauge boson vertex.

The plan of this paper is as follows. In Section 2 we consider the Weinberg-Salam theory of weak and electromagnetic interactions ⁵⁾. We present the results of calculations of β for the coupling of the charged intermediate vector boson, the photon, and the neutral vector boson, in the region of the threshold of the intermediate vector boson masses. As a specific example we consider neutral current events in high energy e-p scattering with momentum transfers of the order of magnitude of the intermediate vector boson mass. We insert the effective couplings and compare them with the tree diagram results in order to display the significance of higher order effects. In Section 3 we consider these threshold effects in the grand unified SU(5) model of strong, weak, and electromagnetic interactions of Georgi and Glashow ⁶⁾. We find that the masses of the super-heavy gauge bosons which split SU(5) down to SU(3)×SU(2)×U(1) are particularly sensitive to threshold effects in the β function. In Section 4 we present our conclusions.

2. - BETA FUNCTIONS IN THE WEINBERG-SALAM MODEL

In the Weinberg-Salam model of weak and electromagnetic interactions ⁴⁾ the gauge group is SU(2)×U(1), which is spontaneously broken by means of a complex doublet of Higgs scalars to U(1), giving rise to a massless photon, A_μ , a massive charged intermediate vector boson, W_μ of mass M_W , and a massive neutral vector boson, Z_μ , of mass $M_Z = M_W \sec \theta$, where θ is the Weinberg angle ⁷⁾. We consider the model with three left-handed lepton doublets (ν_e, e) , (ν_μ, μ) , and a heavy lepton doublet (ν_τ, τ) , and

six quark flavours, each in three colours, arranged in left-handed doublets $(u, \tilde{d})_{i,j,k}$, $(c, \tilde{s})_{i,j,k}$, and $(t, b)_{i,j,k}$, where $\tilde{d}$ and $\tilde{s}$ are orthogonal Cabibbo combinations of d and s quarks. All massive fermions have singlet parts.

In this section we consider the β functions for the following couplings :

- 1) the coupling of the charged vector boson, W_μ ,
- 2) the electromagnetic coupling,
- 3) the coupling of Z_μ to left-handed electrons,
- 4) the coupling of Z_μ to right-handed electrons.

The Feynman diagrams which must be calculated for these β functions are shown in Figs. 1-4, respectively. The derivation of the β functions from these diagrams has been explained in the previous section.

Part a) of Figs. 1-4 contain the diagrams which contribute to the one-particle irreducible vertex corrections. There are two types of diagrams. One which involve only the fermion-antifermion gauge boson vertices, whose contributions we call $V_1(Q^2, M^2)$, where M is the mass of the gauge boson (throughout these calculations we neglect the masses of the fermions). The other involves the gauge boson self-interactions and we call these contributions $V_2(Q^2, M_1^2, M_2^2)$, where M_1 and M_2 are the masses of the two gauge bosons in the internal loop (which are not necessarily equal). In Fig. 4 this second type of diagram is absent because the right-handed electron is a singlet of the non-Abelian part of the gauge group. This, as we will see later, makes a considerable difference to the threshold behaviour of the β function.

Part b) of Figs. 1-4 contains the diagrams which contribute to self-energies. There is only one kind of diagram whose contribution is $S_F(Q^2, M^2)$. In terms of this function the self-energies are given by

$$\Sigma_e(Q^2) = \frac{g^2}{4} \sin^2 \theta \cos^2 \theta S_F(Q^2, M_Z^2) + g^2 \sin^2 \theta S_F(Q^2, 0) + \frac{g^2}{2} S_F(Q^2, M_W^2) \quad (2.1)$$

for the left-handed electron,

$$\Sigma_{e_R}(\vec{q}) = g^2 \sin^2 \theta \tan^2 \theta S_f(\vec{q}, M_Z^2) + g^2 \sin^2 \theta S_f(\vec{q}, 0) \quad (2.2)$$

for the right-handed electron,

$$\Sigma_\nu(\vec{q}) = \frac{g^2}{4} \sec^2 \theta S_f(\vec{q}, M_Z^2) + \frac{g^2}{2} S_f(\vec{q}, M_W^2) \quad (2.3)$$

for the neutrino.

Part c) of Figs. 1-4 contains the contributions from gauge boson self-energies. There are four types of diagrams.

1) Diagrams containing gauge boson loops together with diagrams where the gauge boson loop is replaced by a loop of Faddeev-Popov ghost particles. The contributions from these diagrams are called $P_1(Q^2, M_1^2, M_2^2)$.

2) Diagrams containing a loop of Goldstone bosons. We work in the Feynman gauge in which the Goldstone bosons have the same mass as the gauge boson to which they correspond. The contributions from these diagrams are called $P_2(Q^2, M_1^2, M_2^2)$. We have neglected all diagrams which contain physical Higgs particles (this is equivalent to assuming that the physical Higgs particle has a mass which is much larger than the masses of the gauge bosons). In fact this makes very little difference to the β function since the contribution from these diagrams is very small. Actually the presence of these Higgs particles spoils the asymptotic freedom of the theory because the β function for the self-coupling of the Higgs is positive. However, this only affects the β functions for the gauge bosons in two loops, and assuming that at the threshold for production of physical Higgs particles the Higgs self-coupling is of the same order of magnitude as the gauge coupling constant, g , the deviations from asymptotic freedom will only begin to be felt at values of Q^2 given by

$$\ln\left(\frac{Q^2}{M_H^2}\right) \sim \frac{16\pi^2}{g^2} \quad (2.4)$$

where M_H is the Higgs mass (for $M_H \sim M_W$ this gives $Q^2 \sim 10^{190} \text{ GeV}^2$!).

3) Diagrams containing loops of one Goldstone boson and one gauge boson.

The contributions from these diagrams are called $P_3(Q^2, M_1^2, M_2^2)$. Asymptotically these diagrams do not contribute to the β functions but they give a non-zero contribution in the threshold region.

