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STABILITY IN THE YANG-MILLS SYSTEM
WITH AN EXTERNAL SOURCE

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ABSTRACT

We investigate stability of the Coulomb solution in the Yang-Mills system under finite perturbations below the critical coupling strength. The rôle of recently discovered low-energy solutions is discussed.

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In a previous paper¹⁾ we have shown that the true stable field configuration²⁾ which is created by a sufficiently strong spherically symmetric external source in a Yang-Mills system has to be either spatially asymmetric or genuine time-dependent. We have not been able to decide which possibility actually is realized in the Yang-Mills system. Quite recently, Sikivie and Weiss³⁾ have found time-dependent spherically symmetric solutions of this external source problem. They also argued for the existence of spatially asymmetric, static solutions with spherically symmetric source. The first set contains solutions with energy smaller than any given positive number for a given external source. Very probably, these configurations lead to instability of the external source if the classical system is sufficiently strong perturbed. In quantum theory one would expect a tunnelling transition between the original Coulomb configuration and the energetically much more favoured solutions of the Sikivie-Weiss type unless this tunnelling is suppressed by some unknown selection rule. One has then to be very careful about applying any form of perturbation expansion which uses the Coulomb field of a given source as the starting point, even if the coupling strength is very small. In massless scalar electrodynamics a somewhat similar phenomenon is known to happen only for overcritical coupling strength⁴⁾. Clearly we would like to understand the complete spectrum of small energy solutions for fixed external spherically symmetric source and their meaning for the stability of the Coulomb solution below the critical coupling. Especially we would like to know whether there are static configurations of sufficiently small energy (which are necessarily spatially asymmetric) thus indicating spontaneous breakdown of rotation invariance in the one-source sector. Such a symmetry breakdown is a necessary ingredient of any quark confinement mechanism by colour strings.

If it turns out that the time-dependent spherically symmetric Sikivie-Weiss solutions are the relevant classical configurations in the single source sector, their perfect colour-screening would make it hard to believe in quark confinement for this non-Abelian model. Furthermore, the sequence of solutions with decreasing energy, which does not have a limit at zero energy may cause big problems in a formulation of a consistent perturbation scheme for the quantized theory.

In this paper we argue for the existence of static (spatially asymmetric) solutions for the Yang-Mills system with symmetric source in the neighbourhood of the Coulomb field. We find a source of instability (with respect to small perturbations) for over-critical coupling strength²⁾ which is not directly related to the low-lying time-dependent Sikivie-Weiss solutions. It is due to new static solutions near the Coulomb field. For over-critical coupling strength, these configurations have lower energy than the Coulomb field and are stable under small perturbations (see Fig. 1). For genuine time-dependent fluctuations

around the Coulomb solution we do not know the sign for more than the second derivative of the energy functional.

In the following we are dealing with an $SU(2)$ Yang-Mills system defined by the classical field equations:

$$\begin{aligned} D^i F^{0i}(\vec{r}, t) &= g(\vec{r}, t) \\ D^0 F^{0i}(\vec{r}, t) - D^j F^{ji}(\vec{r}, t) &= 0 \end{aligned} \quad (1)$$

For the fields we use the matrix notation. As in Refs. 1) and 2) we argue within the class of privileged gauges:

$$\begin{aligned} g(\vec{r}, t) &= g(r) T^3 \\ A^0(\vec{r}, t) &= A^0(r, t) T^3 \end{aligned} \quad (2)$$

For small fluctuations around the Coulomb solution

$$A^\mu = \delta^{\mu 0} \phi(r) T^3$$

the field equations are solved by the ansatz

$$\begin{aligned} A^0 &= \phi(r) T^3 \\ \vec{A} &= [T^1 \cos \omega t + T^2 \sin \omega t] \vec{A}(\vec{r}, \omega) \end{aligned} \quad (3)$$

The small vector field $\vec{A}$ has to satisfy the linear fluctuation equation in the Coulomb field background

$$(\omega + \phi(r))^2 \vec{A}(\vec{r}, \omega) = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}(\vec{r}, \omega)) \quad (4)$$

Superposition of any two solutions gives again a solution of the fluctuation equation. Any potential (3) individually defines a static solution in the sense of gauge invariants¹⁾, but a superposition of two solutions with different frequencies is no longer static. We find it easier to work in the gauge with time-independent potentials

$$\begin{aligned} A^0 &= (\phi(r) + \omega) T^3 \\ \vec{A} &= \vec{A}(\vec{r}, \omega) T^1 \end{aligned} \quad (5)$$

The superposition principle does not hold here because ω is not a small parameter which can be neglected on the left-hand side of Eq. (4). This means that the Coulomb solution is embedded in the space of solutions spanned by the small fluctuations (3) which has certain time-independent directions.

