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BARYON STRUCTURE IN QUARK GEOMETRODYNAMICS

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A B S T R A C T

We analyze the problem of baryon structure in the quark-geometrodynamical approach to hadrons. We derive a baryonic structure very similar to the quark-diquark configuration proposed ten years ago. A brief comparison of the calculated spectrum with experiments is carried out and shown to represent experimental knowledge adequately. A discussion of current and meson couplings concludes our analysis.

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1. - INTRODUCTION

Baryon structure is certainly one of the most central problems of hadrodynamics ; and one of the most difficult too. In this paper we want to expand and extend our recently published discussion of the problem in the framework of quark-geometrodynamics ¹⁾.

Before entering the technical part of the paper we want to stress the surprising aspect that our solution of the problem presents. We found that the geometrodynamical equations admit a solution only for an extremely restricted class of quark configurations, i.e., those configurations in which the three quarks are spatially aligned. This fact seems to resolve a puzzle and to raise a very important question at the same time. The puzzle is the non-observation (so far) of the many states predicted by the naïve quark model ²⁾ in the mass region below 2 GeV, and the question refers to the interpretation of quarks as field theoretical objects. For, if quarks are the fields of QCD which inside the hadronic bags are almost free ³⁾, then it is difficult to understand how can it happen that essential degrees of freedom become suppressed.

It is here that the geometrodynamical approach departs from the QCD picture of the world ; should the predicted quark model states fail to show up beyond a reasonable cross-section level, we believe that QCD should be rejected. On the other hand our approach would be untenable should these states definitely be established.

The plan of the paper is as follows : in Section 2 we derive the geometrodynamical equations for baryons starting from the basic geometrodynamical principles ; in Section 3 the space-time structure of the solutions shall be exhibited while the full solution (spin included) as well as the spectrum and the $SU(6)$ structure shall be discussed in Section 4. Section 5 deals with the possible definitions of the direct current coupling and of the baryon-meson coupling.

2. - THE GEOMETRODYNAMICAL EQUATIONS

The principles of the quark-geometrodynamical approach to hadrons have been discussed at length in Ref. 4) ; here we shall only recall that they rest mainly on the idea of simplicity of hadronic wave functions (w.f.'s) and of free-motion for the quark co-ordinates inside the space-time domains (bags) where hadronic matter can be found.

We start from the baryon w.f. which we express, after making use of translation invariance, as

$$\psi_{\alpha a, \beta b, \gamma c} (p; x_1, x_2, x_3) = e^{ipX} \psi_{\alpha a, \beta b, \gamma c} (p; x, y) \quad (2.1)$$

where α, β, γ are Dirac indices ; a, b, c carry flavour and colour, and

$$\begin{aligned} X &= \frac{1}{3} (x_1 + x_2 + x_3) \\ x &= \frac{1}{\sqrt{2}} (x_1 - x_3) \\ y &= \sqrt{\frac{2}{3}} \left(x_1 - \frac{x_2 + x_3}{2} \right). \end{aligned} \quad (2.2)$$

According to the geometrodynamical principles we proceed to characterize the w.f. in the following way ⁵⁾.

i) - Confinement

$$\psi(p; x, y) = 0 \quad \text{for } x, y \notin R^8(p; x, y) \quad (2.3)$$

where $R^8(p; x, y)$ is a compact, eight-dimensional space-time region whose boundary shall be denoted by $B^8(p; x, y)$ ⁶⁾. Its structure shall be discussed later.

ii) - Continuity

Decomposing (2.1) in Lorentz covariants, i.e.,

$$\psi_{\alpha a, \beta b, \gamma c}(p; x, y) = \sum_s P_{\alpha \beta \gamma}^{(s)}(p, -i\partial_x, -i\partial_y) \varphi_{abc}^{(s)}(p; x, y) \quad (2.4)$$

we require that the scalar functions $\varphi^{(s)}(p; x, y)$ be continuous functions, i.e.,

$$\varphi^{(s)}(p; x, y) = 0 \quad \text{for } x, y \in B^8(p; x, y). \quad (2.5)$$

iii) - Wave motion

For $x, y \in R^8(p; x, y)$ ψ obeys the simple differential equation

$$D_1 D_2 D_3 \psi = 0, \quad (2.6)$$

where $D_i = (-i\gamma + m_i)$ is the Dirac operator acting on the i^{th} co-ordinate.

iv) - Maximum freedom

The "distance" ⁷⁾ of ψ from the "free solution" $\psi^{(0)}$ determined by the free equations $[x, y \in R^8]$

$$D_i \psi^{(0)} = 0 \quad (i=1, 2, 3) \quad (2.7)$$

is minimum.

The justification of the above characterizations of the baryon w.f. goes along the same arguments as in Ref. 4), we shall briefly repeat them here :

- i) - embodies confinement in that it states that the space-time domain spanned by the relative quark co-ordinates x and y is finite ; accordingly quarks lose their independent dynamical status, being reduced to the rôle of hadron co-ordinates ;
- ii) - is the strongest form of continuity we can impose on the w.f. and compatible with simple wave equations ;

- iii) - specifies the "wave motion" of the quark degrees of freedom inside the bag in accordance with the requirements of simplicity and freedom ; a simpler set of free motion equations like those in iv) cannot be imposed without clashing against the requirements of continuity ; it is worth noticing that (2.6) is nothing but the three-body Bethe-Salpeter equation with "infinite wall" potentials, and that the boundary conditions (2.5) are those appropriate to such a situation. We thus see that, in agreement with the geometrodynamical principles, the dynamics of the hadronic system has been reduced to its geometry [the bag domain $R^8(p; x, y)$];
- iv) - makes sure that only those solutions of (2.6) are kept which lead to a "quark-motion" very close to the free one ; a really free motion is obviously impossible due to confinement, however, by this requirement we succeed in selecting only those solutions which will look as free as possible ; for this reason, an understanding of the structure of $\psi^{(0)}$ will always be preliminary to the solution of (2.6) ; this shall appear evident very shortly.

