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STRUCTURE OF CONFORMAL SUPERGRAVITY

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ABSTRACT

We give, at the linearized level, the supersymmetric generalization of various tensors of differential geometry, such as the metric tensor and the Riemann, Einstein and Weyl tensors. Using them we write the Lagrangian and the equations of motion of linearized supergravity and conformal supergravity with coupling to sources. Both theories are shown to be globally supersymmetric and invariant under the supersymmetric extension of Einstein transformations. Conformal supergravity is also invariant under supersymmetric Weyl transformations. The Hilbert space of this theory is shown to contain, in addition to physical particles, also dipole and tripole ghost states. Their transformation properties under supersymmetry are discussed.

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1. INTRODUCTION

Supergravity [1,2] and extended supergravity theories [3,4] provide a way of formulating unified theories encompassing all fundamental interactions. However attractive, this approach faces a number of difficulties. Supergravity theories with additional gauge couplings of the vector fields [5] have enormous cosmological terms^{*)}. Furthermore, the largest pure extended supergravity theory, which is the $SO(8)$ theory [4], is apparently too restrictive to be taken as a realistic theory: the group $SU(8)$ is too small [it does not contain the "observed" group $SU(2) \times U(1) \times SU(3)$ for flavour and colour] and the particle multiplets are also not large enough to accommodate all desired quarks, leptons and vector fields [7]. Perhaps the most attractive option left within this framework is $SU(4)$ supergravity coupled to $SU(4)$ supersymmetric matter, which may have as yet unsuspected convergence properties, in spite of the occurrence in it of two separate irreducible supermultiplets. Concerning its matter sector one already knows that the coupling constant renormalization in the $SO(4)$ supersymmetric Yang-Mills theory [8] is finite both in the one and two loop approximation [9].

An interesting class of models which may avoid some of the problems with extended supergravity theories is given by the conformal supergravity theories [10-12]. These are based on the gauging of the superconformal algebra [13,14] and should automatically involve larger internal symmetry groups, which are gauged without the appearance of cosmological terms. At the same time, since these models are conformally invariant, the Lagrangians for the spin 2 and spin 3/2 sectors contain higher derivatives, which leads to the appearance of ghosts in the theory. The conformal supergravity models have been only partially studied up to now, and it is not even known whether they are globally supersymmetric. In the present paper we study in detail the simplest of these models [with internal symmetry $U(1)$] at the linearized level. We show that it is globally supersymmetric as well as invariant under the supersymmetric generalization of Einstein and Weyl transformations. We also study the structure of the Hilbert space, which consists of physical particles, as well as dipole and tripole ghosts.

Our technique consists in extending, in a supersymmetric way, the concepts of Riemannian geometry at the linearized level. Metric, Riemann, Einstein and Weyl tensors are each extended to a supermultiplet. This is done both in terms of explicit components (Section 2) and in superfield notation (Section 3). Having

^{*)} For a possible way out of this difficulty see Ref. [6].

established these geometric concepts the Lagrangian and the equations of motion are easily written, both in explicit component form (Section 4) and in superfield notation (Section 3). The coupling to sources, described by a supercurrent multiplet or superfield [15], is also given. This geometric approach, which includes a minimum set of auxiliary fields, should be of more general utility when considering non-linear supergravity theories.

As a by-product of the geometric developments one can easily give the Lagrangian and the equations for ordinary linearized supergravity coupled to sources (Sections 3 and 4). The resulting superfield equations of motion for ordinary supergravity agree with those given by Ogievetsky and Sokatchev [16]. It is also easy to write additional quadratic invariants and to obtain in this way all possible supersymmetric and gauge invariant linearized gravity theories. In Section 3 we give the supersymmetric version of the Gauss-Bonnet theorem in superfield form. Our result agrees with and completes that of Ref. [17].

The spectrum of states of conformal supergravity involves ghosts, because of the higher derivatives in the Lagrangian. To investigate the situation we study in detail the simpler case of a scalar supermultiplet (chiral superfield) with a supersymmetric higher derivative Lagrangian (Section 5). Because all masses are equal there are no single ghosts but rather dipole and tripole ghosts, in addition to physical particles. To study the structure of the states we use a generalization of the method used by Froissart and Glaser [18] for the case of dipole ghosts. Dipole ghosts consist of two zero norm states, the scalar product of the two being different from zero. For the case of conformal supergravity the spectrum turns out to consist of a physical particle of spin 1, a dipole ghost of spin 2, and a tripole ghost of spin 3/2. The tripole ghost acts effectively like a dipole ghost plus a physical particle. So we have a physical supermultiplet of spins 1 and 3/2 and a dipole ghost supermultiplet of spins 3/2 and 2. In pure conformal supergravity there is no physical graviton state (this could be changed by taking a combination with ordinary supergravity and renouncing conformal invariance). Having seen how ghosts, as well as physical particles, arrange themselves in supermultiplets, provides a tool for the study of the representation content of extended superconformal gravity theories. A naïve counting of states, ignoring the zero norm states, leads to incorrect results. For instance, one can already see in this way that the Lagrangians given so far for extended conformal supergravity are incomplete.

The occurrence of ghosts is of course a serious problem for theories of the kind considered here. Whether it is possible to define a subspace of the complete Hilbert space in which the theory satisfies the required unitarity constraints is

totally unclear at present, but one may try to proceed in the way suggested by Heisenberg [19] in a different context. Even if one succeeds, the resulting theory will have unorthodox causality properties [19].

2. MULTIPLY STRUCTURE OF SUPERGRAVITY THEORIES

In this section we consider several supermultiplets which contain the basic entities needed in the construction of supergravity theories. In particular we shall exhibit the transformation properties under global supersymmetry as well as under gauge transformations of various geometric objects such as the metric tensor, the Riemann tensor and the Einstein tensor. Particular attention will be devoted to the supermultiplet which contains the Weyl conformal tensor. This multiplet is the basic quantity for the construction of any conformal supergravity theory. In the present section we shall present an analysis in terms of component fields but in Section 3 a complete superfield formulation will be presented.

We follow the conventions and definitions of Refs. [13,20]. We call vector supermultiplet that associated with a general real superfield, and use the standard notation $C, \chi_\alpha, M, N, v_\nu, \lambda_\alpha, D$ for its components. Scalar multiplets, corresponding to chiral superfields, will be considered as well and their components denoted by A, B, ψ, F, G . If the scalar or the vector multiplet carries an overall Lorentz index, each component will also carry it as an additional index which does not modify the supersymmetry transformations.

In gravity the basic field is the symmetric tensor $h_{\mu\nu}$, related to the metric tensor by

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu} \quad , \quad (2.1)$$

where κ is the gravitational constant. It belongs to a vector multiplet V_μ with an additional vector which we call the potential. Let us denote its components by $C_\mu, \chi_{\alpha\mu}, M_\mu, N_\mu, v_{\nu\mu}, \lambda_{\alpha\mu}$ and D_μ and split $v_{\nu\mu}$ into its symmetric part $s_{\nu\mu}$ and its antisymmetric part $a_{\nu\mu}$:

$$v_{\nu\mu} = s_{\nu\mu} + a_{\nu\mu} \quad . \quad (2.2)$$

We can define the new fields

$$\begin{aligned}
 h_{\mu\nu} &= s_{\mu\nu} - \frac{1}{3} \eta_{\mu\nu} s_\rho{}^\rho, \\
 \psi_\mu &= \lambda_\mu - \frac{1}{3} \gamma_\mu \gamma \cdot \lambda - \frac{1}{3} \gamma_\mu \partial \cdot \chi, \\
 b_\mu &= D_\mu + \partial_\mu \partial \cdot C + \partial^\nu \hat{a}_{\mu\nu},
 \end{aligned} \tag{2.3}$$

where $\hat{a}_{\mu\nu}$ is the dual tensor

$$\hat{a}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu}{}^{\lambda\rho} a_{\lambda\rho}, \quad \varepsilon_{0123} = 1. \tag{2.4}$$

It is easy to see that $h_{\mu\nu}$ and ψ_μ transform into each other under a global supersymmetry transformation as dictated by the usual supergravity transformations taken to lowest order in κ . As explained below, a supersymmetric Einstein gauge transformation operates on the fields of the vector potential multiplet as

$$\begin{aligned}
 \delta C_\mu &= \tilde{C}_\mu, \\
 \delta \chi_\mu &= \tilde{\chi}_\mu, \\
 \delta M_\mu &= \tilde{M}_\mu, \quad \partial \cdot \tilde{M} = 0, \\
 \delta N_\mu &= \tilde{N}_\mu, \quad \partial \cdot \tilde{N} = 0, \\
 \delta a_{\nu\mu} &= \tilde{a}_{\nu\mu}, \quad \tilde{a}_{\mu\nu} + \tilde{a}_{\nu\mu} = 0, \\
 \delta h_{\nu\mu} &= \partial_\nu \tilde{\xi}_\mu + \partial_\mu \tilde{\xi}_\nu, \\
 \delta \psi_\mu &= \partial_\mu \tilde{\psi}, \\
 \delta b_\mu &= 0,
 \end{aligned} \tag{2.5}$$

where the functions occurring in the right-hand side are arbitrary except for the restrictions explicitly written. Therefore, by choosing a special gauge all the components of the vector potential multiplet can be set equal to zero except for $\partial \cdot M$, $\partial \cdot N$, $h_{\nu\mu}$, ψ_μ and b_μ . One sees that the fields $h_{\mu\nu}$ and ψ_μ undergo the

usual linearized Einstein and Rarita-Schwinger gauge transformations. Actually, it will turn out that the gauge invariant quantities $\partial \cdot M$, $\partial \cdot N$ and b_μ vanish on the mass shell when $h_{\mu\nu}$ and ψ_μ satisfy the linearized Einstein and Rarita-Schwinger equations, respectively. It is interesting to observe that supersymmetry unifies the Einstein transformations, of parameter functions ξ_μ , with the local Lorentz transformations, of parameter functions $\tilde{a}_{\mu\nu}$.

