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INTERMEDIATE STATE STRUCTURE OF DOUBLE SCATTERING

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A B S T R A C T

We study how the phase of double scattering in the Mandelstam diagram is obtained in old-fashioned perturbation theory. We find that, in non-vanishing time ordered contributions, this phase is obtained by putting unphysical states, i.e., hanging chains on the energy shell. Alternatively, the covariant result can be recovered from the contributions obtained by putting only the physical states on energy shell in diagrams in which the two scatterings do not overlap in time.

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1. - INTRODUCTION

We studied in a recent paper [Ref. 1) hereafter referred to as I] the time evolution of the double scattering term represented by the covariant Mandelstam diagram ²⁾. We decompose it into the many time ordered contributions (TOC) according to the rules of old-fashioned perturbation theory (OFPT).

A remarkable result obtained in I was the existence of TOC's that, taken separately, give a real contribution of order $1/s$ two powers of s higher than the covariant result. Therefore different time orderings have to be combined before we obtain physical sense. When they are combined they cancel to such an extent that the total answer is eventually purely imaginary and of order $(\ln s)^N/s^3$. We concluded in I that little physical insight can be obtained by considering one TOC on its own.

The factor i that multiplies the covariant result is obtained in OFPT by putting one intermediate state on the energy shell and allowing it an infinite lifetime. In I we showed in detail for the particularly simple example of the doubly twisted ladder that the covariant result can be recovered by putting on the energy shell the physical four-particle state, that is known to saturate the unitarity equation, in diagrams that correspond to successive (non-overlapping) rescattering (Fig. 1). In these OFPT diagrams other intermediate states can be put on energy shell. The covariant discontinuities associated with these latter states are known to be suppressed due to the softness of the theory. As we pointed out in I this suppression is only obtained in OFPT by the cancellation between different time orderings. Within this given time ordering the different intermediate states cancel ³⁾. We show here that this cancellation is avoided only in time orderings in which very unphysical hanging chains propagate on energy shell.

In Section 2 we study the relation between the two different procedures for the unsignatured, doubly twisted covariant diagram. The structure of cancellations in the signatured case is even more complicated. In Section 3 we first of all study the signatured ladder by itself. In Section 4 we show how the phase of the signatured amplitude is obtained again by putting unphysical states on shell. In Section 5 we set out our conclusions.

2. - THE DOUBLY TWISTED DIAGRAM

The covariant result for the doubly twisted diagram is completely reproduced by the OFPT contribution of Fig. 1 in which the intermediate four-particle state is put on the energy shell. This means we replace the energy denominator $1/E - E_4$ by $i\pi\delta(E - E_4)$. In Fig. 1 states containing more than four particles can be put on the energy shell and therefore there are additional contributions to the imaginary part of the amplitude coming from this specific TOC. In particular putting the five-particle states indicated by the dashed lines on the energy shell amounts to a change of sign due to $\delta(E_4 - E_I)/(E_5 - E_I) \rightarrow \delta(E_5 - E_I)/(E_4 - E_I)$. In addition there is a limitation on the phase space of the fastest particle of the uncut ladder that must be slower than the particles we cut on the other ladder. Otherwise we lose powers of $\log s$. Hence the left and right five-particle cuts give respectively

$$-\frac{i}{p^3} \frac{(\ln p)^{m+n+2}}{m! (n-1)! (m+n+2)(m+n+1)(m+n)} \quad , \quad -\frac{i}{p^3} \frac{(\ln p)^{m+n+2}}{(m-1)! n! (m+n+2)(m+n)(m+n)}$$

and together they cancel the four-particle contribution. Here $m+1, n+1$ are the number of rungs in the two ladders. The contributions obtained by putting six or more particles on shell in Fig. 1 give non-leading contributions. This is because successive intermediate states of the cut ladder provide powers of the smallest cut momentum. These terms cancel the non-leading terms left over from the cancellation of the four and five particle contributions.