3. - THE STRUCTURE OF $\psi^{(0)}$

We now face the task of finding a solution to the problem posed in (2.7), which satisfies the continuity requirement as closely as possible. In order to simplify the problem, we shall deal with scalar quarks, but the surprising results which we mentioned in the Introduction do not depend on this simplification. For scalar quarks, Eq. (2.7) becomes

$$k_i \psi^{(0)}(p; x_1, x_2, x_3) = 0 \quad (i=1,2,3) \quad (3.1)$$

where $k_i = -(\square_i^2 + m_i^2)$ is the Klein-Gordon operator acting on the i^{th} co-ordinate. The structure of the boundary B^8 [see the footnote 6] makes it clear that in the rest frame we must consider factorized functions of the form :

$$\psi^{(0)}(p; x, y) = \phi_T(x_0, y_0) \psi^{(0)}(\vec{x}, \vec{y}) \quad (3.2)$$

and accordingly the Eqs. (3.1) give rise to six second order equations, three for each of the two functions ϕ_T and ψ . Let us write down the three spatial equations ($\vec{x}, \vec{y}$ belong to the six-dimensional bag domain) :

$$-(\vec{\nabla}_x^2 + \vec{\nabla}_y^2) \psi^{(0)} = Q^2 \psi^{(0)}, \quad (3.3)$$

$$2\vec{\nabla}_x \cdot \vec{\nabla}_y \psi^{(0)} = A \psi^{(0)}, \quad (3.4)$$

$$(\vec{\nabla}_x^2 - \vec{\nabla}_y^2) \psi^{(0)} = B \psi^{(0)}, \quad (3.5)$$

where Q^2 , A and B are integration constants.

The solutions of these equations are also required to exhibit definite symmetry properties under the permutation of the constituents. All the elements of the permutation group S_3 can be generated by two of its elements, e.g., by P , interchanging x_2 and x_3 , and by C the cyclic permutation $\begin{pmatrix} 123 \\ 231 \end{pmatrix}$. They can be represented in the two-dimensional space (x, y) as

$$P = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} ; \quad C = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}. \quad (3.6)$$

Were we to solve (3.3) with the symmetric boundary given by (R_s constant)

$$\vec{x}^2 + \vec{y}^2 = R_s^2 \quad (3.7)$$

we would find no problem whatsoever, as we shall see below. But the difficulty rests upon the fact that we cannot satisfy all the three equations at the same time in a finite space-domain of the type (3.7), as we can readily convince ourselves by noticing that the momenta $-i\vec{\nabla}_x$ and $-i\vec{\nabla}_y$ cannot be constant in a finite space-domain due to quantum fluctuations. Thus we must conclude that $\psi^{(0)}$ obeying (3.3), (3.4) and (3.5) and vanishing at the boundary (3.7) does not exist, and consequently that our geometrodynamical problem cannot have in this form any solution.

Is there any possible way out ? Let us look at our problem more closely. It does not have any solution if we consider $\vec{x}$ and $\vec{y}$ as independent dynamical variables, but if $\vec{x}$ and $\vec{y}$ are linearly dependent, i.e., the three quarks are spatially aligned, (3.3), (3.4) and (3.5) collapse to a single equation and our problem admits a very simple solution. Thus we can say that our geometrodynamical problem admits a solution only if the "quark motion" inside the baryon is a "constrained" one. As a consequence the spatial dynamical degrees of freedom of a geometrodynamical baryon are only three rather than six ^{*}). Obviously this will drastically reduce the number of expected states from the one predicted by the naive quark model, and our picture of the baryons will be essentially equivalent to the quark-diquark structure proposed many years ago ⁸⁾.

3.1. - $O(6)$ expansion and the solution of (3.3)

In this subsection we want to find the solutions of (3.3) which vanish at the boundary (3.7). Solving this problem is both instructive and useful ; instructive because it allows us to analyze our system in the $O(6)$ basis (which embodies the representation of S_3), and useful because it will suggest a convenient strategy to obtain the solution of our constrained problem exhibiting a definite symmetry character under S_3 .

We write the Laplacian operator as

$$\vec{\nabla}_x^2 + \vec{\nabla}_y^2 = \frac{d^2}{dR^2} + \frac{5}{R} \frac{d}{dR} - \frac{1}{R^2} \Delta_6 \quad (3.8)$$

where Δ_6 is the $O(6)$ Laplace-Beltrami operator ⁹⁾.

The $O(6)$ variables $\psi, \varphi; \alpha, \beta, \gamma$ are related to $\vec{x}$ and $\vec{y}$ by the following equations :

$$\begin{aligned} \vec{x} &= R \left(\cos \psi \cos \frac{\varphi}{2} \vec{u} + \sin \psi \sin \frac{\varphi}{2} \vec{v} \right) \\ \vec{y} &= R \left(\cos \psi \sin \frac{\varphi}{2} \vec{u} - \sin \psi \cos \frac{\varphi}{2} \vec{v} \right), \end{aligned} \quad (3.9)$$

^{*}) Strictly speaking we have four degrees of freedom left, but as we shall see later, one of them carries no dynamics.

and

$$\begin{aligned}\vec{u} &\equiv (\cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha, -\sin\gamma\cos\beta\cos\alpha - \cos\gamma\sin\alpha, \sin\beta\sin\alpha) \\ \vec{v} &\equiv (\cos\gamma\cos\beta\sin\alpha - \sin\gamma\cos\alpha, -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\sin\alpha, \sin\beta\sin\alpha)\end{aligned}\quad (3.10)$$