A supersymmetric combined Einstein and Weyl gauge transformation operates on the vector potential multiplet as

$$\begin{aligned}
 \delta C_\mu &= \tilde{C}_\mu \quad , \\
 \delta \chi_\mu &= \tilde{\chi}_\mu \quad , \\
 \delta M_\mu &= \tilde{M}_\mu \quad , \\
 \delta N_\mu &= \tilde{N}_\mu \quad , \\
 \delta a_{\nu\mu} &= \tilde{a}_{\nu\mu} \quad , \quad \tilde{a}_{\mu\nu} + \tilde{a}_{\nu\mu} = 0 \quad , \\
 \delta h_{\nu\mu} &= \partial_\nu \xi_\mu + \partial_\mu \xi_\nu + \eta_{\mu\nu} \rho \quad , \\
 \delta \psi_\mu &= \partial_\mu \Psi + \gamma_\mu \chi \quad , \\
 \delta b_\mu &= \partial_\mu \varphi \quad .
 \end{aligned} \tag{2.6}$$

We see that (2.5) can be obtained from (2.6) by imposing some restrictions on the gauge multiplet. These restrictions, which have a very simple form in superspace, will be discussed in a moment in terms of components. We notice that supersymmetry unifies the Weyl transformations, of parameter function ρ , with the electromagnetic type gauge transformations, of parameter function φ .

In order to clarify the supersymmetric structure of the gauge transformations (2.5) and (2.6) we observe that the Weyl gauge multiplet occurring on the right-hand side of (2.6) can be constructed in terms of a vector multiplet Λ_α with an additional Majorana spinor index. Let its components be C_α , $\chi_{\beta\alpha}$, M_α , N_α , $v_{\mu\alpha}$, $\lambda_{\beta\alpha}$, D_α . The bispinors $\chi_{\beta\alpha}$ and $\lambda_{\beta\alpha}$ can be expressed in terms of tensors

$$\begin{aligned}\chi_{\beta\alpha} &= i \left(\gamma^0 A + \gamma_5 \gamma^0 B + \gamma_\mu \gamma^0 A^\mu + \gamma_5 \gamma_\mu \gamma^0 B^\mu + \Sigma_{\mu\nu} \gamma^0 A^{\mu\nu} \right)_{\beta\alpha}, \\ \lambda_{\beta\alpha} &= i \left(\gamma^0 F + \gamma_5 \gamma^0 G + \gamma_\mu \gamma^0 F^\mu + \gamma_5 \gamma_\mu \gamma^0 G^\mu + \Sigma_{\mu\nu} \gamma^0 F^{\mu\nu} \right)_{\beta\alpha}, \quad (2.7) \\ \Sigma_{\mu\nu} &= \frac{1}{4} [\gamma_\mu, \gamma_\nu].\end{aligned}$$

From Λ_α one can construct a real vector multiplet ($i \bar{D}\gamma_5 \gamma_\mu \Lambda$ in superfield language)

$$\begin{aligned}\tilde{C}_\nu &= 4 B_\nu, \\ \tilde{\chi}_{\alpha\nu} &= - \left[\gamma^\mu \gamma_\nu (\psi_\mu + \gamma_5 \partial_\mu C) + \gamma_\nu (M - \gamma_5 N) \right]_\alpha, \\ \tilde{M}_\nu &= -4 F_\nu, \\ \tilde{N}_\nu &= 4 G_\nu, \\ \tilde{\psi}_{\mu\nu} &= \tilde{S}_{\mu\nu} + \tilde{a}_{\mu\nu}, \\ \tilde{\lambda}_\nu &= \gamma_\nu \tilde{\chi} + \partial_\nu \psi, \\ \tilde{D}_\nu &= 4 \partial_\nu G - \partial_\nu \partial \cdot \tilde{C} - \partial^\lambda \hat{a}_{\nu\lambda},\end{aligned} \quad (2.8)$$

where $\hat{a}_{\nu\lambda}$ is the dual of $\tilde{a}_{\nu\lambda}$ and

$$\begin{aligned}\tilde{S}_{\mu\nu} &= 2 (\partial_\mu A_\nu - \partial_\nu A_\mu) - 4 \eta_{\mu\nu} (F + \partial \cdot A), \\ \tilde{a}_{\mu\nu} &= 4 F_{\nu\mu} + 4 \varepsilon_{\nu\mu\lambda\rho} \partial^\rho B^\lambda + 2 (\partial_\nu A_\mu - \partial_\mu A_\nu), \\ \psi &= 2 (M - \gamma_5 N), \\ \tilde{\chi}_\alpha &= \left[\gamma_5 D + \partial^\mu (\psi_\mu + \gamma_5 \partial_\mu C) - \gamma \cdot \partial (M - \gamma_5 N) \right]_\alpha.\end{aligned} \quad (2.9)$$

Observe that only the combinations $(\psi_\mu + \gamma_5 \partial_\mu C)_\alpha$ and $(M - \gamma_5 N)_\alpha$ enter in the gauge multiplet defined by (2.8) while the fields $A, B, A_{\mu\nu}$ do not enter at all. (In superspace language this means that the gauge transformation is unchanged if one changes Λ_α into $\Lambda_\alpha + \Lambda'_\alpha$, where Λ'_α is a chiral superfield, $\bar{D}_\alpha \Lambda'_\alpha = 0$). From the above formulas one can show that the parameters of the Weyl, Einstein, Rarita-Schwinger and Maxwell gauge transformations are related to the components of the spinor superfield by

$$\begin{aligned}
 \tilde{\xi}_\mu &= 2 A_\mu \quad , \\
 \rho &= \frac{4}{3} F \quad , \\
 \psi_\alpha &= 2 (M - \gamma_5 N)_\alpha \quad , \\
 \chi_\alpha &= -\frac{1}{3} (\gamma_5 D + 2 \sum^{\mu\nu} \partial_\mu v_\nu)_\alpha \quad , \\
 \varphi &= 4 G \quad .
 \end{aligned} \tag{2.10}$$

The fields $\tilde{C}_\nu, \tilde{\chi}_\nu, \tilde{M}_\nu, \tilde{N}_\nu, \tilde{a}_{\mu\nu}, \tilde{\xi}_\mu, \rho, \psi, \chi, \varphi$ form an irreducible multiplet.

In order to obtain the Einstein gauge multiplet (2.5) from the Weyl gauge multiplet we must impose the constraints

$$\rho = \varphi = \partial^\mu \tilde{M}_\mu = \partial^\mu \tilde{N}_\mu = \chi = 0 \quad . \tag{2.11}$$

which are equivalent to the supersymmetric conditions

$$\partial^\mu F_\mu = \partial^\mu G_\mu = F = G = (\gamma_5 D + 2 \sum^{\mu\nu} \partial_\mu v_\nu)_\alpha = 0 \tag{2.12}$$

on the components of the multiplet Λ_α .

From the vector potential supermultiplet V_μ one can construct another vector multiplet E_μ which we call the Einstein multiplet. It contains the Einstein tensor and the Rarita-Schwinger field strength and is invariant under an Einstein gauge transformation (2.5). Its components are

$$\begin{aligned}
 C_{1\mu} &= b_\mu \quad , \\
 \chi_{1\mu} &= \frac{3}{2} \left(R_\mu - \frac{1}{3} \gamma_\mu \gamma \cdot R \right) \quad ,
 \end{aligned}$$

$$\begin{aligned}
 M_{1\mu} &= \partial_\mu \partial \cdot M \quad , \\
 N_{1\mu} &= \partial_\mu \partial \cdot N \quad , \\
 a_{1\nu\mu} &= -\frac{1}{2} \varepsilon_{\nu\mu}{}^{\sigma\rho} \partial_\sigma b_\rho \equiv \frac{1}{2} \hat{b}_{\mu\nu} \quad , \quad b_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu \quad (2.13) \\
 s_{1\nu\mu} &= -3 \left(E_{\mu\nu} - \frac{1}{3} \eta_{\mu\nu} E_\rho{}^\rho \right) \quad , \\
 \lambda_{1\mu} &= \partial_\mu \gamma \cdot \chi_1 - \gamma \cdot \partial \chi_{1\mu} \quad , \\
 D_{1\mu} &= \partial^\nu b_{\mu\nu} \quad .
 \end{aligned}$$

Here

$$R_\mu = \varepsilon_{\mu\nu\rho\sigma} \gamma_5 \gamma^\nu \partial^\rho \psi^\sigma = \gamma_5 \gamma^\nu \hat{\psi}_{\mu\nu} \quad (2.14)$$

is the Rarita-Schwinger operator, with $\hat{\psi}_{\mu\nu} = \partial_\mu \psi_\nu - \partial_\nu \psi_\mu$. The linearized

$$\begin{aligned}
 E_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R \\
 &= \frac{1}{2} \left(\partial_\mu \partial_\nu h_\lambda{}^\lambda + \square h_{\mu\nu} - \partial_\mu \partial^\lambda h_{\lambda\nu} - \partial_\nu \partial^\lambda h_{\lambda\mu} - \eta_{\mu\nu} \square h_\lambda{}^\lambda + \eta_{\mu\nu} \partial^\lambda \partial^\rho h_{\lambda\rho} \right) \quad (2.15)
 \end{aligned}$$

is constructed, as well known, from the contractions

$$R_{\mu\rho} = R_{\lambda\mu}{}^\lambda{}_\rho \quad , \quad R = R_\mu{}^\mu \quad , \quad (2.16)$$

of the Riemann tensor

$$R_{\lambda\mu\nu\rho} = \frac{1}{2} \left(\partial_\rho \partial_\mu h_{\lambda\nu} - \partial_\rho \partial_\lambda h_{\mu\nu} - \partial_\nu \partial_\mu h_{\lambda\rho} + \partial_\nu \partial_\lambda h_{\mu\rho} \right) . \quad (2.17)$$