4) Diagrams containing fermion loops. Naively one might think that such diagrams contribute to the β functions even for small values of Q^2 since we are still way above the fermion mass thresholds so that the contributions to the β functions are suppressed by a propagator factor $Q^2/(Q^2+M^2)$ for a gauge boson of mass M . In fact all the previous contributions are subtracted at $Q^2 = -M^2$ (this is a mass renormalization) so that they are all proportional to (Q^2+M^2) ; e.g., $P_1(Q^2, M_1^2, M_2^2)$ is actually given by

$$P_1(Q^2, M_1^2, M_2^2) = P_1^0(Q^2, M_1^2, M_2^2) - P_1^0(-M^2, M_1^2, M_2^2) \quad (2.5)$$

where P_1^0 is the unsubtracted contribution. The functions are still infinite but when the propagator factors are correctly taken into account the infinity turns out to be Q^2 independent and does not, therefore, affect the calculation of β . In the case of fermion loops we get contributions of the form $AQ^2 \ln Q^2$. Performing the subtraction at $Q^2 = -M^2$ this becomes

$$A \left[(Q^2 + M^2) \ln Q^2 - M^2 \ln \left(\frac{Q^2}{M^2} \right) \right] \quad (2.6)$$

The contribution to β from such terms is

$$-Q^2 \frac{\partial}{\partial Q^2} A \left[\ln Q^2 - \frac{M^2}{(Q^2 + M^2)} \ln \left(\frac{Q^2}{M^2} \right) \right] = -A \frac{Q^2}{(Q^2 + M^2)} \left[\frac{M^2}{Q^2 + M^2} \ln \left(\frac{Q^2}{M^2} \right) + 1 \right] \quad (2.7)$$

We see that this is negligible for $Q^2 \ll M^2$. Only the electromagnetic coupling has a non-negligible β for small values of Q^2 . In terms of P_1 , P_2 and P_3 we have for the self-energies of the gauge bosons

$$\begin{aligned} \Pi_w(Q^2) = & g^2 \cos^2 \theta P_1(Q^2, M_w^2, M_z^2) + g^2 \sin^2 \theta P_1(Q^2, M_w^2, 0) + \frac{g^2}{4} P_2(Q^2, M_w^2, M_z^2) \\ & + g^2 \sin^2 \theta \tan^2 \theta M_w^2 P_3(Q^2, M_w^2, M_z^2) + g^2 \sin^2 \theta P_3(Q^2, M_w^2, 0) \\ & - \frac{g^2}{4\pi^2} \left[(Q^2 + M_w^2) \ln Q^2 - M_w^2 \ln \left(\frac{Q^2}{M_w^2} \right) \right] \end{aligned} \quad (2.8)$$

for W_μ ,

$$\Pi_A(Q^2) = g^2 \sin^2 \theta \left[P_1(Q^2, M_W^2, M_W^2) + P_2(Q^2, M_W^2, M_W^2) + 2 M_W^2 P_3(Q^2, M_W^2, M_W^2) - \frac{2}{3\pi^2} Q^2 \ln Q^2 \right] \quad (2.9)$$

for A_μ , and

$$\begin{aligned} \Pi_Z(Q^2) = & g^2 \cos^2 \theta P_1(Q^2, M_W^2, M_W^2) + \frac{g^2}{2} \sec^2 \theta \cos 2\theta P_2(Q^2, M_W^2, M_W^2) \\ & + 2 g^2 \sin^2 \theta \tan^2 \theta M_W^2 P_3(Q^2, M_W^2, M_W^2) \\ & - \frac{g^2}{12\pi^2} [5 \sec^2 \theta - 10 + 8 \cos^2 \theta] \left[(Q^2 + M_Z^2) \ln Q^2 - M_Z^2 \ln \left(\frac{Q^2}{M_Z^2} \right) \right] \end{aligned} \quad (2.10)$$

for Z_μ .

Figures 3 and 4 contain a part d) which comes from the mixing of A_μ and Z_μ in higher order. We have arbitrarily included these diagrams in the renormalization of the coupling to Z_μ and not the electromagnetic coupling. This arbitrariness is justified by the fact that one cannot measure the coupling to Z_μ and A_μ separately. For the scattering of two particles of charge e and e' , respectively, left-handed couplings to Z_μ , g_L and g_L' , respectively, and right-handed Z_μ couplings g_R and g_R' we can only measure the combinations

$$g_L g_L' \frac{Q^2}{(Q^2 + M_Z^2)} + ee' \quad \text{and} \quad g_R g_R' \frac{Q^2}{(Q^2 + M_Z^2)} + ee'$$

In fact for very large values of Q^2 ($Q^2 \gg M_Z^2$) we can no longer talk about Z_μ and A_μ but about W_μ^3 and B_μ which are the gauge bosons corresponding to the third component of the unbroken $SU(2)$ group and the $U(1)$ group, respectively. It is tempting to subtract the Z-A mixing diagrams at $Q^2 = 0$ and $Q^2 = -M_Z^2$ so that when the renormalized propagators are diagonalized, this mixing does not affect the residues of the poles in the photon or Z propagators. Unfortunately this is not possible. The counter terms associated with this mixing are ⁸⁾

$$Z_{AZ} Q^2 + 2 M_Z^2 \delta\theta \quad (2.11)$$

where $\delta\theta$ is the renormalization of the Weinberg angle and Z_{AZ} is related to the wave function renormalization of W_μ and A_μ by

$$Z_{AZ} = \sec^2\theta (Z_A - Z_W) \quad (2.12)$$

This relation is derived in Ref. 8) from the fact that since the original gauge group is $SU(2) \times U(1)$ these can only be two independent wave function renormalization constants (Z_W and Z_B) and so for the broken theory Z_W , Z_Z , Z_A and Z_{AZ} can always be expressed in terms of these. $\delta\theta$ is also fixed from the renormalizations of the W and Z masses

$$\frac{\delta M_Z}{M_Z} - \frac{\delta M_W}{M_W} = \tan\theta \delta\theta \quad (2.13)$$

so the subtractions for the A - Z mixing diagrams are both fixed and the renormalized mixing does not have a zero at $Q^2=0$ or $Q^2=-M_Z^2$. For this reason the β functions for the Z_μ couplings have a small part $\propto 1/Q^2$ and thus never reach an asymptotic value below threshold. However, as explained above, in any physical process this is really a contribution to the electromagnetic coupling which is expected to have a $1/Q^2$ behaviour from the photon propagator. In terms of the unsubtracted functions P_1^0 , P_2^0 , P_3^0 we have for the A - Z mixing

$$\begin{aligned} \Pi_{AZ}(Q^2) = & -g^2 \sin\theta \cos\theta P_1^0(Q^2, M_W^2, M_W^2) - \frac{1}{2} g^2 \tan^2\theta \cos 2\theta P_2^0(Q^2, M_W^2, M_W^2) \\ & + 2g^2 M_W^2 \sin^2\theta \tan\theta P_3^0(Q^2, M_W^2, M_W^2) - \frac{g^2}{16\pi^2} \sin\theta \cos\theta M_W^2 \ln M_W^2 \\ & - 2 M_Z^2 \delta\theta - \frac{g^2}{12\pi^2} (5 \tan\theta - 8 \sin\theta \cos\theta) Q^2 \ln Q^2 \end{aligned} \quad (2.14)$$