The $O(6)$ spherical harmonics are defined as the solutions of the equations

$$\Delta_6 \phi_{KL\mu\omega}(\psi, \varphi; \alpha, \beta, \gamma) = K(K+4) \phi_{KL\mu\omega}(\psi, \varphi; \alpha, \beta, \gamma) \quad (3.11)$$

with K a non-negative integer. Their explicit form is

$$\phi_{KL\mu\omega}(\psi, \varphi; \alpha, \beta, \gamma) = \sum_{\lambda=|\mu|}^K \sum_{\nu=-\lambda}^{\lambda} a_{\nu\lambda}^{K\mu L\omega} \mathcal{D}_{\mu/2, \nu/2}^{\lambda/2}(\varphi, 2\psi - \frac{\pi}{2}, 0) \mathcal{D}_{\nu L_2}^L(\alpha, \beta, \gamma) \quad (3.12)$$

where the $\mathcal{D}$'s are the usual rotation functions, L the total angular momentum and L_z its z component, ω is the eigenvalue of the so-called Racah operator necessary to lift the degeneracy; $a_{\nu\lambda}^{K\mu L\omega}$ are known coefficients whose actual values we shall not need here. The symmetry character of the harmonics is specified by the integer μ , which is defined mod. three and gives rise to the following Young tableaux :

$ \mu = 0$	L even	$\begin{array}{ c c c } \hline & & \\ \hline \end{array}$
$ \mu = 0$	L odd	$\begin{array}{ c } \hline \\ \hline \\ \hline \end{array}$
$ \mu = 1$		$\begin{array}{ c c } \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}$
$ \mu = 2$		$\begin{array}{ c c } \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}$

The action of the operators defined in (3.6) on the $O(6)$ spherical harmonics is given by :

$$\begin{aligned} P \phi_{kL\mu\omega} &= (-1)^L \phi_{kL\mu\omega} \\ C \phi_{kL\mu\omega} &= e^{i\mu\frac{2\pi}{3}} \phi_{kL\mu-\omega}. \end{aligned} \quad (3.13)$$

To get our solution is now very easy : we write

$$\psi = f_k(R) \phi_{kL\mu\omega}(\psi, \varphi; \alpha, \beta, \gamma) \quad (3.14)$$

and using (3.11) we must solve the radial equation

$$\frac{d^2 f_k}{dR^2} + \frac{5}{R} \frac{df_k}{dR} - \frac{K(K+4)}{R^2} f_k + Q^2 f_k = 0, \quad (3.15)$$

whose solution can be readily got in terms of Bessel functions. Thus our solution is given by

$$\psi = \frac{1}{R^2} J_{k+2}(QR) \phi_{kL\mu\omega}(\psi, \varphi; \alpha, \beta, \gamma) \quad (3.16)$$

with

$$Q = \frac{j_{k+2,n}}{R_s} \quad (3.17)$$

$j_{k+2,n}$ being the n^{th} positive zero of the Bessel function $J_{k+2}(z)$.
When $R \rightarrow 0$ (3.16) behaves as

$$\psi \xrightarrow{R \rightarrow 0} R^k. \quad (3.18)$$

The ordering of the levels for increasing values of Q^2 (related to the mass of the state) ¹⁰⁾ is reported in Fig. 1 and coincides with the spectrum one derives in the naïve quark model with a harmonic oscillator potential ²⁾.

3.2. - The constrained quark motion

We know from general arguments, and it can be checked explicitly, that (3.16) does not satisfy the remaining two equations (3.4) and (3.5). The only way to satisfy them at once is to make all of them collapse to the same equation. In order to achieve this we assume that $\vec{x}$ and $\vec{y}$ are not independent degrees of freedom but are given by

$$\begin{aligned}\vec{x} &= \vec{R} \cos \theta_0 \\ \vec{y} &= \vec{R} \sin \theta_0,\end{aligned}\tag{3.19}$$

where $\vec{R}$ is the independent variable, and θ_0 is a constant (we shall argue below that θ_0 should take the values $\pi/2$, and $\pi/2 \pm 2/3 \pi$). As a consequence the momentum operators also depend on the momentum conjugate to $\vec{R}$, through the relations

$$\begin{aligned}-i \vec{\nabla}_x &= \cos \theta_0 (-i \vec{\nabla}_R) \\ -i \vec{\nabla}_y &= \sin \theta_0 (-i \vec{\nabla}_R)\end{aligned}\tag{3.20}$$

By virtue of the previous constraints we can rewrite Eqs. (3.3)-(3.5) as :

$$-\vec{\nabla}_R^2 \psi^{(0)} = Q^2 \psi^{(0)}\tag{3.21}$$

$$-\vec{\nabla}_R^2 \psi^{(0)} \sin 2\theta_0 = A \psi^{(0)}\tag{3.22}$$

$$-\vec{\nabla}_R^2 \psi^{(0)} \cos 2\theta_0 = B \psi^{(0)},\tag{3.23}$$

which clearly collapse to only one equation, provided the integration constants are chosen as

$$\begin{aligned}A &= Q^2 \sin 2\theta_0 \\ B &= Q^2 \cos 2\theta_0.\end{aligned}\tag{3.24}$$

The solutions of (3.21), vanishing at the boundary (3.7) are easy to find, we get :

$$\psi^{(0)} = j_L(QR) Y_L^{L_z}(\Omega_R) \quad (3.25)$$

and

$$Q = \frac{\beta_{nL}}{R_s} ; \quad (3.26)$$

where $j_L(z)$ is the spherical Bessel function and β_{nL} is the position of its n^{th} positive zero. The behaviour of $\psi^{(0)}$ for $R \rightarrow 0$ is

$$\psi^{(0)} \xrightarrow{R \rightarrow 0} R^L \quad (3.27)$$

In order to obtain a solution with a definite symmetry character under the permutation group S_3 we project the solutions of (3.3)-(3.5) $[\vec{p}$ and $\vec{q}$ are two vectors which have a parametrization similar to (3.9) in terms of primed variables]

$$e^{i(\vec{p}'\vec{x} + \vec{q}'\vec{y})} \delta(\vec{q}'^2 - Q^2) \delta(\cos 2\psi' - 1) \delta(\varphi' - 2\theta_0) \quad (3.28)$$

onto a $O(6)$ spherical harmonic $\Phi_{KLL,2\mu\omega}(\psi', \varphi'; \alpha', \beta', \gamma')$. The solution (3.25), on the other hand, depends only on L and L_z , so we must ask ourselves which values of K we should choose for a given L . The answer is very simple when we notice that when $R \rightarrow 0$ our solution should coincide with a solution of the $O(6)$ problem. Thus from (3.18) and (3.27) we must choose