One can rewrite (2.13) in a way which exhibits the physically interesting quantities as

$$\begin{aligned}
 \delta M &= -\frac{i}{2} \bar{\alpha} \gamma \cdot R \quad , \\
 \delta N &= -\frac{i}{2} \bar{\alpha} \gamma_5 \gamma \cdot R \quad , \\
 \delta R_\mu &= -2 \gamma^\lambda E_{\lambda\mu} \alpha + \frac{2}{3} (\gamma \cdot \partial b_\mu - \gamma_\mu \partial \cdot b) \gamma_5 \alpha \\
 &\quad + \frac{1}{3} \gamma^\lambda \hat{b}_{\mu\lambda} \alpha + \frac{2}{3} [\partial_\mu (M + \gamma_5 N) - \gamma_\mu \gamma \cdot \partial (M + \gamma_5 N)] \alpha \quad , \\
 \delta b_\mu &= \frac{3}{2} i \bar{\alpha} \gamma_5 (R_\mu - \frac{1}{3} \gamma_\mu \gamma \cdot R) \quad , \\
 \delta E_{\lambda\mu} &= \frac{i}{4} \bar{\alpha} (\gamma_\lambda \gamma \cdot \partial R_\mu + \gamma_\mu \gamma \cdot \partial R_\lambda - \partial_\lambda R_\mu - \partial_\mu R_\lambda) \quad ,
 \end{aligned} \tag{2.18}$$

where $M = \partial \cdot M$ and $N = \partial \cdot N$. From the Einstein multiplet one can obtain a scalar multiplet R which contains the curvature scalar. Its components A, B, ψ, F and G are given by $M, N, \gamma \cdot R, R, \partial \cdot b$, respectively. As we shall see in Section 4, the equations of motion for free supergravity are equivalent to the vanishing of the Einstein multiplet. In addition to the linearized Einstein and Rarita-Schwinger equations this implies the vanishing of b_μ and, in a special gauge, of M_μ and N_μ .

The Riemann tensor also belongs to a multiplet, which we call the Riemann multiplet, of components $M, N, \psi_{\mu\nu}, b_\mu$ and $R_{\lambda\mu\nu\rho}$. It is Einstein gauge invariant. Under global supersymmetry it transforms as

$$\begin{aligned}
 \delta M &= \frac{i}{2} \bar{\alpha} \gamma^\nu \gamma^\mu \psi_{\mu\nu} \quad , \\
 \delta N &= \frac{i}{2} \bar{\alpha} \gamma_5 \gamma^\nu \gamma^\mu \psi_{\mu\nu} \quad , \\
 \delta \psi_{\mu\nu} &= -R_{\nu\mu\rho\lambda} \gamma^\lambda \gamma^\rho \alpha + b_{\mu\nu} \gamma_5 \alpha - \frac{1}{3} (\gamma_\mu \partial_\nu - \gamma_\nu \partial_\mu) b_\lambda \gamma_5 \gamma^\lambda \alpha \\
 &\quad + \frac{1}{3} (\gamma_\mu \partial_\nu - \gamma_\nu \partial_\mu) (M + \gamma_5 N) \alpha \quad , \\
 \delta R_{\lambda\mu\nu\rho} &= -\frac{i}{4} \bar{\alpha} [(\gamma_\lambda \partial_\mu - \gamma_\mu \partial_\lambda) \psi_{\nu\rho} + (\gamma_\nu \partial_\rho - \gamma_\rho \partial_\nu) \psi_{\lambda\mu}] \quad , \\
 \delta b_\mu &= \frac{3}{2} i \bar{\alpha} \gamma_5 (R_\mu - \frac{1}{3} \gamma_\mu \gamma \cdot R) \quad .
 \end{aligned} \tag{2.19}$$

Observe the identity $\gamma \cdot R = \gamma^\nu \gamma^\mu \psi_{\mu\nu}$. Clearly the transformation law (2.18) for the Einstein multiplet can be derived from (2.19).

We now come to the Weyl multiplet. It contains the Weyl conformal tensor and is invariant under the larger group of combined Einstein and Weyl transformations given by (2.6). The linearized Weyl tensor is defined as

$$C_{\lambda\mu\nu\rho} = R_{\lambda\mu\nu\rho} - \frac{1}{2} (\eta_{\lambda\nu} R_{\mu\rho} - \eta_{\mu\nu} R_{\lambda\rho} - \eta_{\lambda\rho} R_{\mu\nu} + \eta_{\mu\rho} R_{\lambda\nu}) \\ + \frac{1}{6} R (\eta_{\lambda\nu} \eta_{\mu\rho} - \eta_{\mu\nu} \eta_{\lambda\rho}) \quad (2.20)$$

and is invariant under a linearized Weyl transformation $\delta h_{\mu\nu} = \eta_{\mu\nu} \rho$. It is completely traceless, e.g., $C_{\lambda\mu}{}^{\lambda\rho} = 0$, and satisfies the usual identities

$$\partial^\lambda C_{\lambda\mu\nu\rho} = \frac{1}{2} \left[\partial_\nu (E_{\mu\rho} - \frac{1}{3} \eta_{\mu\rho} E_\lambda{}^\lambda) - \partial_\rho (E_{\mu\nu} - \frac{1}{3} \eta_{\mu\nu} E_\lambda{}^\lambda) \right] \quad (2.21)$$

and

$$\varepsilon^{\mu\nu\rho\sigma} \partial_\nu \partial^\tau C_{\tau\lambda\rho\sigma} = 0 \quad (2.22)$$

The Weyl multiplet $W_{\mu\nu\alpha} = -W_{\nu\mu\alpha}$ is a scalar supermultiplet with the additional Lorentz indices of a skewsymmetric tensor and of a Majorana spinor. Its components will be denoted $\Phi_{\mu\nu}$, $C_{\mu\nu\rho\sigma}$, $b_{\mu\nu}$, $\Delta_{\mu\nu}$. Only the two tensor spinors $\Phi_{\mu\nu}$ and $\Delta_{\mu\nu}$ have not yet been defined. They are given, in terms of the Rarita-Schwinger field strength $\psi_{\mu\nu} = \partial_\mu \psi_\nu - \partial_\nu \psi_\mu$, by

$$\Phi_{\mu\nu} = \frac{1}{3} \left(-\gamma_5 \hat{\psi}_{\mu\nu} + \psi_{\mu\nu} + \Sigma_\nu{}^\lambda \psi_{\lambda\mu} - \Sigma_\mu{}^\lambda \psi_{\lambda\nu} \right) \quad (2.23)$$

and

$$\Delta_{\mu\nu} = -\Xi_{\mu\nu} + \gamma_5 \hat{\Xi}_{\mu\nu} \quad , \quad (2.24)$$

where

$$\Xi_{\mu\nu} = \frac{1}{2} (\gamma \cdot \partial \psi_{\nu\mu} + \gamma_\mu \partial^\lambda \psi_{\lambda\nu} - \gamma_\nu \partial^\lambda \psi_{\lambda\mu}) + \gamma_5 \gamma \cdot \partial \hat{\psi}_{\mu\nu} \quad . \quad (2.25)$$

We note the following identities (not independent of each other)

$$\hat{\Phi}_{\mu\nu} = \gamma_5 \Phi_{\mu\nu} \quad , \quad \gamma^\mu \Phi_{\mu\nu} = 0 \quad , \quad (2.26)$$

$$\gamma_\lambda \Phi_{\mu\nu} + \gamma_\mu \Phi_{\nu\lambda} + \gamma_\nu \Phi_{\lambda\mu} = 0 \quad .$$

Relations identical to these are also satisfied by $\Delta_{\mu\nu}$. Also

$$3 \gamma \cdot \partial \Phi_{\mu\nu} = \Xi_{\mu\nu} + \gamma_5 \hat{\Xi}_{\mu\nu} \quad , \quad (2.27)$$

$$\partial^\mu \Xi_{\mu\nu} = 0 \quad ,$$

and therefore

$$\begin{aligned} \partial^\mu \Delta_{\mu\nu} &= 3 \gamma \cdot \partial \partial^\mu \Phi_{\mu\nu} \quad , \\ \gamma^\mu \Xi_{\mu\nu} &= 3 \partial^\mu \Phi_{\mu\nu} \quad , \\ \gamma^\mu \gamma^\nu \Xi_{\mu\nu} &= 0 \quad . \end{aligned} \quad (2.28)$$

Observe that if one introduces a matrix $T_{\mu\nu\rho\tau}$ such that $3\Phi_{\mu\nu} = T_{\mu\nu\rho\tau}\psi^{\rho\tau}$, one has $\Delta_{\mu\nu} = T_{\mu\nu\rho\tau}\gamma^{\rho}\partial\psi^{\sigma\tau}$. The spinors $\Phi_{\mu\nu}$, $\Xi_{\mu\nu}$ and $\Delta_{\mu\nu}$ are all invariant under the transformation $\delta\psi_{\mu} = \gamma_{\mu}\chi$.