In terms of $V_1, V_2, \Sigma_{eL}, \Sigma_{eR}, \Sigma_\nu, \Pi_W, \Pi_A, \Pi_Z, \Pi_{AZ}$ we have the following expressions for the four β functions

$$\begin{aligned} \beta_W = & \frac{g}{2} Q^2 \frac{\partial}{\partial Q^2} \left\{ -\frac{g^2}{4} \sec^2 \theta \cos 2\theta V_1(Q^2, M_Z^2) + g^2 \sin^2 \theta V_1(Q^2, 0) \right. \\ & + \frac{g^2}{2} \cos 2\theta V_2(Q^2, M_W^2, M_Z^2) + \frac{g^2}{2} V_2(Q^2, M_Z^2, M_W^2) \\ & \left. - \frac{1}{2} \Sigma_{eL} - \frac{1}{2} \Sigma_\nu - \frac{1}{2(Q^2 + M_W^2)} \Pi_W(Q^2) \right\} \end{aligned} \quad (2.15)$$

$$\begin{aligned} \beta_A = & g \sin \theta Q^2 \frac{\partial}{\partial Q^2} \left\{ g^2 \sin^2 \theta V_1(Q^2, 0) + \frac{g^2}{4} \sec^2 \theta (\cos 2\theta + 4 \sin^4 \theta) \right. \\ & \left. V_1(Q^2, M_Z^2) + \frac{1}{2} V_2(Q^2, M_W^2, M_Z^2) - \frac{1}{2} \Sigma_{eL} - \frac{1}{2} \Sigma_{eR} - \frac{1}{2Q^2} \Pi_A(Q^2) \right\} \end{aligned} \quad (2.16)$$

$$\begin{aligned} \beta_{Z_L} = & \frac{g}{2} \sec \theta \cos 2\theta Q^2 \frac{\partial}{\partial Q^2} \left\{ g^2 \sin^2 \theta V_1(Q^2, 0) + \frac{g^2}{4} \sec^2 \theta \cos 2\theta V_1(Q^2, M_Z^2) \right. \\ & - \sec 2\theta V_1(Q^2, M_W^2) + g^2 \cos^2 \theta \sec 2\theta V_2(Q^2, M_W^2, M_Z^2) - \Sigma_{eL} \\ & \left. - \frac{1}{2(Q^2 + M_Z^2)} \Pi_Z(Q^2) - \frac{2 \tan \theta \sec 2\theta}{Q^2} \Pi_{AZ}(Q^2) \right\} \end{aligned} \quad (2.17)$$

$$\begin{aligned} \beta_{Z_R} = & g \tan \theta \sin \theta Q^2 \frac{\partial}{\partial Q^2} \left\{ g^2 \sin^2 \theta V_1(Q^2, 0) + \tan^2 \theta \sin^2 \theta V_1(Q^2, M_Z^2) \right. \\ & \left. - \Sigma_{eR} - \frac{1}{2(Q^2 + M_Z^2)} \Pi_Z(Q^2) + \frac{\cot \theta}{Q^2} \Pi_{AZ}(Q^2) \right\} \end{aligned} \quad (2.18)$$

The exact expressions for these β functions are too long and complicated to be given here, so in Figs. 5-8 we have plotted their numerical values against $\ln(Q^2/4M_W^2)$ in the threshold region, i.e., in the range

$$-3 \leq \ln\left(\frac{Q^2}{4M_W^2}\right) \leq 7$$

We have used the current experimental value for $\sin^2\theta$ of 0.24⁹⁾. We note that, except for β_{Z_R} which does not involve the function V_2 , the β functions rise fairly slowly above their threshold up to their asymptotic value, reaching 90% of this value at $\ln(Q^2/4M_W^2) \sim 4$ ^{*}). Below threshold, on the other hand, β_W and β_A reach their asymptotic values quite rapidly. It is also interesting to observe that β_W is positive in the threshold region whereas its asymptotic value is of course negative, SU(2) being an asymptotically free theory. All this shows us that the θ function approximation of Eq. (1.4) is not very good. To exemplify this we have plotted in Fig. 9 $\bar{\alpha}_W(Q^2) = \bar{g}_W^2(Q^2)/4\pi$ against $\ln(Q^2/4M_W^2)$ using the β function of Fig. 5, and the β function using the θ function approximation. We see that for $\ln(Q^2/4M_W^2) \gtrsim 6$ there is a discrepancy in $\bar{\alpha}_W(Q^2)$ of about 3%. This discrepancy will turn out to be important in the next section where we discuss a grand unified theory. The effective Weinberg angle defined by

$$\sin^2\theta(Q^2) = \frac{\bar{\alpha}(Q^2)}{\bar{\alpha}_W(Q^2)} \quad (2.19)$$

changes by 8% (in $\sin^2\theta$) over the range considered due to the difference between β_W and β_A . For the θ function approximation this change would have been 10% [the threshold effects delay the fall of $\bar{\alpha}_W(Q^2)$ but have a very small effect on the increase of $\bar{\alpha}(Q^2)$].

