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ON THE UNIQUE SOLUTION OF THE
QUANTUM MECHANICAL PROBLEM OF FOUR AND MORE PARTICLES

Comment on a letter by Komarov, Popova and Shablov

B.R. Karlsson

CERN -- Geneva

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A general formulation of the N body problem in quantum mechanics along the lines of the Faddeev three-body theory was first given by Yakubovskii¹⁾. An alternative formulation by Alt, Grassberger and Sandhas²⁾ for $N = 4$ was subsequently generalized to arbitrary N ³⁾ and was shown to be equivalent to that of Yakubovskii. Below we shall make use of the minimal hierarchy formulation as outlined in Ref. 3). This version is closely related to a wave function formulation, as shown in Ref. 4), in terms of which the N body theory is particularly simple and transparent.

We first recall that the Faddeev-Yakubovskii-type theories have a hierarchical structure in the sense that the N body equations cannot be solved until a hierarchy of subsystem equations has been solved. For instance, the four-body equations are formulated in terms of operators (or their matrix elements) defined in the subsystems of two non-interacting clusters (of the type 3+1 or 2+2). These operators can, in turn, be obtained from operators defined in the three non-interacting cluster subsystems, i.e., from two-body operators. In the 3+1 case this amounts to solving a three-body Faddeev-type equation, while in the 2+2 case a somewhat different equation is often quoted for the 2+2 subsystem operators in terms of the two-body t matrices.

Komarov, Popova and Shablov⁵⁾ have recently argued that this last equation might not have a unique solution, implying that the four- and N -body Faddeev-Yakubovskii-type equations would suffer from the same shortcoming.

It is the purpose of this note to point out that the equation considered in Ref. 5) is superfluous in the four-body theory (in very much the same way as the three-body Lippmann-Schwinger equation is superfluous in the three-body theory once the Faddeev equations have been written down). The investigation in Ref. 5) therefore has no implication for the possible (non)-uniqueness of the solutions to the four- and N -body equations in the Faddeev-Yakubovskii-type theories.

To see this in detail, consider the derivation of the four-body wave function equations. Starting with the Schrödinger equation, the four-body wave function is first split into six components, defined through

$$(H_0 - E) \psi_\beta = -V_\beta \psi \quad (1)$$

Clearly, $\sum_\beta \psi_\beta = \psi$ so that

$$(H_0 + V_\beta - E) \psi_\beta = -V_\beta \sum_\gamma \delta_{\beta\gamma} \psi_\gamma \quad (2)$$

where $\bar{\delta}_{\beta\gamma} = 1 - \delta_{\beta\gamma}$. The index β labels pairs, or more generally the different ways in which the four-body system can be split into three non-interacting clusters, e.g., (12)(3)(4). Let σ label the various ways in which the system can be split into two such clusters. σ can either be of the 3+1 type, e.g., (123)(4), or of the 2+2 type, e.g., (12)(34). The notation $\beta \subset \sigma$ or $\sigma \supset \beta$ is used to indicate that β is the result of a further splitting of σ [if σ is (123)(4), β can, for instance, be (12)(3)(4) but not (1)(2)(34)].

A further splitting of the wave function into 18 components can now be defined through

$$(H_0 + V_\beta - E)\psi_\beta^\sigma = -V_\beta \sum_{\gamma \subset \sigma} \bar{\delta}_{\beta\gamma} \psi_\gamma \quad (3)$$

and it is easy to verify that

$$\sum_{\sigma \supset \beta} \psi_\beta^\sigma = \psi_\beta.$$

It is convenient to write Eq. (3) in the form $(\psi^\sigma = \sum_{\beta \subset \sigma} \psi_\beta^\sigma)$

$$(H_0 - E)\psi_\beta^\sigma + V_\beta \psi^\sigma = -V_\beta \sum_{\gamma \subset \sigma} \bar{\delta}_{\beta\gamma} \sum_{\rho \supset \gamma} \bar{\delta}^{\sigma\rho} \psi_\rho^\rho \quad (4)$$

where $\bar{\delta}^{\sigma\rho} = 1 - \delta^{\sigma\rho}$ excludes the term $\rho = \sigma$ in the sum over ρ , and to sum over all $\beta \subset \sigma$ in (4),

$$(H_0 + \sum_{\beta \subset \sigma} V_\beta - E)\psi^\sigma = - \sum_{\beta, \gamma \subset \sigma} V_\beta \bar{\delta}_{\beta\gamma} \sum_{\rho \supset \gamma} \bar{\delta}^{\sigma\rho} \psi_\rho^\rho \quad (5)$$