Finally we record the transformation properties of the Weyl multiplet

$$\begin{aligned}\delta\Phi_{\mu\nu} &= C_{\mu\nu\lambda\rho}\gamma^{\rho}\gamma^{\lambda}\alpha + \frac{1}{3}(b_{\mu\nu}\gamma_5\alpha + \hat{b}_{\mu\nu}\alpha \\ &\quad + \Sigma_{\nu}{}^{\lambda}\hat{b}_{\lambda\mu}\alpha - \Sigma_{\mu}{}^{\lambda}\hat{b}_{\lambda\nu}\alpha) \\ \delta b_{\mu\nu} &= 3i\bar{\alpha}\gamma_5\gamma\cdot\partial\Phi_{\mu\nu} - i\bar{\alpha}\gamma_5\Xi_{\mu\nu} \\ \delta C_{\lambda\mu\nu\rho} &= \frac{i}{4}[\bar{\alpha}(\gamma_{\lambda}\partial_{\mu} - \gamma_{\mu}\partial_{\lambda})\Phi_{\rho\nu} + \bar{\alpha}(\gamma_{\nu}\partial_{\rho} - \gamma_{\rho}\partial_{\nu})\Phi_{\mu\lambda}] \\ &\quad - \frac{i}{6}(\bar{\alpha}\Sigma_{\lambda\mu}\Xi_{\rho\nu} + \bar{\alpha}\Sigma_{\nu\rho}\Xi_{\mu\lambda})\end{aligned}\tag{2.29}$$

$$\begin{aligned}\delta\Xi_{\mu\nu} &= 3\varepsilon_{\mu\nu\rho\sigma}\partial_{\tau}C^{\tau\lambda\rho\sigma}\gamma_{\lambda}\gamma_5\alpha \\ &\quad + \frac{1}{2}(\gamma\cdot\partial b_{\nu\mu} + \gamma_{\mu}\partial^{\rho}b_{\rho\nu} - \gamma_{\nu}\partial^{\rho}b_{\rho\mu})\gamma_5\alpha + \gamma\cdot\partial\hat{b}_{\mu\nu}\alpha.\end{aligned}$$

Note that $\Delta_{\mu\nu}$, and not $\Xi_{\mu\nu}$, is the last component of the chiral superfield $W_{\mu\nu\alpha}$; so, in order to establish a closer relation with superfields, it is convenient to write the variations of $C_{\lambda\mu\nu\rho}$ and $b_{\mu\nu}$ as follows

$$\begin{aligned}\delta C_{\lambda\mu\nu\rho} &= \frac{i}{4}\bar{\alpha}\gamma\cdot\partial(\Sigma_{\lambda\mu}\Phi_{\nu\rho} + \Sigma_{\nu\rho}\Phi_{\lambda\mu}) \\ &\quad - \frac{i}{12}\bar{\alpha}(\Sigma_{\lambda\mu}\Delta_{\nu\rho} + \Sigma_{\nu\rho}\Delta_{\lambda\mu}) \\ \delta b_{\mu\nu} &= \frac{3}{2}i\bar{\alpha}\gamma_5\gamma\cdot\partial\Phi_{\mu\nu} + \frac{i}{2}\bar{\alpha}\gamma_5\Delta_{\mu\nu}\end{aligned}\tag{2.30}$$

3. SUPERFIELD FORMULATION OF LINEARIZED SUPERGRAVITY AND CONFORMAL SUPERGRAVITY

In this section we give a formulation in terms of superfields. We use two component spinor notation. Our conventions are explained in Refs. [15, 21, 22] where one finds also a number of useful identities.

The supermultiplet which contains the spin 2 and the spin 3/2 fields is described by a superfield $V_{\alpha\dot{\beta}}$ which is a Hermitean 2×2 matrix in the indices α and $\dot{\beta}$. It corresponds to the potential supermultiplet of Section 2. From it one can construct by differentiation a superfield

$$\begin{aligned} E_{\alpha\dot{\beta}} = & 2DD\bar{D}\bar{D}V_{\alpha\dot{\beta}} + 4i\partial_{\gamma\dot{\gamma}}D^{\gamma}\bar{D}^{\dot{\gamma}}V_{\alpha\dot{\beta}} \\ & - 4i\partial_{\gamma\dot{\gamma}}D^{\gamma}\bar{D}_{\dot{\beta}}V_{\alpha\dot{\gamma}} - 4i\partial_{\alpha\dot{\alpha}}D^{\gamma}\bar{D}_{\dot{\gamma}}V_{\gamma\dot{\beta}} - 8\partial_{\alpha\dot{\beta}}\partial^{\gamma\dot{\gamma}}V_{\gamma\dot{\gamma}} . \end{aligned} \quad (3.1)$$

It can be checked that $E_{\alpha\dot{\beta}}$ is Hermitean, when $V_{\alpha\dot{\beta}}$ is. We recall the definitions

$$\begin{aligned} D_{\alpha} = \frac{\partial}{\partial\theta^{\alpha}} + i\partial_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}} , \quad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^{\beta}\partial_{\beta\dot{\alpha}} \\ \partial_{\alpha\dot{\beta}} = (\sigma^{\mu})_{\alpha\dot{\beta}}\partial_{\mu} , \end{aligned} \quad (3.2)$$

$$DD = D^{\alpha}D_{\alpha} , \quad \bar{D}\bar{D} = \bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}$$

Among the components of $E_{\alpha\dot{\beta}}$ one finds the Einstein tensor. It corresponds to the Einstein supermultiplet of Section 2.

We also define the chiral superfield

$$\begin{aligned} R^{*} = 2iDD\partial^{\alpha\dot{\beta}}V_{\alpha\dot{\beta}} , \\ R = -2i\bar{D}\bar{D}\partial^{\alpha\dot{\beta}}V_{\alpha\dot{\beta}} , \end{aligned} \quad D_{\alpha}R^{*} = 0 . \quad (3.3)$$

Among its components one finds the totally contracted curvature tensor. It is not hard to verify the identity

$$\mathcal{D}^\alpha E_{\alpha\dot{\beta}} = \bar{\mathcal{D}}_{\dot{\beta}} R^* \quad (3.4)$$

and its complex conjugate. They, in turn, imply

$$\bar{\mathcal{D}}_{\dot{\beta}} \mathcal{D} \mathcal{D} E_{\gamma\dot{\tau}} = 4i \partial_{\gamma\dot{\tau}} \mathcal{D}^\alpha E_{\alpha\dot{\beta}} \quad (3.5)$$

and

$$\partial_\alpha{}^{\dot{\tau}} \mathcal{D}^\alpha E_{\gamma\dot{\tau}} = -\partial_\gamma{}^{\dot{\tau}} \mathcal{D}^\alpha E_{\alpha\dot{\tau}} \quad (3.6)$$

The gauge transformation

$$V_{\alpha\dot{\beta}} \longrightarrow V_{\alpha\dot{\beta}} + \bar{\mathcal{D}}_{\dot{\beta}} \Lambda_\alpha - \mathcal{D}_\alpha \bar{\Lambda}_{\dot{\beta}} \quad (3.7)$$

is the superspace equivalent of the combined Weyl and Einstein transformations and corresponds to (2.6). Under (3.7) $E_{\alpha\dot{\beta}}$ and R^* transform as

$$E_{\alpha\dot{\beta}} \longrightarrow E_{\alpha\dot{\beta}} - 2i \partial_{\alpha\dot{\beta}} (\bar{\mathcal{D}} \mathcal{D} \mathcal{D}^\gamma \Lambda_\gamma - \mathcal{D} \mathcal{D} \bar{\mathcal{D}}_{\dot{\gamma}} \bar{\Lambda}^{\dot{\gamma}}) \quad (3.8)$$

and

$$R^* \longrightarrow R^* - \frac{1}{2} \mathcal{D} \mathcal{D} \bar{\mathcal{D}} \mathcal{D}^\alpha \Lambda_\alpha \quad (3.9)$$

Therefore, if Λ_α satisfies the restriction

$$\bar{D}\bar{D}D^\alpha\Lambda_\alpha = 0 \quad , \quad (3.10)$$

both $E_{\alpha\beta}$ and R^* are invariant. Gauge transformations (3.7) with the restriction (3.10) are the supersymmetric generalization of Einstein transformations and correspond to (2.5).

The geometry is completed by the introduction of the chiral superfield

$$W_{\alpha\beta\gamma} = \bar{D}\bar{D}\left[D_\alpha\partial_\beta{}^{\dot{\tau}}V_{\dot{\gamma}\dot{\tau}} + (\alpha\beta\gamma)\right] \quad , \quad (3.11)$$

$$\bar{D}_{\dot{\tau}}W_{\alpha\beta\gamma} = 0 \quad .$$

Here $(\alpha\beta\gamma)$ denotes the five additional terms which make $W_{\alpha\beta\gamma}$ totally symmetric in the three indices. The superfield (3.11) is invariant under the gauge transformation (3.7) without need for the restriction (3.10). In other words, it is invariant under the supersymmetric generalization of both Weyl and Einstein transformations. It is not difficult to verify the identity

$$D^\alpha W_{\alpha\beta\gamma} = \partial_\beta{}^{\dot{\tau}}E_{\dot{\gamma}\dot{\tau}} + \partial_\gamma{}^{\dot{\tau}}E_{\beta\dot{\tau}} \quad . \quad (3.12)$$

In order to identify the component fields we expand $W_{\alpha\beta\gamma}$ in "basis one", where the covariant derivative $\bar{D}_{\dot{\alpha}} \equiv -\partial/\partial\bar{\theta}^{\dot{\alpha}}$:

$$W_{\beta\gamma\delta} = \Phi_{\beta\gamma\delta} + \theta^\alpha W_{\alpha\beta\gamma\delta} + \theta^\alpha\theta_\alpha \Delta_{\beta\gamma\delta} \quad . \quad (3.13)$$

The multispinor $W_{\alpha\beta\gamma\delta}$, symmetric in $\beta\gamma\delta$ can be decomposed as

$$W_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + \varepsilon_{\alpha\beta} b_{\gamma\delta} + \varepsilon_{\alpha\gamma} b_{\delta\beta} + \varepsilon_{\alpha\delta} b_{\beta\gamma} \quad , \quad (3.14)$$

where the totally symmetric spinor $C_{\alpha\beta\gamma\delta}$ is the spinorial equivalent of the Weyl conformal tensor^{*)}. The symmetric spinor $b_{\gamma\delta}$ corresponds to the anti-symmetric tensor $b_{\mu\nu}$ of Section 2 and the symmetric spinors $\phi_{\beta\gamma\delta}$ and $\Delta_{\beta\gamma\delta}$ correspond to the antisymmetric tensor spinors $\phi_{\mu\nu}$ and $\Delta_{\mu\nu}$ of that same section.