We close this section by displaying the effects of higher order weak and electromagnetic corrections in neutral current e-p scattering for values of Q^2 between 500 GeV² and 2500 GeV² which may soon become accessible to experiment¹⁰⁾. In Fig. 10 we have plotted $F_2(x, Q^2)$ for $x=0.4$. The strong interaction scaling violations have been included using a parametrization derived from QCD calculations¹¹⁾. We see that the effects of higher order corrections increase F_2 by an amount which may be measurable, although the difference between using the correct β function

*) In plotting these numerical values we have assumed that g remains constant over the region considered. This is of course not correct but because of the smallness of g the errors introduced by this approximation are negligible.

and the Θ function approximation is extremely small. In Fig. 11 we have divided out the strong interaction scaling violations to show the effect of higher order weak and electromagnetic corrections and the interference of photon and Z exchanges which is responsible for the rise of a factor of 2 throughout the range considered. In Fig. 12, we have plotted $F_3(x, Q^2)$ (which could be obtained from $\sigma_{e-p} - \sigma_{e+p}$) in order to display the parity violation of the Z interaction, which is predicted in the Weinberg-Salam theory. Once again we see that this is enhanced by the higher order corrections and in this case the difference between using the correct β function and the Θ function approximation is not so small. In these plots the higher order effects have been included by simply replacing the coupling constants by their effective values at each Q^2 . This is not a full analysis of the higher order corrections since the effective values of the coupling constants are calculated with all legs at equal Euclidean momenta, which is not the case, and diagrams involving the exchange of two gauge bosons have not been considered. Nevertheless we believe that these graphs give a good idea of what the higher order effects are (they would, for example, affect a calculation of M_Z from this experiment) and that they indicate that a more careful and fuller calculation of all the higher order effects merits consideration.

3. - APPLICATION TO GRAND UNIFIED THEORIES (GUTS)

In this section we consider the threshold effects on calculations in the $SU(5)$ model of Georgi and Glashow⁶⁾ which unites strong, weak and electromagnetic interactions. The $SU(5)$ gauge group is broken by means of an adjoint 24-plet representation of Higgs scalars to $SU(3) \times SU(2) \times U(1)$. This produces 12 superheavy charged gauge bosons $X_\mu^{i\pm}$ (charge $\frac{4}{3}$) and $Y_\mu^{i\pm}$ (charge $\frac{1}{3}$), where i runs over the three colours. These gauge bosons couple to colour changing currents and in particular to currents which violate boson and lepton number, e.g., we have interaction terms like

$$\mathcal{L}_I = \frac{g}{\sqrt{2}} \bar{d}_i \gamma^\mu \left(\frac{1+\gamma^5}{2} \right) e^+ X_\mu^{i-} + \dots \quad (3.1)$$

The group $SU(3) \times SU(2) \times U(1)$ is further broken by means of a complex 5-plet of Higgs scalars to $SU(3) \times U(1)$ giving the usual massive intermediate vector bosons of weak interactions. There is only one coupling constant in the Lagrangian. However, because of renormalization effects the effective couplings for the strong, weak and electromagnetic interactions, as well as the exotic interactions of X_μ and Y_μ , only become equal for values of Q^2 way above the mass of X_μ and Y_μ . For Q^2 in the threshold region and below, loops containing X_μ and Y_μ give diminished contributions to the β functions and since diagrams containing such loops occur with different weights in the renormalization of the four types of interaction mentioned above, the four couplings separate as one decreases Q^2 and this is responsible for the differences in the strong, weak and electromagnetic coupling constants seen at presently available energies. Starting from the known values of the couplings at low Q^2 , Buras et al. ¹²⁾ have calculated the masses of X_μ and Y_μ (which are almost equal) required to give these couplings. Taking the QCD renormalization subtraction point to be 300 MeV they have found for these masses

$$M_X = 3.7 \times 10^{16} \text{ GeV} \quad (3.2)$$

They have found furthermore that although the value of the Weinberg angle for $Q^2 \gg M_X^2$ is fixed by the group theory to be $\sin^2 \theta = 3/8$, renormalization effects bring this value down to 0.2 at low Q^2 .

The above-mentioned calculations were all performed using the Θ function approximation. Since it is actually $\ln M_X^2$ which is calculated this way we see that M_X is particularly sensitive to errors in calculations of the β functions, for both the threshold regions $Q^2 \sim 4M_X^2$, and $Q^2 \sim 4M_W^2$. We have repeated the calculations taking these threshold effects into account. For the strong, weak $SU(2)$, $U(1)$ and exotic interactions we have four effective coupling constants $\bar{g}_V(Q^2)$, $\bar{g}_W(Q^2)$, $\bar{g}_U(Q^2)$, $\bar{g}_X(Q^2)$, respectively. In general the β function for each of these effective couplings can depend on all of the other couplings as well as its own. We have calculated the higher order corrections to the following interactions

$$\frac{g_V}{\sqrt{2}} V_\mu^{12} \bar{d}' \gamma_\mu \left(\frac{1+\gamma_5}{2} \right) d^2$$

$$\frac{g_W}{2} W_\mu^0 \bar{e}^+ \gamma_\mu \left(\frac{1+\gamma_5}{2} \right) e^+$$

$$\frac{3}{5} g_u B_\mu \bar{e}^+ \gamma_\mu \frac{(1+\gamma^5)}{2} e^+$$

$$\frac{g_x}{\sqrt{2}} \chi'_\mu \bar{d}^+ \gamma_\mu \frac{(1+\gamma^5)}{2} e^+$$

The diagrams contributing to the renormalization of these four interactions are shown in Figs. 13-16, respectively. As in the previous section each figure has three sections of diagrams corresponding to vertex corrections, fermion self-energies, and gauge boson self-energies. We have treated the scalar particles by making the following assumptions :

- 1) the physical Higgs scalars from the 5-plet all have masses which are much smaller than M_x ;
- 2) the physical Higgs scalars from the 24-plet all have masses equal to M_x .

This is different from the treatment of scalars in Ref. 12), but since the scalar loops, treated in this way, have the same effect on all the β functions they do not change the value of M_x and have only a small effect on the values of the couplings at $Q^2 = 4M_x^2$. In terms of the functions $V_1(Q^2, M^2)$, $V_2(Q^2, M_1^2, M_2^2)$, $S_f(Q^2, M^2)$, $P_1(Q^2, M_1^2, M_2^2)$, $P_2(Q^2, M_1^2, M_2^2)$ and $P_3(Q^2, M_1^2, M_2^2)$ defined in the previous section we find it convenient to define the following six functions

$$A_1(Q^2) = -Q^2 \frac{\partial}{\partial Q^2} \left[\frac{1}{Q^2} P_4(Q^2, M_x^2, M_x^2) + S_f(Q^2, M_x^2) \right] \quad (3.3)$$

$$A_2(Q^2) = Q^2 \frac{\partial}{\partial Q^2} \left[V_1(Q^2, M_x^2) - S_f(Q^2, M_x^2) \right] \quad (3.4)$$