$$K = L. \quad (3.29)$$

Performing now the projection $[\vec{Q}]$ is the momentum conjugate to $\vec{R}]$

$$\psi_{LL\mu}(\vec{R}, \theta_0) = \int \vec{Q}' dQ' d(\cos 2\psi') d\varphi' d\alpha' d(\cos \beta') d\gamma' e^{i\vec{Q}' \cdot \vec{R}} \delta(Q'^2 - Q^2) \delta(\cos 2\psi' - 1) \delta(\varphi' - \theta_0) \phi_{LL\mu\omega} \quad (3.30)$$

we get easily :

$$\psi_{LL\mu}(\vec{R}, \theta_0) \sim j_L(QR) Y_L^{L_2}(Q\vec{R}) e^{i\mu\theta_0} A_{L\mu} \quad (3.31)$$

where

$$A_{L\mu} = \sum_{\lambda=|\mu|}^L \sum_{\nu=-\lambda}^{\lambda} a_{\nu\lambda}^{L\mu\omega} d_{\frac{\mu}{2} \frac{\nu}{2}}^{\frac{\lambda}{2}}(\pi/2) d_{\omega}^L(\pi/2). \quad (3.32)$$

From which one can immediately deduce the following consequences :

- i) the states with $L=0$ are totally symmetric ;
- ii) there is no totally antisymmetric state ;
- iii) states with odd L have mixed symmetry ;
- iv) even L states with $L \geq 2$ exhibit both the totally symmetric and the mixed symmetric character.

The spectrum of the constrained quark motion gets drastically reduced from the unconstrained one, as we can check by comparing Fig. 2 with Fig. 1. It will be now argued that the only reasonable choices for θ_0 are

$$\theta_0 = \frac{\pi}{2} \quad \text{or (by symmetry)} \quad \theta_0 = \frac{\pi}{2} \pm \frac{2\pi}{3}. \quad (3.33)$$

In fact as we have remarked above, we can look at the bag structure as arising from the dynamics of "infinite wall" type potentials. In our three body problem we have two types of potentials $[\vec{x}_{ij} = \vec{x}_i - \vec{x}_j]$:

- one three-body potential V_{123} whose argument is a symmetric function of the antiquark distances $|\vec{x}_{12}|$, $|\vec{x}_{13}|$ and $|\vec{x}_{23}|$;
- three two-body potentials V_{12} , V_{13} and V_{23} whose arguments are $|\vec{x}_{12}|$, $|\vec{x}_{13}|$ and $|\vec{x}_{23}|$, respectively.

If we now want to recover a constrained motion by means of these potentials it is clear that the spatial bag region $B_s(\vec{x}, \vec{y})$ must be given by

$$B_s(\vec{x}, \vec{y}) = \{|\vec{x}_{12}| = R_s(\eta); |\vec{x}_{23}| = 0\} \cup \{|\vec{x}_{13}| = R_s(\eta); |\vec{x}_{12}| = 0\} \cup \{|\vec{x}_{23}| = R_s(\eta); |\vec{x}_{13}| = 0\} \quad (3.34)$$

thus we can interpret the constrained quark motion as arising from an "infinite wall" two-body potential confining the quarks in a bag of vanishing extension ¹¹⁾. From (3.34), Eqs. (3.33) trivially follow.

The discussion of this subsection leads us to the conclusion that the geometrodynamical baryon exists only in the three spatial configurations depicted in Fig. 3. One should note, however, that these configurations are similar but not identical with the ones appearing in the already mentioned quark-diquark model ; here the "diquark" still consists of two quarks as it should be clear from the symmetry properties of the w.f., and shall become more evident from the structure of the time-w.f. (see below).

4. - THE FULL SOLUTION SPIN AND SU(6)

In this section we shall construct first the solutions of the problem for scalar quarks, and after having introduced spin we shall analyze the SU(6) properties of the spectrum and of the w.f.'s.

4.1. - The full scalar solution

We must now find the solutions of the geometrodynamical equation :

$$K_1 K_2 K_3 \psi = 0 \quad (4.1)$$

which vanishes outside the bag region $R^8(p;x,y)$ and whose space boundary in the rest frame is given in (3.34). In the constrained quark motion situation ψ is the superposition of three configurations $[(1), (2) \text{ and } (3)]$ of Fig. 3 which are factorized as

$$\psi^{(1)} = \psi_s^{(1)}(\vec{R}) \varphi^{(1)}(x_0, y_0). \quad (4.2)$$

We shall consider only the configuration (1), the results being easily extended to the other two configurations. Thus using (3.21)-(3.23), (4.1) becomes a sixth order differential equation for $\varphi^{(1)}(x_0, y_0)$ $[k_0 = -i(\partial/\partial x_0), q_0 = -i(\partial/\partial y_0); \vec{k}^2 = \frac{2}{3}Q^2]$

$$\begin{aligned} & \left[\left(\frac{M}{3} + \sqrt{\frac{2}{3}} q_0 \right)^2 - (\vec{k}^2 + m_1^2) \right] \left[\left(\frac{M}{3} - \frac{1}{2} \sqrt{\frac{2}{3}} q_0 + \frac{k_0}{\sqrt{2}} \right)^2 - \left(\frac{\vec{k}^2}{4} + m_2^2 \right) \right] \\ & \cdot \left[\left(\frac{M}{3} - \frac{1}{2} \sqrt{\frac{2}{3}} q_0 - \frac{k_0}{\sqrt{2}} \right)^2 - \left(\frac{\vec{k}^2}{4} + m_3^2 \right) \right] \varphi^{(1)}(x_0, y_0) = 0. \end{aligned} \quad (4.3)$$