Taken together, the superfields $W_{\alpha\beta\gamma}$, $E_{\alpha\beta}$ and R , subject to the identities (3.4) and (3.12) (and their complex conjugates) contain all components of the uncontracted Riemann tensor and their supersymmetric generalization described in Section 2. Using them, one can give the supersymmetric version of the Gauss-Bonnet theorem.

We first observe that one can write

$$W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} = 6 \bar{D}\bar{D} (D_{\alpha} \partial_{\beta}^{\dagger} V_{\gamma}^{\dagger} W^{\alpha\beta\gamma}) \quad . \quad (3.15)$$

Now, up to a divergence in superspace,

$$\begin{aligned} D_{\alpha} \partial_{\beta}^{\dagger} V_{\gamma}^{\dagger} W^{\alpha\beta\gamma} &= V_{\gamma}^{\dagger} (\square E^{\gamma\ddagger} + \partial^{\beta\ddagger} \partial^{\gamma\ddagger} E_{\beta\ddagger}) \\ &= V_{\gamma}^{\dagger} (2 \square E^{\gamma\ddagger} + \partial^{\gamma\ddagger} \partial_{\alpha\ddagger} E^{\alpha\beta} E^{\alpha\beta}) \quad . \end{aligned} \quad (3.16)$$

On the other hand, using the definition (3.1) and the identities (3.4) to (3.6), one can check that

$$E_{\alpha\beta} E^{\alpha\beta} = 4 R R^* - 12 V_{\gamma}^{\dagger} (2 \square E^{\gamma\ddagger} + \partial^{\gamma\ddagger} \partial_{\alpha\ddagger} E^{\alpha\beta} E^{\alpha\beta}) \quad , \quad (3.17)$$

*) A very clear description of the spinorial notation for Riemannian quantities has been given by Penrose [23].

again up to total derivatives. Therefore

$$\int d_4x d_4\theta \left[E_{\alpha\dot{\beta}} E^{\alpha\dot{\beta}} - 4 R^* R + 12 (D_\alpha \partial_{\dot{\beta}} \bar{V}_{\gamma\dot{\delta}}) W^{\alpha\beta\gamma\delta} \right] = 0 \quad (3.18)$$

which is the supersymmetric Gauss-Bonnet theorem. From the integrand of (3.18) one can construct chiral superfield

$$\mathbb{D}\bar{\mathbb{D}} (E_{\alpha\dot{\beta}} E^{\alpha\dot{\beta}} - 4 R^* R) + 2 W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} . \quad (3.19)$$

Its F component gives, up to an over-all normalization factor, the supersymmetric version of the Gauss-Bonnet integrand. It contains, in addition to the terms already given in Ref. [17], terms involving the vector potential b_μ of Section 2. In tensor notation these are proportional to

$$2 b^\mu \partial^\nu b_{\mu\nu} - \partial^\mu b^\nu \partial_\mu b_\nu + (\partial^\mu b_\mu)^2 - \frac{1}{2} b^{\mu\nu} b_{\mu\nu} . \quad (3.20)$$

This expression can be written as a divergence. Observe that (3.20) is not gauge invariant, corresponding to the fact that (3.19) is not invariant under Weyl transformations but only under Einstein transformations, which leave b_μ unchanged.

In superspace the Lagrangian of conformal supergravity is simply proportional to the chiral superfield $1/12 W_{\alpha\beta\gamma} W^{\alpha\beta\gamma}$. According to (3.15) the action can be taken proportional to

$$I = \frac{1}{2} \int d_4x d_4\theta (D_\alpha \partial_{\dot{\beta}} \bar{V}_{\gamma\dot{\delta}}) W^{\alpha\beta\gamma\delta} . \quad (3.21)$$

Varying $V_{\gamma\tau}$ one finds

$$\delta I = \int d_4x d_4\theta \delta V_{\gamma\tau} D_\alpha \partial_\beta{}^\tau W^{\alpha\beta\gamma} \quad (3.22)$$

One can add a source term

$$-g \int d_4x d_4\theta V_{\alpha\beta} J^{\alpha\beta} \quad , \quad (3.23)$$

where $J^{\alpha\beta}$ is the supercurrent of the matter system. The differential equations of motion following from (3.21) plus (3.23) are

$$D_\alpha \partial_\beta{}^\tau W^{\alpha\beta\gamma} = g J^{\gamma\tau} \quad (3.24)$$

These equations are consistent if the supercurrent satisfies the equation

$$D_\gamma J^{\gamma\tau} = 0 \quad , \quad (3.25)$$

appropriate to a conformally invariant matter system (see Ref. [15]). Using the identity (3.12) they can be written in the equivalent form

$$\square E^{\gamma\tau} + \partial^{\gamma\beta} \partial^{\alpha\tau} E_{\alpha\beta} = -g J^{\gamma\tau} \quad (3.26)$$

The action of ordinary supergravity is

$$\int d_4x d_4\theta \left(\frac{1}{2} V^{\alpha\beta} E_{\alpha\beta} - c \kappa V^{\alpha\beta} J_{\alpha\beta} \right) \quad , \quad (3.27)$$

where c is a numerical coefficient which depends on the convention chosen for the gravitational constant κ . It is invariant under (3.7) with the restriction

(3.10). The resulting equations of motion are

$$E_{\alpha\beta} = c \kappa J_{\alpha\beta} \quad . \quad (3.28)$$

Observe that, for a general matter system, the supercurrent satisfies [15]

$$\bar{D}^\alpha J_{\alpha\beta} = \bar{D}_\beta S^* \quad , \quad D_\alpha S^* = 0 \quad , \quad (3.29)$$

with an appropriate superfield S^* which is known for each matter supermultiplet.

Combining (3.28), (3.29), (3.3) and (3.4) one finds

$$\begin{aligned} \bar{D}_\beta (R^* - c \kappa S^*) &= 0 \\ D_\beta (R^* - c \kappa S^*) &= 0 \end{aligned} \quad . \quad (3.30)$$

Therefore we have the derived equations

$$R^* = c \kappa S^* \quad . \quad (3.31)$$

4. EQUATIONS OF MOTION IN COMPONENT FORM

In this section we write explicitly in component form the equations of motion given in superfield notation in the preceding section. The matter system is described by its supercurrent multiplet whose components we denote by $C_{2\mu}$, $\chi_{2\mu}$, $M_{2\mu} = \partial_\mu A$, $N_{2\mu} = \partial_\mu B$, $v_{2\lambda\mu} = s_{2\lambda\mu} + a_{2\lambda\mu}$, $\lambda_{2\mu}$ and $D_{2\mu}$. According to Ref. [15]

the spinor current, the (improved) energy momentum tensor and the axial vector current of matter are given respectively by

$$J_\mu = -\frac{4}{3} (\chi_{2\mu} - \gamma_\mu \gamma \cdot \chi_2) , \quad (4.1)$$

$$\Theta_{\lambda\mu} = -\frac{2}{3} (s_{2\lambda\mu} - \eta_{\lambda\mu} s_2 \rho^\rho) \quad (4.2)$$

and

$$J_\mu^{(5)} = 2 C_{2\mu} . \quad (4.3)$$

For the case of Einstein supergravity we have derived equations (3.28). Written out in component form they give, using (4.1), (4.2) and (4.3),

$$\begin{aligned} D_{2\mu} - \partial_\mu \partial \cdot C_2 + \square C_{2\mu} &= 0 , \\ \lambda_{2\mu} - \partial_\mu \gamma \cdot \chi_2 + \gamma \cdot \partial \chi_{2\mu} &= 0 , \\ a_{2\lambda\mu} + \frac{1}{2} \varepsilon_{\lambda\mu}{}^{\nu\rho} \partial_\nu C_{2\rho} &= 0 , \\ \partial \cdot M &= 2 \kappa A , \\ \partial \cdot N &= 2 \kappa B , \\ E_{\lambda\mu} &= \kappa \Theta_{\lambda\mu} , \\ R_\mu &= -\kappa J_\mu , \\ b_\mu &= \kappa J_\mu^{(5)} . \end{aligned} \quad (4.4)$$

These equations are invariant under the gauge transformations (2.5). The first three equations are just constraints which the components of the supercurrent multiplet always satisfy (see Ref. [15]). The equations for $\partial \cdot M$ and $\partial \cdot N$ determine the gauge invariant part of the fields M_μ and N_μ ; in writing them we have already used the fact that the components $M_{2\mu}$ and $N_{2\mu}$ of the supercurrent multiplet are gradients. Clearly (4.4) imply the conservation of the energy momentum tensor and of the spinor current.