$$A_3(Q^2) = -\frac{5}{4} Q^2 \frac{\partial}{\partial Q^2} \left\{ \frac{1}{(Q^2 + M_x^2)} \left[P_1(Q^2, M_x^2, 0) + 2M_x^2 P_3(Q^2, M_x^2, 0) \right. \right. \\ \left. \left. + 2M_x^2 P_3(Q^2, M_x^2, M_x^2) + P_2(Q^2, M_x^2, M_x^2) \right] - S_f(Q^2, M_x^2) \right\} \quad (3.5)$$

$$A_4(Q^2) = Q^2 \frac{\partial}{\partial Q^2} V_2(Q^2, M_X^2, 0) \quad (3.6)$$

$$A_5(Q^2) = -\frac{5}{4} Q^2 \frac{\partial}{\partial Q^2} \frac{1}{Q^2} \left[P_2(Q^2, M_X^2, M_X^2) + 2 M_X^2 P_3(Q^2, M_X^2, M_X^2) \right. \\ \left. + 2 M_X^2 P_3(Q^2, M_X^2, 0) \right] \quad (3.7)$$

$$A_6(Q^2) = Q^2 \frac{\partial}{\partial Q^2} V_3(Q^2, M_X^2, M_X^2) \quad (3.8)$$

The full expressions for the functions A_1 to A_6 are given in the Appendix. We present here their asymptotic values :

$$A_1(Q^2) \xrightarrow{Q^2 \rightarrow \infty} -\frac{1}{24\pi^2} \quad (3.9)$$

$$A_2(Q^2) \xrightarrow{Q^2 \rightarrow \infty} 0 \quad (3.10)$$

$$A_3(Q^2) \xrightarrow{Q^2 \rightarrow \infty} -\frac{5}{32\pi^2} \quad (3.11)$$

$$A_4(Q^2) \xrightarrow{Q^2 \rightarrow \infty} -\frac{3}{16\pi^2} \quad (3.12)$$

$$A_5(Q^2) \xrightarrow{Q^2 \rightarrow \infty} \frac{5}{192\pi^2} \quad (3.13)$$

$$A_6(Q^2) \xrightarrow{Q^2 \rightarrow \infty} -\frac{3}{16\pi^2} \quad (3.14)$$

All of these functions vanish as $Q^2 \rightarrow 0$. In the six-flavour model we have the following expressions for β_{g_V} , β_{g_W} , β_{g_U} and β_{g_X} in terms of A_1 to A_6

$$\beta g_v = \frac{-41}{192\pi^2} g_v^3 + g_x^2 g_v A_1(Q^2) + g_x^2 g_v A_5(Q^2) + g_x^3 A_6(Q^2) \quad (3.15)$$

$$\beta g_w = \frac{-19}{192\pi^2} g_w^3 + \frac{3}{2} g_x^2 g_w A_1(Q^2) + g_x^2 g_w A_5(Q^2) + \frac{3}{2} g_x^3 A_6(Q^2) \quad (3.16)$$

$$\begin{aligned} \beta g_u = & \frac{25}{192\pi^2} g_u^3 + \frac{5}{2} g_x^2 g_u A_1(Q^2) - g_x^2 g_u A_2(Q^2) \\ & + g_x^2 g_u A_5(Q^2) + \frac{5}{2} g_x^3 A_6(Q^2) \end{aligned} \quad (3.17)$$

$$\begin{aligned} \beta g_x = & \frac{25}{192\pi^2} \frac{Q^2 g_x^3}{(Q^2 + M_x^2)} \left\{ 1 + \frac{M_x^2}{Q^2 + M_x^2} \ln\left(\frac{Q^2}{M_x^2}\right) \right\} + g_x^3 A_3(Q^2) + \frac{4}{3} g_x^2 g_v A_4(Q^2) \\ & + \frac{3}{4} g_x^2 g_w A_4(Q^2) + \frac{5}{12} g_x^2 g_u A_4(Q^2) + \frac{5}{384\pi^2} g_x g_u^2 \\ & + \frac{g_x g_v^2}{24\pi^2} + \frac{3}{128\pi^2} g_x g_w^2 \end{aligned} \quad (3.18)$$

These four equations constitute a set of coupled, first order differential equations which have to be solved numerically. In Fig. 17 we plot the solutions for $\bar{\alpha}_s(Q^2) = \bar{g}_s^2(Q^2)/4\pi$, $\bar{\alpha}_w(Q^2) = \bar{g}_w^2(Q^2)/4\pi$, and $\bar{\alpha}_u(Q^2) = \bar{g}_u^2(Q^2)/4\pi$ against $\ln(Q^2/4M_x^2)$ in the range $-5 \leq \ln(Q^2/4M_x^2) \leq 5$.

We see that, as expected, the couplings are not equal at $Q^2 = 4M_x^2$ but vary between 0.0229 and 0.0234, and only become indistinguishable for $\ln(Q^2/4M_x^2) \gtrsim 3$, where they have the value 0.022, which is the value for α_{GUM} given in Ref. 12). This has the effect of lowering the value of M_x required to produce the unification of couplings at this value. This effect is diminished by the fact that the β functions do not reach their below threshold asymptotic values immediately below the threshold. This decreases the rate of splitting between the couplings. If we look at the splitting at $\ln(Q^2/4M_x^2) = -5$ we find values which would correspond to $\ln(Q^2/4M_x^2) = -6.5$ in the θ function approximation. This has the effect of lowering M_x by a factor of 2 from the calculation of Ref. 12). When we take into account the threshold effects at $Q^2 \sim 4M_w^2$

we get a further reduction of a factor of 3 in M_x . This is because the $Q^2 \sim 4M_W^2$ threshold effects delay the fall in g_W so the difference between g_W and g_V in the region $4M_W^2 \ll Q^2 \ll 4M_x^2$ is actually smaller than one would expect from Θ function calculations. More precisely we find the value

$$M_x = 5.7 \times 10^{15} \text{ GeV} \quad (3.19)$$

The $Q^2 \sim 4M_W^2$ threshold effects increase $\sin^2\theta$ by 2% but the $Q^2 \sim 4M_x^2$ threshold effects decrease it again by almost the same amount so we find a value of $\sin^2\theta = 0.2$ in agreement with Ref. 12).