We seek those solutions of (4.3) which vanish at the boundary of a closed region $R^{(1)}(x_0, y_0)$ in the x_0-y_0 plane, and are "as close as possible", in the sense of point iv) of Section 2, to the "free solutions"

$$\varphi_0^{(1)}(x_0, y_0) = e^{i(E_2 + E_3 - \frac{2M}{3}) \sqrt{\frac{2}{3}} y_0} e^{i(E_3 - E_2) \frac{x_0}{\sqrt{2}}} \quad (4.4)$$

where

$$E_1 = (\vec{k}^2 + m_1^2)^{\frac{1}{2}}, \quad E_2 = \left(\frac{\vec{k}^2}{4} + m_2^2 \right)^{\frac{1}{2}}, \quad E_3 = \left(\frac{\vec{k}^2}{4} + m_3^2 \right)^{\frac{1}{2}}$$

are the quark energies in the rest frame, and the baryon mass is given by

$$M = E_1 + E_2 + E_3. \quad (4.5)$$

In general we have many possibilities among which to choose for the structure of $R^{(1)}(x_0, y_0)$. Our freedom of choice is reduced if we maintain the picture of the time "bag" as arising from two-body "infinite wall-potentials", furthermore requiring a domain $R^{(1)}(x_0, y_0)$ which is symmetric for the exchange $2 \leftrightarrow 3$ (i.e., for $x_0 \rightarrow -x_0$) we are led to a region having the simple boundary

$$B^{(0)}(x_0, y_0) \equiv \begin{cases} |t_1 - t_2| = \left| \sqrt{\frac{3}{2}} y_0 - \frac{x_0}{\sqrt{2}} \right| = T \\ |t_1 - t_3| = \left| \sqrt{\frac{3}{2}} y_0 + \frac{x_0}{\sqrt{2}} \right| = T \end{cases} \quad (4.6)$$

The solution is then straightforward, and we can write (N is a normalization factor)

$$\begin{aligned} \varphi^{(0)}(x_0, y_0) = N & e^{i(E_2 + E_3 - \frac{2M}{3})\sqrt{\frac{2}{3}}y_0} e^{i(E_3 - E_2)\frac{x_0}{\sqrt{2}}} \\ & \times e^{i\sqrt{6}\omega y_0} \cos \omega \left(\sqrt{\frac{3}{2}} y_0 - \frac{x_0}{\sqrt{2}} \right) \cos \omega \left(\sqrt{\frac{3}{2}} y_0 + \frac{x_0}{\sqrt{2}} \right), \end{aligned} \quad (4.7)$$

where

$$\omega = \frac{2\pi}{T} \quad (4.8)$$

and the condition

$$M = E_1 + E_2 + E_3 - \omega \quad (4.9)$$

must be satisfied.

We must notice the strict similarity between (4.9) and the equation of Ref. 4) giving the spectrum for the meson $q\bar{q}$ system. This, however, should not come as a surprise due to the great similarity of the space-time structure of the two systems, originating from the constrained nature of the spatial motion of the three quarks discussed in the preceding section.

In view of this it appears natural to choose for the quantized momentum $k_{nL} = \sqrt{\frac{2}{3}} Q_{nL}$ [see Eq. (3.26)] and for the "time-bag energy" $\omega_{nL} = 2\pi/T_{nL}$ the same expressions found in Ref. 4). In this fashion we obtain for the baryon system Regge trajectories similar in structure to the mesonic Regge trajectories, as experimentally observed.

4.2. - Introducing spin and flavour

Having solved (4.1), it is now very easy to solve our geometrodynamical problem in the presence of spinning quarks, that is Eq. (2.6) with the conditions discussed in Section 2. We write for the configuration (1) ¹³⁾

$$\psi_{\alpha\beta\gamma}^{(1)}(p; x_1, x_2, x_3) = (p_1 + m_a)^{\alpha'} (p_2 + m_b)^{\beta'} (p_3 + m_c)^{\gamma'} \psi_{\alpha'\beta'\gamma', abc}^{(1)}(p; x, y) \quad (4.10)$$

where p_i 's are the differential operators ($p = -i\partial_x$, $q = -i\partial_y$ ¹⁴⁾)

$$\begin{aligned} p_1 &= \frac{p}{3} + \sqrt{\frac{2}{3}} q \\ p_2 &= \frac{p}{3} - \frac{1}{2}\sqrt{\frac{2}{3}} q + \frac{1}{\sqrt{2}} p \\ p_3 &= \frac{p}{3} - \frac{1}{2}\sqrt{\frac{2}{3}} q - \frac{1}{\sqrt{2}} p \end{aligned} \quad (4.11)$$

and the function $\psi_{\alpha'\beta'\gamma', abc}^{(1)}(p; x, y)$ can be written as the product of a spin-function and a tensor in flavour indices :

$$\psi_{\alpha'\beta'\gamma', abc}^{(1)} = I_{abc}^{(1)} S_{\alpha'\beta'\gamma'}^{(1)} \psi^{(1)}(p; x, y). \quad (4.12)$$

Furthermore, the scalar function $\psi^{(1)}$ is the solution (4.2) of the scalar problem ; the factor $S_{\alpha'\beta'\gamma'}^{(1)}$ is given by

$$S_{\alpha'\beta'\gamma'}^{(1)} = S_{\alpha'}^i(L_p) S_{\beta'}^j(L_p) S_{\gamma'}^k(L_p) \psi_{ijk} \quad (4.13)$$

where L_p^{-1} is the Lorentz transformation bringing the vector p_μ at rest ; ψ_{ijk} is a $SU(2)$ tensor ($i, j, k = 1, 2$) and $S_\alpha^\beta(L)$ is the matrix representing a Lorentz transformation L acting on a Dirac spinor. Thus we can write (4.12) as

$$\psi_{a\beta\gamma,abc}^{(i)} = \psi^{(i)}(p;x,y) S_a^i(L_p) S_\beta^j(L_p) S_\gamma^k(L_p) B_{ia,bj,ck}^{(i)} \quad (4.14)$$

where $B_{ai,bj,ck}^{(1)}$ is a tensor of $SU(2N)$ if N is the number of flavours. Hence we see that in the case of $SU(3)$ as flavour-group, $SU(6)$ emerges in a very natural way. The physical baryon states can in fact be classified according to the representations of a $SU(2N)$ group. Collecting the results of this subsection, the complete form of the wave function is

$$\begin{aligned} \psi_{a\alpha,b\beta,c\gamma}^{(i)}(p;x_1,x_2,x_3) &= (x_1+m_a)_{\alpha}^{a'} (x_2+m_b)_{\beta}^{b'} (x_3+m_c)_{\gamma}^{c'} \\ &\quad S_a^i(L_p) S_{\beta}^j(L_p) S_{\gamma}^k(L_p) B_{ai,bj,ck}^{(i)} \psi^{(i)}(p;x,y). \end{aligned} \quad (4.15)$$