The action for conformal supergravity (3.21), written in components, corresponds to the Lagrangian

$$-\frac{i}{4} \bar{\Delta}^{\mu\nu} \Phi_{\mu\nu} + \frac{3}{2} C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} - \frac{1}{4} b_{\mu\nu} b^{\mu\nu} \quad (4.5)$$

Note that the fields ψ_μ , $h_{\mu\nu}$ and b_μ have mass dimensions $\frac{1}{2}$, 0 and 1, respectively. Only the pseudovector field b_μ is canonical, its kinetic term is the usual Maxwell Lagrangian. The other two terms contain higher derivatives. Using the identities

$$\begin{aligned} \bar{\Phi}^{\mu\nu} \gamma \cdot \partial \Phi_{\mu\nu} &= 0, \\ \bar{\Delta}^{\mu\nu} \Phi_{\mu\nu} &= -2 \bar{\Xi}^{\mu\nu} \Phi_{\mu\nu} = -3 \bar{\psi}^{\mu\nu} \gamma \cdot \partial \Phi_{\mu\nu}, \\ \gamma \cdot \partial \partial^\mu \Phi_{\mu\nu} &= \frac{1}{3} (\square \eta_{\mu\nu} - \partial_\mu \partial_\nu) (\gamma_\lambda \psi^{\lambda\mu} + \frac{1}{2} \gamma_5 \gamma_\lambda \hat{\psi}^{\lambda\mu}), \end{aligned} \quad (4.6)$$

the kinetic term for the Rarita-Schwinger field can be written in the more familiar form

$$-\frac{i}{2} \bar{\psi}^\nu (\gamma \cdot \partial \square \eta_{\mu\nu} - \gamma \cdot \partial \partial_\mu \partial_\nu - \frac{1}{2} \varepsilon_{\nu\mu\rho\sigma} \gamma_5 \gamma^\sigma \partial^\rho \square) \psi^\mu. \quad (4.7)$$

The field equations following from (4.5) plus a term coupling the potential to the matter supercurrent are the component form of (3.26) (with a slightly different normalization of the coupling constant g)

$$\begin{aligned}
 \mathbb{D}_{2\mu} &= -\frac{1}{2} \square J_{\mu}^{(5)} , \\
 \lambda_{2\mu} &= \frac{3}{4} \gamma \cdot \partial J_{\mu} , \\
 a_{2\lambda\mu} &= -\frac{1}{4} \varepsilon_{\lambda\mu}{}^{\nu\rho} \partial_{\nu} J_{\rho}^{(5)} , \\
 A &= B = 0
 \end{aligned} \tag{4.8}$$

$$\begin{aligned}
 (\square \eta_{\mu\nu} - \partial_{\mu} \partial_{\nu}) (E^{\nu}{}_{\lambda} - \frac{1}{3} \delta^{\nu}{}_{\lambda} E_{\rho}{}^{\rho}) &= g \Theta_{\mu\lambda} , \\
 (\square \eta_{\mu\nu} - \partial_{\mu} \partial_{\nu}) (R^{\nu} - \frac{1}{3} \gamma^{\nu} \gamma \cdot R) &= -g J_{\mu} , \\
 (\square \eta_{\mu\nu} - \partial_{\mu} \partial_{\nu}) b^{\nu} &= g J_{\mu}^{(5)} .
 \end{aligned}$$

The last three equations are the conformal invariant equations for fields of spin 2, 3/2 and 1 coupled to conserved currents. The system is invariant under the combined Einstein and Weyl gauge transformations (2.6). This implies the additional constraints

$$\Theta_{\lambda}{}^{\lambda} = 0 , \quad \gamma^{\mu} J_{\mu} = 0 , \quad \partial^{\mu} J_{\mu}^{(5)} = 0 . \tag{4.9}$$

which correspond to (3.25). They are a simple consequence of the same properties which are satisfied by the left-hand sides of (4.8). Alternative forms are obtained by using the identities

$$R^{\nu} - \frac{1}{3} \gamma^{\nu} \gamma \cdot R = \frac{2}{3} \left(\gamma_{\lambda} \psi^{\lambda\nu} + \frac{1}{2} \gamma_5 \gamma_{\lambda} \hat{\psi}^{\nu\lambda} \right) \tag{4.10}$$

and

$$\begin{aligned}
& (\Box \eta_{\mu\nu} - \partial_\mu \partial_\nu) (E^\nu{}_\lambda - \frac{1}{3} \delta^\nu{}_\lambda E^\rho{}_\rho) \\
&= \frac{1}{2} \Box^2 h_{\mu\lambda} - \frac{1}{6} \eta_{\mu\lambda} \Box^2 h^\rho{}_\rho \\
&+ \frac{1}{6} \partial_\mu \partial_\lambda \Box h^\rho{}_\rho + \frac{1}{3} \partial_\lambda \partial_\mu \partial_\nu \partial_\rho h^{\nu\rho} - \frac{1}{2} \Box \partial_\lambda \partial_\nu h_\mu{}^\nu \\
&- \frac{1}{2} \Box \partial_\mu \partial_\nu h^\nu{}_\lambda + \frac{1}{6} \eta_{\mu\lambda} \partial_\nu \partial_\rho \Box h^{\nu\rho} .
\end{aligned} \tag{4.11}$$

Comparing (4.8) with (4.4) we see that in conformal supergravity there is an additional spin 1 (pseudovector) propagating field. Moreover the equations for the spin 3/2 field have third order derivatives and those for the spin 2 field fourth order derivatives. The coupling constant g , unlike the κ of Einstein's theory, is dimensionless and coincides with that of the pseudovector coupling of the field b_μ , which is the gauge field of the chiral $U(1)$ group of the superconformal algebra. As expected from the analysis of Refs. [10,11] there are no additional gauge fields related to dilations, special conformal transformations and special supersymmetry transformations. Due to the higher derivative structure of the equations of conformal supergravity, the field b_μ , which was an auxiliary field in Einstein supergravity, becomes a propagating field. At the same time the fields $h_{\mu\nu}$ and ψ_μ are now accompanied by ghosts.

5. A SUPERSYMMETRIC MODEL WITH GHOSTS

In this section we study the Hilbert space structure of a supersymmetric theory with a Lagrangian with higher derivatives. For simplicity we take the case of a scalar supermultiplet, but the results are perfectly analogous in the case of conformal supergravity.

Instead of the usual free Lagrangian for the scalar multiplet [13] we take

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \partial_\nu A)^2 + \frac{1}{2} (\partial_\mu \partial_\nu B)^2 + \frac{i}{2} \partial_\nu \bar{\psi} \gamma^\nu \partial^\nu \psi - \frac{1}{2} (\partial_\nu F)^2 - \frac{1}{2} (\partial_\nu G)^2 . \tag{5.1}$$

Clearly, since the derivatives of the fields of the scalar supermultiplet transform like the fields themselves, the action integral of (5.1) is invariant under the usual supersymmetry transformation of the scalar multiplet. The equations of motion are

$$\begin{aligned} \square^2 A = 0 \quad , \quad \square^2 B = 0 \quad , \\ \square \gamma \cdot \partial \psi = 0 \quad , \\ \square F = 0 \quad , \quad \square G = 0 \quad . \end{aligned} \tag{5.2}$$

In order to quantize this system, we use a generalization of the method of Froissart and Glaser [18], which consists in replacing the higher derivative Lagrangian with a Lagrangian containing certain auxiliary fields and only first derivatives, so that canonical quantization can be applied. Therefore, instead of (5.1) we take

$$\begin{aligned} L = & -\partial A_1 \cdot \partial A - \frac{\lambda^2}{2} A_1^2 - \partial B_1 \cdot \partial B - \frac{\lambda^2}{2} B_1^2 \\ & - \frac{i}{2} \bar{\psi}_1 \gamma \cdot \partial \psi_1 - i \bar{\psi} \gamma \cdot \partial \psi_2 + i \lambda \bar{\psi}_2 \psi_1 \\ & - \frac{1}{2} (\partial_\mu F)^2 - \frac{1}{2} (\partial_\mu G)^2 \quad . \end{aligned} \tag{5.3}$$

The equations of motion are now

$$\begin{aligned} \square A = \lambda^2 A_1 \quad , \quad \square A_1 = 0 \quad , \\ \square B = \lambda^2 B_1 \quad , \quad \square B_1 = 0 \quad , \\ \gamma \cdot \partial \psi = \lambda \psi_1 \quad , \quad \gamma \cdot \partial \psi_1 = \lambda \psi_2 \quad , \quad \gamma \cdot \partial \psi_2 = 0 \quad , \\ \square F = 0 \quad , \quad \square G = 0 \end{aligned} \tag{5.4}$$

and are equivalent to (5.2) upon elimination of the auxiliary fields A_1, B_1, ψ_1, ψ_2 . The constant λ has the dimensions of a mass and is introduced so that

all fields have canonical dimensions. It could be eliminated by a redefinition of the fields. We have not introduced a real mass in order to proceed in closest analogy with the case of conformal supergravity, but in the present case of a scalar supermultiplet the massive case presents no difficulties. It is easy to verify that (5.3) changes by a divergence under the supersymmetry transformation

$$\begin{aligned}
 \delta A &= i \bar{\alpha} \psi & , & & \delta A_1 &= i \bar{\alpha} \psi_2 & , \\
 \delta B &= i \bar{\alpha} \gamma_5 \psi & , & & \delta B_1 &= i \bar{\alpha} \gamma_5 \psi_2 & , \\
 \delta \psi &= \gamma \cdot \partial (A + \gamma_5 B) \alpha + \lambda (F + \gamma_5 G) \alpha & , & & & & \\
 \delta \psi_1 &= \gamma \cdot \partial (F + \gamma_5 G) \alpha + \lambda (A_1 + \gamma_5 B_1) \alpha & , & & & & (5.5) \\
 \delta \psi_2 &= \gamma \cdot \partial (A_1 + \gamma_5 B_1) \alpha & , & & & & \\
 \delta F &= i \bar{\alpha} \psi_1 & , & & \delta G &= i \bar{\alpha} \gamma_5 \psi_1 & .
 \end{aligned}$$

These transformations agree with the usual transformations for the scalar multiplet only if one uses the equations of motion (5.4). With (5.5) the supersymmetry algebra closes only on the mass shell.