The interaction of X_μ and Y_μ with baryon number and lepton number violating currents gives rise to the decay of a proton into lepton and a meson. In Ref. 12) the lifetime has been estimated to be

$$\tau_p \sim (10^3 - 10^4) M_x^4 / m_p^5 \quad (3.20)$$

Using their values of M_x this gives a lifetime of $5 \times 10^{37} - 5 \times 10^{38}$ years which is way above the experimental lower limit of 10^{30} years. However, using our value of M_x the lifetime could be as low as 3×10^{34} years. Furthermore it has been shown that by taking a lower value for the QCD subtraction point, Λ , the value of M_x is reduced by a factor of 2, reducing the proton lifetime by an order of magnitude and bringing it dangerously close to the experimental limit.

A further consequence of threshold effects in grand unified theories is a correction to the fermion masses. The "effective mass" of the $I_3 = -\frac{1}{2}$ quarks (d,s,b) is equal to that of the leptons (e, μ , τ) for $Q^2 \gg M_x^2$. For lower values of Q^2 the "effective masses" split due to the different contributions to the mass renormalizations for leptons and quarks. The effective mass is related to the physical mass by the equation

$$\frac{m(Q^2)}{m_{phys}} = \exp - \sum_i \int_{\bar{g}(4m_{phys}^2)}^{\bar{g}(Q^2)} dg \frac{\gamma_m^i(g, Q^2)}{\bar{g}(g, Q^2)} \quad (3.21)$$

where the sum runs over all the types of interactions - strong, weak, and electromagnetic and for sufficiently large Q^2 the interaction of the superheavy gauge bosons - and γ_m^i are the contributions to the anomalous dimension of the mass operator from these different interactions. In Ref. 12) the ratio of the physical masses of the $I_3 = -\frac{1}{2}$ quarks to those of the charged leptons has been calculated using Eq. (3.21) with the Θ function approximation for β_i , neglecting the Q^2 dependence of the γ_m^i due to gauge boson mass thresholds and assuming that the interactions of the superheavy gauge bosons only enter after the fermion masses have unified. This last approximation gives rise to the largest correction since $\gamma_m(g, Q^2)$ for superheavy gauge bosons is given by

$$\gamma_m = -\frac{9}{64\pi^2} g^2 (2I_3 - q) \left[3 - \frac{2M_x^2}{Q^2} \left\{ 1 + \left(1 - \frac{M_x^2}{Q^2}\right) \ln\left(1 + \frac{Q^2}{M_x^2}\right) \right\} \right] \quad (3.22)$$

where q is the charge of the fermion. Although $\gamma_m \propto Q^2$ for $Q^2 \ll M_x^2$ it has the value $(-17.1/64\pi^2)q(2I_3 - q)$ at $Q^2 = 4M_x^2$ and as we can see from Fig. 17 the unification does not occur until $Q^2 \gg 4M_x^2$. This contribution reduces the ratio $M_q(I_3 = -\frac{1}{2})/M_\ell$ by 3%. There is a further reduction of 1.5% from the threshold behaviour of β_{g_v} and 0.8% from the threshold behaviour of β_{g_u} . Any correction to the fermion mass from the threshold behaviour of β_{g_w} or the weak interaction contribution to γ_m at $Q^2 \sim M_w^2$ would be the same for quarks and leptons (with a negligible difference arising from the different couplings of the Z). Therefore the total reduction of the ratio is just over 5%.

4. - CONCLUSION

In this paper we have shown that treating the threshold effects on the β functions in asymptotically free gauge theories with Θ functions is not a good approximation. The β functions of the couplings in the Weinberg-Salam model only reach 0.9 of their above threshold asymptotic values for $Q^2/4M_w^2 \sim 50$ (with the exception of the coupling of right-handed electrons to Z_μ). The below threshold damping set in somewhat quicker but 90% damping is still only achieved for $4M_w^2/Q^2 \sim 10$. Moreover the β function for g_w is positive at threshold and only above threshold does it turn round and become negative. The explanation for this is very simple. The positive

contributions from fermion loops to the β functions are given by Eq. (2.7). These reach 0.9 of their asymptotic value at $Q^2/4M_W^2 \sim 0.5$ whereas the negative contributions from the self interactions of the gauge bosons are still heavily damped by factors like $Q^2/(Q^2 + 4M_W^2)$ and $M_W^2/4Q^2 \ln((Q^2 + 4M_W^2)/4M_W^2)$, so that they only dominate for much larger values of Q^2 . In fact the very simplified formula

$$\beta_W = \frac{g^3}{16\pi^2} \left\{ \frac{2Q^2}{(Q^2 + M_W^2)} \left(1 + \frac{M_W^2}{(Q^2 + M_W^2)} \ln\left(\frac{Q^2}{M_W^2}\right) \right) - \frac{29}{8} \left[\frac{Q^2}{(Q^2 + 4M_W^2)} \left(1 - \frac{M_W^2}{Q^2} \ln\left(\frac{Q^2 + 4M_W^2}{4M_W^2}\right) \right) - \frac{M_W^4 Q^2}{4(Q^2 + M_W^2)^3} \right] \right\} \quad (4.1)$$

reproduces the curve of Fig. 5 with remarkable accuracy. The first term in Eq. (4.1) comes from the fermion loops and asymptotically gives the known result for six flavours ; each in three colours, and six leptons. For very small Q^2 this term becomes negative before vanishing, which accounts for the dip in Fig. 5 for $\ln(Q^2/4M_W^2) < -2.5$. The second term comes from the gauge boson interactions and asymptotically gives the known result for an SU(2) gauge theory with a scalar triplet that couples with half the usual strength (the Goldstone bosons).

In Fig. 10 we have shown that threshold effects can make a difference of 3% in $\bar{\alpha}_W(Q^2)$ above the threshold region. They also have a 2% effect on the renormalization of $\sin^2\theta$. For deep inelastic electron proton scattering with values of Q^2 up to 25000 GeV² we have shown that replacing the coupling constants by their effective value has a significant effect on the differential cross-sections so that we expect a full calculation of higher order weak and electromagnetic effects in this region to yield measurable corrections. On the other hand, because of the smallness of the weak and electromagnetic coupling constants the difference between using the effective couplings calculated exactly and those calculated in the θ function approximation has very little effect on these cross-sections.