4.3. - The spectrum

In order to derive the spectrum we must now select those solutions which in the symmetric limit ($m_a = m_b = m_c$) exhibit a complete symmetry under permutations of the quark degrees of freedom. This means that, writing

$$\psi_{a\alpha,b\beta,c\gamma}(p;x_1,x_2,x_3) = \sum_i \psi_{a\alpha,b\beta,c\gamma}^{(i)}(p;x_1,x_2,x_3) \quad (4.16)$$

we must impose that $\psi_{a\alpha,b\beta,c\gamma}$ be invariant under any permutation of the quark labels. It is straightforward to verify that this can be achieved if and only if the $SU(2N)$ tensor $B_{ai,bj,ck}^{(i)}$ is

- i) the completely symmetric representation $\square\square\square$ if the function $\psi^{(i)}(p;x,y)$ corresponds to $\mu = 0$;
- ii) the mixed symmetry representation $\square\square$ if $\psi^{(i)}(p;x,y)$ corresponds to $\mu = 1, 2$.

In terms of the $SU(6)$ classification this means that :

all the physical baryon states can be organized in $[SU(6), L^P]$ multiplets following the pattern in Fig. 4. The representations $(20, L^P)$ and $(70, 0^+)$ are absent, and the parity of the states is always given by $P = (-1)^L$.

In the symmetric limit the spectrum is given by (4.9), but if we introduce $SU(N)$ symmetry breaking by giving different masses to the quarks, Eq. (4.9) should be written as

$$M_{nL}^{(\mu)} = E_{1a} + E_{2b} + E_{3c} - \omega_{abc,nL}^{(\mu)} \quad (4.17)$$

where

$$E_{1a} = (\vec{k}_{nL}^2 + m_a^2)^{1/2}, \quad E_{2b} = E_{3b} = (\frac{\vec{k}_{nL}^2}{4} + m_b^2)^{1/2}.$$

Clearly $\omega_{abc}^{(\mu)}$ must depend on the particular wave function configuration if we want a unique mass value to be attributed to the physical baryon state. We assume that the average of $\omega_{abc,nL}^{(\mu)}$ is independent of the particular $SU(2N)$ state, i.e.,

$$\sum_{\substack{abc \\ ijk}} |B_{ia,jb,kc}^{(1)}|^2 \omega_{abc,nL}^{(\mu)} = \omega_{nL}^{(\mu)} \quad (4.18)$$

where $B^{(1)}$ is a normalized $SU(2N)$ tensor, i.e.,

$$\sum_{\substack{abc \\ ijk}} |B_{ia,jb,kc}^{(1)}|^2 = 1. \quad (4.19)$$

Accordingly we can write for the mass of a given state with w.f. $B^{(1)}$ the following mass formula

$$M_{nL}^{(\mu)} = \sum_{\substack{abc \\ ijk}} |B_{ia,jb,kc}^{(1)}|^2 (E_{1a} + E_{2b} + E_{3c}) - \omega_{nL}^{(\mu)} \quad (4.20)$$

Following exactly the same line of reasoning as in Ref. 4), we can determine

$$|\vec{k}|_{nL} = \frac{p_{nL}}{R_{3,nL}} \quad \text{and} \quad \omega_{nL}^{(\mu)} = \frac{\pi}{R_{t,nL}^{(\mu)}}$$

by setting

$$R_{s,nl}^2 = R_0^2 \beta_{nl} \quad (4.21)$$

$$R_{t,nl}^{(\mu)^2} = R_0^2 \beta_{nl} - c^{(\mu)}, \quad (4.22)$$

where R_0 and $c^{(\mu)}$ are parameters to be determined. In the Table and Fig. 4 we give the mean values of the masses of the various SU_6 multiplets setting in (4.21) and (4.22) $R_0^2 = 4 \text{ GeV}^{-2}$, as in the meson case, and $c^{(\mu)} = 0$.

4.4. - A brief experimental comparison

Looking at the Table, we see that besides the $(56, 0^+)$ and $(70, 1^-)$ which are all well-established, for masses below 2 GeV there remain two multiplets $(56, 2^+)$ and $(70, 2^+)$.

In terms of (SU_3, J^P) multiplets we have

$$\begin{aligned} (56, 2^+) &\equiv (8, \frac{3}{2}^+) \oplus (8, \frac{5}{2}^+) \oplus (10, \frac{1}{2}^+) \\ &\oplus (10, \frac{3}{2}^+) \oplus (10, \frac{5}{2}^+) \oplus (10, \frac{7}{2}^+) \end{aligned} \quad (4.23)$$

and

$$\begin{aligned} (70, 2^+) &\equiv (1, \frac{3}{2}^+) \oplus (1, \frac{5}{2}^+) \oplus (8, \frac{1}{2}^+) \oplus 2(8, \frac{3}{2}^+) \\ &\oplus 2(8, \frac{5}{2}^+) \oplus (8, \frac{7}{2}^+) \oplus (10, \frac{3}{2}^+) \oplus (10, \frac{5}{2}^+). \end{aligned} \quad (4.24)$$

Experimentally it appears that all the above multiplets have candidates with the exception of two $(8, \frac{5}{2}^+)$ and one $(8, \frac{3}{2}^+)$. We regard this situation as very encouraging, especially in view of the fact that a symmetric quark model would expect, below 2 GeV, in addition to our states the following multiplets :

$$\begin{aligned}
 (70, 0^+) &\equiv (1, \frac{1}{2}^+) \oplus (8, \frac{1}{2}^+) \oplus (8, \frac{3}{2}^+) \oplus (10, \frac{1}{2}^+) \\
 (20, 1^+) &\equiv (1, \frac{1}{2}^+) \oplus (1, \frac{3}{2}^+) \oplus (1, \frac{5}{2}^+) \oplus (8, \frac{1}{2}^+) \oplus (8, \frac{3}{2}^+) .
 \end{aligned} \tag{4.25}$$