Note that the operator on the left-hand side of this equation is the total Hamiltonian in the subsystem σ . Suppose now for definiteness that we want a solution to (4) corresponding to an initial state of two bound pairs, e.g., $\sigma_0 = (12)(34)$. Equation (5) can, in this case, be turned into an integral equation

$$\psi^\sigma = \phi^{\sigma_0} \delta^{\sigma\sigma_0} - G^\sigma \sum_{\beta, \gamma \subset \sigma} V_\beta \bar{\delta}_{\beta\gamma} \sum_{\rho \supset \gamma} \bar{\delta}^{\sigma\rho} \psi_\rho^\rho \quad (6)$$

where ϕ^{σ_0} is the initial state wave function (an eigenfunction of the Hamiltonian in σ_0), and where $G^\sigma = (H_0 + \sum_{\beta \subset \sigma} V_\beta - E - i0)^{-1}$ is the subsystem Green's

function. If Eq. (6) is used to replace ψ^σ in Eq. (4) we finally obtain
 $(G_0 = (H_0 - E - i0)^{-1})$

$$\psi_\beta^\sigma = \phi_\beta^\sigma \delta^{\sigma\sigma_0} - G_0 \sum_{\gamma < \sigma} K_{\beta\gamma}^\sigma \sum_{\gamma > \gamma} \bar{\delta}^{\sigma\gamma} \psi_\gamma^\sigma \quad (7)$$

where $\phi_\beta^\sigma \equiv -G_0 V_\beta \phi^\sigma$ is a "component" of ϕ^σ in the sense that $\sum_{\beta \in \sigma} \phi_\beta^\sigma = \phi^\sigma$, and where $K_{\beta\gamma}^\sigma$ is defined through

$$K_{\beta\gamma}^\sigma = \sum_{\lambda < \sigma} (V_\beta \delta_{\beta\lambda} - V_\beta G^\sigma V_\lambda) \bar{\delta}_{\lambda\gamma} \quad (8)$$

The equation (7) is the Faddeev-Yakubovskii equation for the four-body wave function components.

It now remains to discuss the subsystem operators $K_{\beta\gamma}^\sigma$. When σ is of the 3+1 type, $K_{\beta\gamma}^\sigma$ is a three-body operator for which the Faddeev equations take the form^{3),4)} ($\alpha, \beta, \gamma \in \sigma$)

$$K_{\beta\gamma}^\sigma = t_\beta \bar{\delta}_{\beta\gamma} - t_\beta G_0 \sum_{\lambda < \sigma} \bar{\delta}_{\beta\lambda} K_{\lambda\gamma}^\sigma \quad (9)$$

If on the other hand σ is of the 2+2 type, $K_{\beta\gamma}^\sigma$ can be obtained directly from the definition Eq. (8) using the fact that G^σ is diagonal in the basis $\{\vec{s}, \psi_{\vec{q}}, \psi_{\vec{q}}^\dagger\}$, where $\{\psi_{\vec{q}}\}$ and $\{\psi_{\vec{q}}^\dagger\}$ are the complete sets of bound and scattering states of the two pairs, and $\vec{s}$ is the momentum associated with the free relative motion of the pairs. With this observation, the construction of the four-body Faddeev-Yakubovskii theory is completed.

Where, then, does the equation considered in Ref. 5) come in? In most presentations of the four-body theory it is found convenient to use an integral equation identity for $K_{\beta\gamma}^\sigma$ not only in the $\sigma = 3+1$ case, as above, but also in the 2+2 case: formally one obtains again Eq. (9). This identity is the equation considered in detail by Komarov, Popova and Shablov, and found not to define $K_{\beta\gamma}^\sigma$ uniquely. However, since $K_{\beta\gamma}^\sigma$ is already uniquely defined by Eq. (8), this property of Eq. (9) in the 2+2 case is really of no consequence.

The above arguments have been given for the special case $N = 4$; suitably modified, they should apply to the general case $N > 4$ as well.

It might be added that the point of view emphasized in this note is not new; it was, for instance, exploited in the detailed formulation of the four-body theory in Ref. 6).

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6