In the grand unified SU(5) model of strong, weak, and electromagnetic interactions we find that the threshold effects causes the coupling constants to be split at $Q^2 = 4M_X^2$. They in fact only become equal at infinite Q^2 but the difference between them becomes negligibly small at $Q^2/4M_X^2 > 20$. This has the effect of reducing the value of M_X required to give unification by about a factor of 2. A larger effect comes from the weak interaction threshold which delays the fall of $\bar{\alpha}_w(Q^2)$ so that the difference between $\bar{\alpha}_w(Q^2)$ and $\bar{\alpha}_s(Q^2)$ is smaller than one would expect from calculating $\bar{\alpha}_w(Q^2)$ in the θ function approximation. This suppresses M_X by a factor of 3. Together the suppression of M_X by a factor of 6 has the effect of decreasing the calculated lifetime of the proton by three orders of magnitude, although it is still comfortably above the experimental limit. We have also calculated the effects of these thresholds on the fermion masses, and found that the ratio of the mass of the $I_3 = -\frac{1}{2}$ quarks to the mass of the charged leptons (which are equal in the grand unified limit) is decreased by 5%.

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APPENDIX

In this Appendix we present the expressions for the functions $A_1(Q^2) \dots A_6(Q^2)$, used in Section 3. We leave the expressions in terms of definite integrals over one Feynman parameter, α . Most of these integrals can be performed analytically but the results are very long and cumbersome.

$$A_1(Q^2) = \frac{-1}{48\pi^2} \left\{ \frac{2Q^2}{(Q^2+M_x^2)} + \frac{M_x^2}{Q^2} \int_0^1 d\alpha \left[3 \ln \left(\frac{Q^2(1-\alpha)+M_x^2}{M_x^2} \right) + 7 \ln \left(\frac{Q^2\alpha(1-\alpha)+M_x^2}{M_x^2} \right) \right] \right. \\ \left. - \frac{M_x^2}{(Q^2+M_x^2)} \int_0^1 d\alpha \left[\frac{7M_x^2\alpha(1-\alpha)+Q^5(5+2\alpha-\alpha^2)}{Q^2\alpha(1-\alpha)+M_x^2} + \frac{3M_x^2-6Q^2(1-2\alpha)}{Q^2(1-\alpha)+M_x^2} \right] \right\} \quad (A.1)$$

$$A_2(Q^2) = \frac{-1}{12\pi^2} \left\{ \frac{M_x^2}{(Q^2+M_x^2)} + \frac{M_x^2}{Q^2} \int_0^1 d\alpha \ln \left(\frac{Q^2(1-\alpha)+M_x^2}{M_x^2} \right) \right. \\ - \int_0^1 d\alpha \frac{M_x^2+Q^2(1-2\alpha)}{(Q^2+M_x^2)(Q^2\alpha(1-\alpha)+M_x^2)} + \frac{M_x^2}{Q^2} \int_0^1 d\alpha \frac{\ln \left(\frac{Q^2\alpha(1-\alpha)+M_x^2}{Q^2+M_x^2\alpha} \right)}{(1-\alpha+\alpha^2)} \\ \left. + \frac{M_x^2}{(Q^2+M_x^2)} \int_0^1 d\alpha \left[\frac{M_x^2+Q^2(1-2\alpha)}{(Q^2+M_x^2\alpha)} - \frac{M_x^2+Q^2(1-2\alpha)}{(Q^2(1-\alpha)+M_x^2)} \right] \right\} \quad (A.2)$$

$$A_3(Q^2) = \frac{-5}{64\pi^2} \left\{ \frac{2Q^4+12M_x^2Q^2+3M_x^4}{(Q^2+M_x^2)^2} + \frac{M_x^2}{Q^2} \int_0^1 d\alpha \ln \left(\frac{Q^2(1-\alpha)+M_x^2}{M_x^2} \right) \right. \\ - \frac{2M_x^2Q^2}{(Q^2+M_x^2)^2} \left[\int_0^1 d\alpha \ln \left(\frac{Q^2\alpha(1-\alpha)+M_x^2}{M_x^2(1-\alpha+\alpha^2)} \right) - \int_0^1 d\alpha (3+2\alpha) \ln \left(\frac{Q^2(1-\alpha)+M_x^2}{M_x^2} \right) \right] \\ + \int_0^1 d\alpha \frac{M^2(2Q^2\alpha^2+(M_x^2+3Q^2)\alpha-4M_x^2-7Q^2)}{Q^2(Q^2+M_x^2)(Q^2(1-\alpha)+M_x^2)} \\ \left. + \frac{M_x^2Q^2}{(Q^2+M_x^2)} \int_0^1 d\alpha \frac{2\alpha(1-\alpha)}{(Q^2\alpha(1-\alpha)+M_x^2)} \right\} \quad (A.3)$$

$$\begin{aligned}
 A_4(Q^2) = & \frac{-1}{96\pi^2} \left\{ \frac{18Q^2 + 2M_X^2}{(Q^2 + M_X^2)} - \frac{M_X^2}{Q^2} \ln\left(\frac{Q^2}{M_X^2}\right) + 3 \frac{M_X^2}{Q^2} \int_0^1 d\alpha \ln\left(\frac{Q^2(1-\alpha) + M_X^2}{M_X^2}\right) \right. \\
 & + \frac{M_X^2}{Q^4} (4M_X^2 - Q^2) \int_0^1 \frac{d\alpha}{(1-\alpha+\alpha^2)} \ln\left(\frac{Q^2\alpha(1-\alpha) + M_X^2\alpha}{Q^2 + M_X^2\alpha}\right) \\
 & - \int_0^1 d\alpha \frac{M_X^2 (3M_X^2 + 23Q^2 - 3M_X^2\alpha)}{(Q^2 + M_X^2)(Q^2(1-\alpha) + M_X^2)} - \int_0^1 d\alpha \frac{M_X^2 (2Q^4 - M_X^2Q^2 + 2M_X^4 - Q^4\alpha)}{(1-\alpha+\alpha^2)Q^2(Q^2 + M_X^2)(Q^2 + M_X^2\alpha)} \\
 & \left. + \int_0^1 d\alpha \frac{M_X^2 ((Q^4 - M_X^2Q^2 + 2M_X^4) - (2Q^4 - M_X^2Q^2 + 2M_X^2)\alpha)}{(1-\alpha+\alpha^2)Q^2(Q^2 + M_X^2)(Q^2(1-\alpha) + M_X^2)} \right\} \quad (A.4)
 \end{aligned}$$