It is difficult to undervalue the great importance that an experimental clarification of the baryon spectrum below 2 GeV has in view of the deviation of the geometrodynamical predictions from the quark model expectations. As supporting arguments for the presence of a $(70, 2^+)$ representation, we should recall that a SU_6 analysis of the decay of the assumed members of the $(56, 2^+)$ needs substantial mixing to a $(70, 2^+)$ representation¹⁵⁾. As for the breaking of the flavour symmetry, we get for the octet a Gell-Mann-Okubo formula with pure F coupling, and attributing quarks the same masses as in the meson case

$$\begin{aligned}
 m_p &= 100 \text{ MeV} \\
 m_\lambda &= 460 \text{ MeV}
 \end{aligned} \tag{4.26}$$

we get

$$M_\Lambda - M_p = 160 \text{ MeV} \tag{4.27}$$

which is very reasonable. The origin of the D type coupling (which splits Σ and Λ) should in our opinion be traceable to the neglected unitarity corrections¹⁶⁾, which must also be responsible for the extremely intricate fine structure of the spectrum. It will require considerable effort to work this all out. For the decuplet we derive the equi-spacing rule with a mass spacing of 160 MeV, which also works quite well. Giving the charmed quark a mass $m_c = 1860 \text{ MeV}$ we obtain for the charmed baryon $(\frac{1}{2}^+, pnc)$:

$$m_{c^+} - m_p = 1360 \text{ MeV} \tag{4.28}$$

and for the baryon B made out of a hypothetical new quark with mass $m_H = 5.2 \text{ GeV}$ we get

$$m_B \approx 5.650 \text{ MeV} \quad (4.29)$$

5. - THE CURRENT MATRIX ELEMENTS AND THE MESON BARYON VERTEX

In order to complete our theory of baryons we must introduce the currents and the meson-baryon vertex ; this shall be done in this last section. We must stress that our proposal is still quite tentative due to the latitude of possible assumptions ; extensive calculations and comparison with experiments shall, we hope, pin down the adequate prescriptions.

5.1. - The direct current coupling

The problem we must face in introducing the direct coupling of the currents to the quarks constituting the baryons has to do with the "constrained" motion which, we have seen, characterizes our baryons in contrast with a naïve quark model. This implies that the definition of w.f. overlaps, which in the meson case was immediate, requires much more care.

The outstanding feature of the baryon w.f.'s derived in Section 3 is that the quarks span in the x variable a domain of vanishing extension which depends on the four-momentum p_μ of the baryon. Thus for different momenta the overlap vanishes, and in order to have a non-zero amplitude the quarks which comprise the "diquark system", must first tunnel out of the diquark bag. A natural candidate for this tunnelling amplitude is the qq Green's function (Fig. 5) which can be obtained from the $q\bar{q}$ Green's function discussed in the meson case ¹⁷⁾, whose singularities correspond to the meson physical states.

Thus in the configuration $i=1$, the overlap integral is represented in Fig. 6, where integration on the variables x_1, x_2, x_3, x_2^1 and x_3^1 is implied.

In momentum space the Green's function $G(k_2, k_3; k_2', k_3')$ can be shown to take the approximate form

$$G(k_2, k_3; k_2', k_3') \sim \lambda \delta^4(k_2 - k_2') \delta_{R^2}(k_2^2 - m_1^2) \delta_{R^2}(k_3^2 - m_3^2) \times (2\pi)^4 \delta^4(k_2 + k_3 - k_2' - k_3') \quad (5.1)$$

where λ is a numerical constant which can be determined, and $\delta_{R^2}(z)$ is the already introduced "fat δ function" $[\delta_{R^2} = 1/\pi (\sin R^2 z/z)]^{18}$. For each configuration i the direct current coupling can be expressed as the sum of the six contributions depicted in Fig. 7, where the legs with arrow indicate the application of the operator $(+i\gamma_1 + m_1)$ to the w.f. the arrow points towards ¹⁹. As an illustration, using (5.1), the overlap integral for the scalar part of the w.f. for the first two diagrams reads

$$\lambda \int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 q}{(2\pi)^4} \varphi_{p'}^{*(1)}(q - \sqrt{\frac{2}{3}}(p - p'), k) \varphi_p^{(0)}(q, k) \times \delta_{R^2}((\frac{p}{3} - \frac{1}{2}\sqrt{\frac{2}{3}}q + \frac{1}{2}k)^2 - m_2^2) \delta_{R^2}((\frac{p}{3} - \frac{1}{2}\sqrt{\frac{2}{3}}q - \frac{k}{2})^2 - m_3^2) \quad (5.2)$$

where $\varphi_p(q, k)$ is the Fourier transform of $\varphi_p(x, y)$, which is given by $[\tau = x_0$ in the rest frame $p \equiv (M, \vec{0})$

$$\varphi_p(x, y) = \int d\tau \delta^4(x - \frac{p}{M}\tau) \varphi_p(y; \tau). \quad (5.3)$$

One can show that when $Q^2 = (p - p')^2$ becomes large (5.2) behaves for real Q^2 as $(1/Q^2)^2$, giving rise to a "dipole" behaviour.

Conservation and partial conservation can be enforced for the currents by adding suitable surface terms, as discussed in Ref. 20).

5.2. - The meson-baryon vertex

For each configuration we have two possible contributions to the meson-baryon vertex, according to whether the meson is emitted by a quark or a diquark system. In the first case we have a diagram similar to the current

matrix element, with the pointlike current vertex replaced by a meson w.f., and the adjacent quark lines multiplied by the tunnelling amplitude discussed in Ref. 4). In the second case the qq Green's function is no more needed to insure a non-vanishing overlap, and we have the diagrams reported in Fig. 8 where the dotted lines indicate the multiplication by the quark tunnelling amplitude. The other two configurations ($i=2,3$) can be trivially obtained by appropriate permutations.