We now consider quantization. Let us look at the sector A, A_1 . They form what is called a dipole ghost. The momentum conjugate to A_1 is $\partial A_1 / \partial t$, that conjugate to A is $\partial A / \partial t$. So we have the equal time commutators

$$\begin{aligned}
 \left[\frac{\partial A_1(t, \underline{x})}{\partial t} , A(t, \underline{x}') \right] &= -i \delta_3(\underline{x} - \underline{x}') & , \\
 \left[\frac{\partial A(t, \underline{x})}{\partial t} , A_1(t, \underline{x}') \right] &= -i \delta_3(\underline{x} - \underline{x}') & ,
 \end{aligned} \tag{5.6}$$

while all other equal time commutators vanish, e.g.,

$$\left[A_1(t, \underline{x}) , A_1(t, \underline{x}') \right] = \left[\frac{\partial A_1(t, \underline{x})}{\partial t} , A_1(t, \underline{x}') \right] = 0 \quad \text{etc.} \tag{5.7}$$

Using the differential equations we can find the commutators for any two space time points

$$\begin{aligned} [A_1(x), A_1(x')] &= 0, \\ [A_1(x), A(x')] &= -i \Delta(x-x'), \\ [A(x), A(x')] &= -i \lambda^2 \Delta'(x-x'), \end{aligned} \quad (5.8)$$

where Δ is the usual commutator function for zero mass and the function Δ' satisfies^{*)}

$$\begin{aligned} \square \Delta' &= \Delta, \\ \Delta'(x-x') &= 0 \quad \text{for} \quad (x-x')^2 > 0. \end{aligned} \quad (5.9)$$

In order to study the structure of the Hilbert space we must define creation and destruction operators. We satisfy the differential equations by writing

$$A_1(x) = \frac{1}{(2\pi)^{3/2}} \int d_3 k e^{i \underline{k} \cdot \underline{x}} \left[A_1(\underline{k}) e^{-i\omega t} + A_1^*(-\underline{k}) e^{i\omega t} \right] \quad (5.10)$$

and

$$\begin{aligned} A(x) &= \frac{1}{(2\pi)^{3/2}} \int d_3 k e^{i \underline{k} \cdot \underline{x}} \left[A(\underline{k}) e^{-i\omega t} + A^*(-\underline{k}) e^{i\omega t} \right. \\ &\quad \left. - \lambda^2 \left\{ \left(\frac{1}{4\omega^2} + \frac{it}{2\omega} \right) A_1(\underline{k}) e^{-i\omega t} + \left(\frac{1}{4\omega^2} - \frac{it}{2\omega} \right) A_1^*(-\underline{k}) e^{i\omega t} \right\} \right] \quad (5.11) \end{aligned}$$

*) For finite mass

$$(\square - m^2) \Delta = 0, \quad \left[\frac{\partial \Delta}{\partial t} \right]_{t=0} = \delta_3,$$

from which it follows $(\square - m^2) \frac{\partial \Delta}{\partial m^2} = \Delta$;

one defines $\Delta' \equiv \frac{\partial \Delta}{\partial m^2}$.

where $\omega = |\underline{k}|$. The explicit time dependence in $A(x)$ is typical of dipole ghosts. It can be shown that neither translational nor Lorentz covariance are violated. The creation and destruction operators in (5.11) have been separated so as to obtain simple commutation relations. One obtains easily

$$[A_1(\underline{k}), A^*(\underline{k}')] = [A(\underline{k}), A_1^*(\underline{k}')] = \delta_3(\underline{k} - \underline{k}') \quad , \quad (5.12)$$

while all other commutators vanish, e.g.,

$$[A_1(\underline{k}), A_1(\underline{k}')] = [A_1(\underline{k}), A_1^*(\underline{k}')] = 0 \quad . \quad (5.13)$$

The total energy and total momentum operators can be constructed in the usual canonical way. In momentum space they take the form

$$\begin{aligned} P^i &= \int k^i (A_1^* A + A^* A_1) d_3 k \quad , \\ P^0 &= \int \omega \left(A_1^* A + A^* A_1 + \frac{\lambda^2}{2\omega^2} A_1^* A_1 \right) d_3 k \quad . \end{aligned} \quad (5.14)$$

It is easy to verify that they are the generators of translations in co-ordinate space

$$[P_\mu, A] = i \frac{\partial A}{\partial x^\mu} \quad , \quad [P_\mu, A_1] = i \frac{\partial A_1}{\partial x^\mu} \quad . \quad (5.15)$$

At this point the Fock space can be constructed easily in the usual way. For instance, the vacuum is defined by

$$A_1(\underline{k}) |0\rangle = A(\underline{k}) |0\rangle = 0 \quad (5.16)$$

and the one-particle states by

$$|\underline{k}\rangle_1 = A_1^*(\underline{k}) |0\rangle, \quad |\underline{k}\rangle = A^*(\underline{k}) |0\rangle \quad (5.17)$$

Using the commutation relations for creation and destruction operators one verifies that the one-particle states have zero norm

$$\langle \underline{k} | \underline{k}' \rangle_1 = \langle \underline{k} | \underline{k}' \rangle = 0 \quad (5.18)$$

and that

$$\langle \underline{k} | \underline{k}' \rangle_1 = \delta_3(\underline{k} - \underline{k}') \quad (5.19)$$

Also, one of them is an eigenstate of the energy

$$P^0 |\underline{k}\rangle_1 = \omega |\underline{k}\rangle_1, \quad (5.20)$$

while the other satisfies

$$P^0 |\underline{k}\rangle = \omega |\underline{k}\rangle + \frac{\lambda^2}{2\omega} |\underline{k}\rangle_1 \quad (5.21)$$

Equations (5.18) to (5.21) are typical of a dipole ghost. Observe that the energy operator cannot be diagonalized: with a linear transformation one could replace $|\underline{k}\rangle_1$ and $|\underline{k}\rangle$ with two states one having positive and the other negative norm, however then neither would be an eigenstate of the energy operator.

The case of the ψ , ψ_1 and ψ_2 sector is slightly more complicated. They form what we may call a "tripole" ghost. The canonical anticommutation relations

are easily written and, together with the differential equations, give rise to the relativistic relations

$$\begin{aligned} \{\psi_2(x), \bar{\psi}_2(x')\} &= \{\psi_2(x), \bar{\psi}_1(x')\} = 0, \\ \{\psi_1(x), \bar{\psi}_1(x')\} &= \{\psi_2(x), \bar{\psi}_2(x')\} = \gamma \cdot \partial \Delta(x-x'), \\ \{\psi(x), \bar{\psi}_2(x')\} &= \lambda \Delta(x-x'), \\ \{\psi(x), \bar{\psi}(x')\} &= \lambda^2 \gamma \cdot \partial \Delta'(x-x'). \end{aligned} \quad (5.22)$$

For the spinor ψ_2 , which satisfies the Dirac equation, the transition to creation and destruction operators is accomplished by the formula

$$\psi_2(x) = \frac{1}{(2\pi)^{3/2}} \int d_3 k e^{i \underline{k} \cdot \underline{x}} \left[\psi_2(\underline{k}) e^{-i\omega t} + \psi_2^*(-\underline{k}) e^{i\omega t} \right], \quad (5.23)$$

where

$$\gamma_0 \underline{\gamma} \cdot \underline{k} \psi_2(\underline{k}) = \omega \psi_2(\underline{k}), \quad \gamma_0 \underline{\gamma} \cdot \underline{k} \psi_2^*(\underline{k}) = \omega \psi_2^*(\underline{k}). \quad (5.24)$$

In order to satisfy its equation of motion, the spinor ψ_1 must have the form

$$\begin{aligned} \psi_1(x) &= \frac{1}{(2\pi)^{3/2}} \int d_3 k e^{i \underline{k} \cdot \underline{x}} \left[\psi_1(\underline{k}) e^{-i\omega t} + \psi_1^*(-\underline{k}) e^{i\omega t} \right. \\ &\quad \left. + \frac{i\lambda \gamma_0}{2\omega} \left(\psi_2(\underline{k}) e^{-i\omega t} - \psi_2^*(-\underline{k}) e^{i\omega t} \right) \right], \end{aligned} \quad (5.25)$$

where $\psi_1(\underline{k})$ and $\psi_1^*(\underline{k})$ satisfy equations identical to (5.24). Finally, we have

$$\begin{aligned} \psi(x) = & \frac{1}{(2\pi)^{3/2}} \int d_3 k e^{i\underline{k} \cdot \underline{x}} \left[\psi(\underline{k}) e^{-i\omega t} + \psi^*(-\underline{k}) e^{i\omega t} \right. \\ & + \frac{i\lambda\gamma_0}{2\omega} \left(\psi_1(\underline{k}) e^{-i\omega t} - \psi_1^*(-\underline{k}) e^{i\omega t} \right) \\ & \left. - \left\{ \left(\frac{\lambda^2}{8\omega^2} + \frac{i\lambda^2 t}{2\omega} \right) \psi_2(\underline{k}) e^{-i\omega t} + \left(\frac{\lambda^2}{8\omega^2} - \frac{i\lambda^2 t}{2\omega} \right) \psi_2^*(-\underline{k}) e^{i\omega t} \right\} \right] \end{aligned} \quad (5.26)$$

where $\psi(\underline{k})$ and $\psi(\underline{k})$ also satisfy equations identical to (5.24). The canonical anticommutation relations imply

$$\{\psi_1(\underline{k}), \psi_1^*(\underline{k}')\} = \{\psi_2(\underline{k}), \psi_2^*(\underline{k}')\} = \delta_3(\underline{k} - \underline{k}') , \quad (5.27)$$

while all other anticommutators vanish, e.g.,

$$\{\psi_2(\underline{k}), \psi_2(\underline{k}')\} = \{\psi_2(\underline{k}), \psi_2^*(\underline{k}')\} = 0 . \quad (5.28)$$