$$\begin{aligned}
 A_5(Q^2) = & \frac{5}{142\pi^2} \left\{ \frac{Q^2}{(Q^2 + M_X^2)} + \frac{6M_X^2}{Q^2} \int_0^1 d\alpha \ln\left(\frac{Q^2(1-\alpha) + M_X^2}{M_X^2}\right) + \frac{2M_X^2}{Q^2} \int_0^1 d\alpha \right. \\
 & \ln\left(\frac{Q^2\alpha(1-\alpha) + M_X^2}{M_X^2}\right) + \int_0^1 d\alpha \frac{M_X^2 (Q^2(3\alpha^2 - 3\alpha - 1) + M_X^2\alpha(1-\alpha))}{(Q^2 + M_X^2)(Q^2\alpha(1-\alpha) + M_X^2)} \\
 & \left. + 6 \int_0^1 d\alpha \frac{M_X^2(\alpha - 1)}{(Q^2(1-\alpha) + M_X^2)} \right\} \quad (A.5)
 \end{aligned}$$

$$\begin{aligned}
 A_6(Q^2) = & \frac{-1}{48\pi^2} \left\{ \frac{9Q^2}{(Q^2 + M_X^2)} + \frac{M_X^2}{Q^2} \int_0^1 d\alpha \ln\left(\frac{Q^2(1-\alpha) + M_X^2}{M_X^2}\right) \right. \\
 & - \frac{M_X^2}{Q^2} \int_0^1 \frac{d\alpha}{(1-\alpha+\alpha^2)} \left[\ln\left(\frac{Q^2\alpha(1-\alpha) + M_X^2}{Q^2 + M_X^2}\right) + \frac{Q^2}{Q^2 + M_X^2} \right] \\
 & + \frac{M_X^2 Q^2}{(Q^2 + M_X^2)} \int_0^1 d\alpha \frac{11\alpha - 11\alpha^2 - 12}{Q^2\alpha(1-\alpha) + M_X^2} + \int_0^1 d\alpha \frac{M_X^2\alpha(1-\alpha)}{(1-\alpha+\alpha^2)(Q^2\alpha(1-\alpha) + M_X^2)} \\
 & \left. + \frac{M_X^2}{(Q^2 + M_X^2)} \int_0^1 d\alpha \frac{(3Q^2 + M_X^2)\alpha - M_X^2}{Q^2(1-\alpha) + M_X^2} \right\} \quad (A.6)
 \end{aligned}$$

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FIGURE CAPTIONS

- Figure 1 Diagrams contributing to the renormalization of g_w . In this and the next three figures wavy lines are gauge bosons, dotted lines are Faddeev-Popov ghosts, broken lines are Goldstone bosons and continuous straight lines are fermions.
- Figure 2 Diagrams contributing to the renormalization of the electromagnetic coupling.
- Figure 3 Diagrams contributing to the renormalization of g_{Z_L} .
- Figure 4 Diagrams contributing to the renormalization of g_{Z_R} .
- Figure 5 Graph of β_w against $\ln(Q^2/4M_w^2)$ in the threshold region.
- Figure 6 Graph of β_A against $\ln(Q^2/4M_w^2)$ in the threshold region.
- Figure 7 Graph of β_{Z_L} against $\ln(Q^2/4M_w^2)$ in the threshold region.
- Figure 8 Graph of β_{Z_R} against $\ln(Q^2/4M_w^2)$ in the threshold region.
- Figure 9 Graph of $\bar{\alpha}_w(Q^2)$ against $\ln(Q^2/4M_w^2)$ in the threshold region using the exact β function of Fig. 5 (continuous line) and the Θ function approximation (dotted line).
- Figure 10 Graph of $F_2(Q^2, x)$ for $x=0.4$ and Q^2 up to 25000 GeV^2 for neutral current events containing photon or Z exchange. In this and the next two figures the broken line is the tree diagram value, the continuous line contains the effect of coupling constant renormalization and the dotted line contains the effect of coupling constant renormalization in the Θ function approximation.
- Figure 11 Graph of $F_2(Q^2, x)/\tilde{F}_2(Q^2, x)$ for $x=0.4$ and Q^2 up to 25000 GeV^2 , where $F_2(Q^2, x)$ is as in the previous figure and $\tilde{F}_2(Q^2, x)$ contains only photon exchange without higher weak or electromagnetic corrections but with the strong interaction scaling violations calculated in QCD.

- Figure 12 Graph of $F_3(Q^2, x)$ for $x=0.4$ and Q^2 up to 25000 GeV^2 , showing the parity violation effects of Z exchange.
- Figure 13 Diagrams contributing to the renormalization of g_v . In this and the next three figures wavy lines are gauge bosons, dotted lines are Faddeev-Popov ghosts, broken lines are scalar particles (either Goldstone bosons or physical Higgs particles) and continuous straight lines are fermions. In part c) the loops are either gluon loops, V , or superheavy gauge bosons, X or Y (and their associated scalars).
- Figure 14 Diagrams contributing to the renormalization of g_w . In part c) the loops are either weak gauge bosons, W , or loops containing superheavy gauge bosons, one X and one Y (and their associated scalars).
- Figure 15 Diagrams contributing to the renormalization of g_u . In part c) the loops are superheavy gauge bosons X or Y (and their associated scalars).
- Figure 16 Diagrams contributing to the renormalization of g_x . In part c) the loops are one X and a gluon, V (and their associated scalars).
- Figure 17 Graph of $\bar{\alpha}_v(Q^2)$ (continuous line), $\bar{\alpha}_w(Q^2)$ (dotted line) and $\bar{\alpha}_u(Q^2)$ (broken line) against $\ln(Q^2/4M_x^2)$ in the threshold region.

FIG.1

FIG.2

(a)

(b)

(c)

(d)

FIG.3

(a)

(b)

(c)

(d)

FIG.4

FIG. 5

FIG. 6

FIG.7

FIG. 8

FIG.9

FIG.10

FIG.11

FIG. 12

FIG.13

(a)

(b)

(c)

FIG.14

(a)

(b)

(c)

FIG.15

(a)

(b)

(c)

FIG.16

FIG.17