5.3. - The vector dominated current matrix element

Once we have defined the meson-baryon vertex we can immediately write the expression of the vector dominated piece of the current which corresponds to approximating the singularities of an unsubtracted dispersion relation by vector (axial vector) meson poles. The adequacy of such a description of the nucleon form factors is at present still unclear ²⁰⁾, it is possible that we shall have to change radically our views about baryon form factors and their analytic structure when a more precise knowledge of the vector meson spectrum and couplings shall be available.

n [SU(6) L^P]	1	2	3	4	5
(56, 0^+)	1.14	2.07	2.71	3.24	3.69
(70, 1^-)	1.60	2.39	2.97	3.46	3.89
$\begin{pmatrix} 56, 2^+ \\ 70, 2^+ \end{pmatrix}$	1.94	2.65	3.20	3.66	4.07
(70, 3^-)	2.23	2.89	3.41	3.85	4.24
$\begin{pmatrix} 56, 4^+ \\ 70, 4^+ \end{pmatrix}$	2.48	3.10	3.60	4.03	4.41
(70, 5^-)	2.70	3.30	3.78	4.19	4.56
$\begin{pmatrix} 56, 6^+ \\ 70, 6^+ \end{pmatrix}$	2.91	3.48	3.94	4.34	4.71
(70, 7^-)	3.10	3.65	4.10	4.49	
$\begin{pmatrix} 56, 8^+ \\ 70, 8^+ \end{pmatrix}$	3.27	3.81	4.25	4.64	

TABLE : The mean masses of the [SU(6), L^P] multiplets.

REFERENCES AND FOOTNOTES

- 1) G. Preparata and K. Szegö - Phys.Letters 68B, 239 (1977).
- 2) See, for instance : R. Morgan and R.H. Dalitz - Nuclear Phys. 66B, 135 (1973).
- 3) This is the picture arising in the MIT bag approach proposed by :
A. Chodos, R.L. Jaffe, K. Johnson, C.B. Thorn and V.F. Weisskopf -
Phys.Rev. D9, 3471 (1974).
- 4) G. Preparata - CERN Preprint TH. 2271 (1977), and also Proceedings
of the EPS Budapest Conference (1977) (to be published).
- 5) We are paralleling the treatment of the meson states carried out in
Ref. 4).
- 6) In the rest frame $B^8(p_i, x, y)$ must necessarily decompose into a closed
contour in the plane (x_0, y_0) and a rotationally invariant
surface in the six-dimensional space $(\vec{x}, \vec{y})$.
- 7) The notion of distance can be introduced as the norm of the differ-
ence ; i.e., as $\|\psi - \psi^{(0)}\|$ when $\|\psi\| = \|\psi^{(0)}\| = 1$. The norm
can be defined in a number of different ways ; a suitable defi-
nition is, for instance :
$$\|\psi\|^2 = \int_{R^8} (\bar{\psi}(x,y) \psi(x,y)) d^4x d^4y$$
- 8) D.B. Lichtenberg and L.J. Tassie - Phys.Rev. 155, 1601 (1967).
- 9) In this subsection we make ample use of the results of the interesting
paper by :
F. Buccella, A. Sciarrino and P. Sorba - Ann.Phys. 100, 536
(1976).
- 10) We are assuming for simplicity that R_s increases approximately
linearly with Q , as in the meson case [see Ref. 4].
- 11) Quantum mechanically, one can always interpret in this way a constrained
motion.
- 12) It turns out that we cannot enforce more symmetry without obtaining the
eigenvalue equation of the "free motion" (4.5), which cannot be
accepted in the presence of the symmetry breaking induced by the
different quark masses (see later).
- 13) The other configurations can be trivially obtained by the appropriate
permutation of co-ordinates.

- 14) According to the "constrained quark motion" $-i(\partial/\partial x_\mu)$ must be interpreted as $(p_\mu/M)(-i \partial/\partial \tau)$ where $\tau = x_0$ in the baryon rest frame.
- 15) P.J. Litchfield, R.J. Cashmore and A.J.G. Hey - "SU(6)_w and Decays of Baryon Resonances", Rutherford Laboratory Preprint 76-111 (1976).
- 16) For mesons, unitarity corrections have been computed in a few cases in : G. Fogli and G. Preparata - CERN Preprint TH. 2322 (1977).
- 17) See : G. Preparata - Nuclear Phys. B122, 29 (1977).
- 18) The parameter R^2 is tied with the asymptotic behaviour of the meson bag radii $R_{s,t} \rightarrow R^2 M$ for large mass, see Ref. 17).
- 19) The necessity of applying the operator $(+i\not{x} + m)$ can be determined by considering our theory in the context of a Bethe-Salpeter equation and introducing the electromagnetic coupling via a minimal substitution.
- 20) F. Csikor and G. Preparata - "Geometrodynamics for Quarks and Hadrons : The Definition of Currents", CERN Preprint TH 2396 (1977).
- 21) See, for example : G. Höhler - in Hadron Structure, Proceedings of the 2nd Adriatic Meeting, M. Martinis and N. Zovko Eds. (1976).

FIGURE CAPTIONS

<u>Figure 1</u>	The spectrum of the solutions of Eq. (3.3).
<u>Figure 2</u>	The spectrum of the solutions of Eq. (3.21).
<u>Figure 3</u>	The three inequivalent quark-diquark spatial configurations.
<u>Figure 4</u>	$[SU(6), L^P]$ pattern of physical baryon states.
<u>Figure 5</u>	The qq Green's function.
<u>Figure 6</u>	The w.f. overlap in the calculation of current matrix elements.
<u>Figure 7</u>	The diagrams contributing to the direct current coupling.
<u>Figure 8</u>	The diagrams contributing to the emission of a meson by a "diquark system".

FIG.1

FIG.2

FIG.3

FIG. 4

FIG.5

FIG. 6

FIG. 7

FIG. 8