In order to avoid problems with a divergent energy integral at large distances, we enclose the whole system in a ball with a large radius R . For any given value j of the angular momentum, there is a pair of solutions²⁾

$$\begin{aligned}\vec{A}(\vec{r}, \omega) &= h_j(r, \omega) \vec{r} \times \vec{\nabla} Y_{j,m}(\vartheta, \varphi) \\ \vec{A}(\vec{r}, \omega) &= f_j(r, \omega) \frac{\vec{r}}{r} Y_{j,m}(\vartheta, \varphi) + g_j(r, \omega) r \vec{\nabla} Y_{j,m}(\vartheta, \varphi)\end{aligned}\quad (6)$$

The radial functions are solutions of second-order differential equations. We have to select those solutions with a good behaviour at the origin and with the following surface integral vanishing

$$\int_{|\vec{r}|=R} d\vec{f} \vec{A}(\vec{r}, \omega) \times (\vec{\nabla} \times \vec{A}(\vec{r}, \omega)) = 0 \quad (7)$$

With this condition on ω the increase in energy of solution (3), with respect to the undeformed Coulomb solution is given by

$$W - W_C = \omega \cdot \int_{|\vec{r}| < R} dV (\omega + \phi(r)) \vec{A}(\vec{r}, \omega) \vec{A}(\vec{r}, \omega) \quad (8)$$

The solution $\omega = 0$ does not satisfy (7).

For the first of the solutions (6) the eigenvalue condition amounts to finding the zeros of Whittaker's function on the imaginary axis

$$M_{-iq, \mu}(2i\omega R) = 0 \quad \left(\mu = \sqrt{(j + \frac{1}{2})^2 - q^2} \right) \quad (9)$$

For simplicity we have taken here a point source

$$g(r) = 4\pi q \delta(\vec{r}) \quad (10)$$

The qualitative dependence of these zeros $\omega_n(q)$ ($n = 1, 2, \dots$) on the strength of the external source is shown for large n and $j = 1$ in Fig. 2. It is important to note that all these curves have vertical tangent for $q \rightarrow 3/2$. This proves the sign change for the energy difference (8)⁵⁾ which is the statement of classical instability of the Coulomb field with respect to small perturbations. For any source with connected support the sign of ρ and ϕ are the same, whereas ω has to be chosen with the opposite sign in order to have a chance for instability. Therefore we have shown negative ω values only in Fig. 2.

Now we discuss higher order corrections for the small fluctuations $\vec{A}(\vec{r}, \omega)$. These corrections tell us whether exactly at the critical point the Coulomb field is still stable under small perturbations or whether it is already unstable in the directions of these modes. Consequently, two physically different situations can arise:

- i) For instability under small perturbations at the critical point we would conclude that there is a solution with lower energy than the Coulomb field in the neighbourhood of the Coulomb configuration already for undercritical coupling. Only some barrier, which vanishes if the coupling approaches the critical value, prevents the Coulomb configuration from decaying into this lower-lying state.
- ii) If the Coulomb field is stable under a small perturbation at the critical point, a new stable configuration near the Coulomb field has to build up for over-critical couplings. Under the slightest perturbation the Coulomb field would classically decay into this new configuration (see Fig. 1).

Starting with a small fluctuation (5) around the Coulomb field, the solution of the exact field equations up to third order terms in the small fluctuation $\vec{A}_1(\vec{r})$ is given by

$$\begin{aligned} A^0 &= T^3 [\phi(r) + \omega + \Lambda(\vec{r})] \\ \vec{A} &= T^1 [\vec{A}_1(\vec{r}) + \vec{A}_3(\vec{r})] \end{aligned} \quad (11)$$

The correction Λ to the Coulomb potential is determined by

$$\Delta \Lambda(\vec{r}) = (\omega + \phi(r)) \vec{A}_1(\vec{r}) \vec{A}_1(\vec{r}) \quad (12)$$