The momentum operator

$$P^i = \int k^i (\psi^* \psi_2 + \psi_2^* \psi + \psi_1^* \psi_1) d_3 k \quad (5.29)$$

and the energy operator

$$P^0 = \int \omega \left(\psi^\dagger \psi_2 + \psi_2^\dagger \psi + \psi_1^\dagger \psi_1 + \frac{\lambda^2}{2\omega^2} \psi_2^\dagger \psi_2 \right) d_3 k \quad (5.30)$$

are the generators of space-time translations, e.g.,

$$[P_\mu, \psi(x)] = i \frac{\partial \psi}{\partial x^\mu} \quad (5.31)$$

Let us look at Fock space. The vacuum is defined from

$$\psi(\underline{k}) |0\rangle = \psi_1(\underline{k}) |0\rangle = \psi_2(\underline{k}) |0\rangle = 0 \quad (5.32)$$

and the one-particle states from

$$\begin{aligned} \psi^\dagger(\underline{k}) |0\rangle &= |\underline{k}\rangle, \\ \psi_1^\dagger(\underline{k}) |0\rangle &= |\underline{k}\rangle_1, \\ \psi_2^\dagger(\underline{k}) |0\rangle &= |\underline{k}\rangle_2. \end{aligned} \quad (5.33)$$

One finds the relations

$$\begin{aligned} \langle \underline{k} | \underline{k}' \rangle &= \langle \underline{k} | \underline{k}' \rangle_2 = 0, \\ \langle \underline{k} | \underline{k}' \rangle_1 &= \langle \underline{k} | \underline{k}' \rangle_1 = 0, \\ \langle \underline{k} | \underline{k}' \rangle_1 &= \langle \underline{k} | \underline{k}' \rangle_2 = \delta_3(\underline{k} - \underline{k}') \end{aligned} \quad (5.34)$$

and

$$\begin{aligned} P^0 |\underline{k}\rangle_1 &= \omega |\underline{k}\rangle_1, \\ P^0 |\underline{k}\rangle_2 &= \omega |\underline{k}\rangle_2, \\ P^0 |\underline{k}\rangle &= \omega |\underline{k}\rangle + \frac{\lambda^2}{2\omega} |\underline{k}\rangle_2. \end{aligned} \quad (5.35)$$

We see that the Fock space of a "tripole" is the same as that of a dipole plus a physical particle.

One can now ask what is the effect of a supersymmetry transformation on the Fock space (clearly the fields F and G are quantized like ordinary physical scalar fields). It is not difficult to rewrite (5.5) in terms of creation and destruction operators and then to see how the states are transformed. Instead of writing the explicit formulas, we describe qualitatively the result in terms of a picture. The structure of Fock space is represented in the Figure. The symbols represent particle states, not fields. The states inside each ellipse transform into each other under supersymmetry. Thus A_1 transforms into ψ_2 while F transforms into a linear combination of ψ_1 and ψ_2 ; A transforms into a linear combination of ψ , ψ_1 and ψ_2 . The states A_1 , B_1 , ψ_2 have zero norm and so do A , B and ψ , while F , G and ψ_1 are ordinary physical states. The generator of supersymmetry commutes with the energy operator.

So we have a physical supermultiplet with spins $(0,0,\frac{1}{2})$ and a dipole ghost supermultiplet also with spins $(0,0,\frac{1}{2})$. The situation for the case of conformal supergravity is perfectly analogous to that treated here (see Introduction).

Can one make physical sense of a theory with dipole ghosts? This problem has been discussed in some detail in a different context [19]. One must find a subspace of the total Hilbert space which contains all the physical particles and some of the zero norm states, but no negative norm states. This subspace must be transformed into itself by the scattering matrix. This problem is very hard to solve, in general, and it is possible that in a supersymmetric theory its solution is simpler. Even so, the resulting theory would have very weak causality properties, because the fields satisfy locality conditions only as operators in the entire Hilbert space.

6. CONCLUDING REMARKS

A number of questions [10,11] concerning conformal supergravity are left unanswered by our investigation. We have already mentioned the problems coming from the occurrence of ghosts in the theory. Our analysis of the Fock space and of its supersymmetry properties is only a beginning. There remains the more difficult problem of defining the interacting theory so that unitarity is satisfied. Another serious difficulty, also related to the higher order differential equations, is that conformal supergravity, in its present form, does not reproduce Einstein's gravity at the classical level. This shortcoming is a consequence of the conformal invariance of the theory and can only be remedied by breaking it, either explicitly or spontaneously. Finally we mention that it is crucial to verify that the full non-linear theory is locally supersymmetric. Only local supersymmetry can ensure that the characteristic inconsistencies of higher spin conformal equations will be absent from the theory.

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APPENDIX

The superfield formulation of Section 3 can be used to give a discussion of the possible perturbation theory divergences in supergravity. One obtains in this way a simple and rigorous derivation of the results of Ref. [17]. As explained there, a knowledge of the linearized theory is sufficient for the argument. The problem reduces to the construction of supersymmetric invariants which are also invariant under the gauge transformation (2.5), or (3.7) with the restriction (3.10).

The equations of motion of ordinary (linearized) supergravity are (3.28) and (3.31) with $\kappa = 0$,

$$E_{\alpha\dot{\beta}} = 0 \quad , \quad R = 0 \quad . \quad (A.1)$$

Therefore the only gauge invariant superfields available for the construction of invariants on the mass shell are the chiral superfield $W_{\alpha\beta\gamma}$, given by (3.11), and its complex conjugate. Making use of the identity (3.12) we can write

$$D^{\alpha} W_{\alpha\beta\gamma} = 0 \quad , \quad \bar{D}_{\dot{\alpha}} W_{\alpha\beta\gamma} = 0 \quad . \quad (A.2)$$

The fields b_{μ} and $b_{\mu\nu}$ vanish on the mass shell, so the term linear in θ in the expansion of $W_{\alpha\beta\gamma}$ contains just the totally symmetric spinor $C_{\alpha\beta\gamma\delta}$ which corresponds to the Weyl conformal tensor.

One loop divergences vanish on the mass shell. This follows from the fact that the only invariant of the right dimension is the ordinary integral of the last component of the chiral superfield $W_{\alpha\beta\gamma} W^{\alpha\beta\gamma}$. This integral vanishes on the mass shell as a consequence of the supersymmetric Gauss-Bonnet theorem (3.18), (3.19).

The leading two loop divergence also vanishes on the mass shell. It should be a supersymmetric invariant having as spin 2 part the product of three Riemann tensors^{*)}

*) See P. van Nieuwenhuizen and C.C. Wu, Journal of Math. Phys. 18 (1977) 182.

$$R_{\mu\nu}{}^{\lambda\rho} R_{\lambda\rho}{}^{\sigma\tau} R_{\sigma\tau}{}^{\mu\nu} . \quad (\text{A.3})$$

On the mass shell the Riemann tensor and the Weyl tensor are identical. In two component spinor notation (A.3) corresponds to

$$C_{\alpha\beta}{}^{\gamma\delta} C_{\gamma\delta}{}^{\varepsilon\varphi} C_{\varepsilon\varphi}{}^{\alpha\beta} . \quad (\text{A.4})$$

Now, it is easy to see that (A.4) cannot be completed to a supersymmetric invariant. It would require multiplying by itself three times the superfield $W_{\alpha\beta\gamma}$. This gives nine spinor indices which can be saturated only if one applies at least one supercovariant derivative D_δ . In this way one can construct a superfield containing (A.4), but not in its last component.

At the three loop level there is an invariant which does not vanish on the mass shell. The superfield

$$T_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}} = W_{\alpha\beta\gamma} W_{\dot{\alpha}\dot{\beta}\dot{\gamma}}^* \quad (\text{A.5})$$

contains the Bel-Robinson tensor, which in spinor notation corresponds to [23]

$$C_{\alpha\beta\gamma\delta} C_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}^* . \quad (\text{A.6})$$

Observe that (A.2) and their complex conjugates imply

$$D^\alpha T_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}} = 0 , \quad \bar{D}^{\dot{\alpha}} T_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}} = 0 . \quad (\text{A.7})$$

Applying $\bar{D}^{\dot{\alpha}}$ to the first of (A.7) and D^α to the second and adding, we get

$$\partial^{\alpha\dot{\alpha}} T_{\alpha\dot{\alpha}\beta\dot{\beta}} \gamma\dot{\gamma} = 0 \quad (\text{A.8})$$

which, in particular, gives the conservation of the Bel-Robinson tensor. The product of (A.5) by itself

$$W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} W_{\dot{\alpha}\dot{\beta}\dot{\gamma}}^* W^{*\dot{\alpha}\dot{\beta}\dot{\gamma}} \quad (\text{A.9})$$

is a supersymmetric invariant which contains as part of its last component the square of the Bel-Robinson tensor

$$C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} C_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}^* C^{*\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}, \quad (\text{A.10})$$

an expression quartic in the Riemann tensor. Therefore, as observed in Ref. [17], there can be a divergence in supergravity at the three loop level.

We thank M. Grisaru for a discussion on the renormalization of supergravity.

FIG. 1

Fock space of a supersymmetric model with ghosts.

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A