The third order term $\vec{A}_3$ is a solution of the following equation:

$$(\phi(r) + \omega)^2 \vec{A}_3(\vec{r}) + 2 \Lambda(\vec{r}) (\phi(r) + \omega) \vec{A}_4(\vec{r}) = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}_3(\vec{r})) \quad (13)$$

One notices that the field equations up to third order terms are of the same structure as the linear equations (4). Only the Coulomb background field has got a correction term which amounts to the following change in the original source

$$g(r) \longrightarrow g(r) - (\phi(r) + \omega) \vec{A}_4(\vec{r}) \cdot \vec{A}_4(\vec{r}) \quad (14)$$

Formula (8) tells us that the global contribution of the correction term to the source vanishes at the critical point. From what we have said about the relative sign of ρ , ϕ and ω it becomes clear that (14) tends to screen the original source thereby weakening the background field. This means that the energy functional evaluated around a slightly deformed Coulomb configuration has positive second derivatives at the critical point. Therefore the energy is a concave functional with respect to these static deformations of the Coulomb field at the critical coupling (Fig. 1). One expects then for over-critical coupling a stable solution (with respect to small perturbations) in the neighbourhood of the Coulomb field. A mechanism like that was found in massless scalar electrodynamics⁴⁾. The analogue of the solutions found by Mandula in that system has to be spatially asymmetric in the Yang-Mills case. An ansatz for these solutions is given by

$$\begin{aligned} A^0 &= A^0(\vec{r}) T^3 \\ \vec{A} &= \vec{A}(\vec{r}) T^1 \end{aligned} \quad (15)$$

The exact field equations (1) reduce to the still non-linear system of partial differential equations

$$\begin{aligned} (A^0(\vec{r}))^2 \vec{A}(\vec{r}) &= \vec{\nabla} \times (\vec{\nabla} \times \vec{A}(\vec{r})) \\ -\Delta A^0(\vec{r}) + A^0(\vec{r}) (\vec{A}(\vec{r}) \cdot \vec{A}(\vec{r})) &= g(r) \end{aligned} \quad (16)$$

In Ref. 6) it was pointed out that the first of the equations (16) could be interpreted (up to a sign) as the Maxwell equation

$$\vec{\nabla} \times \vec{H} = - \vec{j}$$

where $\vec{j}$ is the quantum mechanical current

$$\vec{j} = \text{Re} \left\{ A^0(\vec{r}) [-i \vec{\nabla} - \vec{A}(\vec{r})] A^0(\vec{r}) \right\}$$

Using $\vec{\nabla} \cdot \vec{j} = 0$, the second of Eqs. (16) can be written in a form similar to Schrödinger's equation:

$$[-i \vec{\nabla} - \vec{A}(\vec{r})]^2 A^0(\vec{r}) = g(r)$$

It is not difficult to find solutions of (16) with spatially asymmetric source [see also Ref. 3)]. One possible ansatz is $A^0 = \Omega = \text{const.}$ The first of the Eqs. (16) becomes then simply the wave equation for light with frequency Ω . The second of the Eqs. (16) can be taken to define the external source. It seems somewhat harder to find finite energy solutions of Eqs. (16) with a spherically symmetric source.

Our analysis has not covered the higher derivatives of the energy functional for genuine time-dependent fluctuations around the Coulomb field. This problem will be attacked in a later paper.

Finally, we would like to comment further on the very interesting time-dependent solutions of Sikivie and Weiss. We emphasize the fact that given the invariant field $\text{Tr}\{\rho(\vec{r})\rho(\vec{r})\}$ the source distribution ρ in the privileged gauge (2) is unique if the support of the source is connected. Actually, the source in the example of Ref. 3) vanishes nowhere, nevertheless there are solutions to this source with energies smaller than any given positive number. Therefore, we do not think the simple sign flip mechanism for the external source as proposed by the authors of Ref. 3) explains the existence of their solutions. From this mechanism, one also hardly understands the unavoidable time dependence of their spherically symmetric solution. Nevertheless, the mere existence of these configurations presents a challenge to our understanding of the physical field of a heavy quark source, probably with important consequences for the consistent formulation of QCD.

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FIGURE CAPTIONS

- Fig. 1 : Energy for small physically static fluctuations around the Coulomb field.
- Fig. 2 : Dependence of frequencies ω_n on the coupling strength q for small fluctuations around the Coulomb field enclosed in a ball of radius R .

FIG. 1

FIG. 2