The mechanism of successive cancellations of powers of $\log p$ is similar to the one responsible for the AFS cancellation³⁾. There is, however, an essential difference in that covariantly it is known that the five-particle discontinuity of the Mandelstam diagram is suppressed by one power of $\log p$ and therefore cannot cancel the four-particle discontinuity. In the AFS case the covariant two and three-particle discontinuities are comparable and in fact cancel. The covariant suppression of the five-particle contribution is obtained by summing the TOC's of Figs. 1 and 2.

It is straightforward to see that in Fig. 2 also the six-particle contributions obtained on cutting along the dashed lines contribute an imaginary part in the leading log approximation. Again this contribution is obtained from the five-particle one on exchanging the rôles of an energy denominator and a δ function. They therefore again cancel in leading log.

The contribution of this six-particle, hanging chain, state is cancelled by a TOC in which one more vertex of the right ladder is pulled across the left ladder. The sequential cancellation continues till we reach the configuration of Fig. 3. In this TOC only the intermediate states containing one complete hanging chain for a total of $m+n+2$ particles (Fig. 4) can be put on energy shell. There is no further term to cancel this contribution to the imaginary part. This therefore, by itself, reproduces the covariant result.

The real part coming from Fig. 3 is $\sim 1/p$. Therefore we have to add to Fig. 3 the TOC's which cancel this wrong power behaviour as we did in I. However, only the TOC of Fig. 3 has a leading imaginary part.

To compute it we observe that in order to have the intermediate state on the energy shell one of the two momenta q_1, q_2 has to be small. Furthermore q_3 has to be larger than any other internal integration momentum. At the bottom of the diagram the larger of q_1, q_2 has to be smaller than any other flowing down the ladders. Hence upon integration we obtain an answer

$$\frac{i\pi (\ln p)^{m+n+2}}{n! n! (m+n+2)(m+n+1) p^3}$$

The structure of imaginary contributions coming from different TOC's and different states on the energy shell is as follows

$$A(\text{covariant}) = \overbrace{A_4^0 - A_5^0} + \overbrace{A_5^1 - A_5^1} + \dots - \overbrace{A_{m+n+2}^{m+n-3}} + \overbrace{A_{m+n+2}^{m+n+2}}$$

The lower index identifies the state that is put on energy shell, the upper index the number of crossed vertices in time, i.e., the specific TOC. In leading log they are all equal and the series may be summed by lower or upper brackets. The upper brackets select one TOC at a time which is a dangerous procedure as we know from our experience with real parts. According to this order of summation the absorptive phase characteristic of double scattering is generated by the propagation on energy shell of the hanging chains that are the most unphysical states imaginable. The summation according to the lower brackets reproduces the contribution from the physical four-particle state computed in I.

We wish to stress that the vanishing of the global contribution of the TOC of Fig. 1, for example, occurs through the cancellation of a physical and an unphysical state. In the language of Ref. 4), Fig. 1 would be a wrong time ordering but we see that its vanishing is not due to any relation between states that contribute to the saturation of physical unitarity.

3. - SIGNATURED LADDERS

Because of the more complicated cancellation structure implicit in the fact that the signatured diagrams are powers of $\log s$ down in the unsignatured ones we find it convenient to go to a more general frame. Here we take the two incident particles to have momenta $p, -r$, ($p, r > 0$) with $p \gg r \gg m$. Then the invariant energy $s \simeq 4pr$ and so we may distinguish easily the covariant $(\log s)^N$ terms from non-covariant terms $\sim (\log p)^N$ which clearly must cancel on summing all diagrams. We use this trick to avoid tedious cancellation calculations.

We start by deriving how the signatured box diagram, i.e., the sum of covariant planar and twisted box diagrams becomes pure imaginary in OFPT.

We start with the twelve diagrams of Fig. 5 where the external lines are not crossed. Each diagram has three intermediate states and therefore three denominators. We write these out below for $|q| \leq \epsilon r \ll p$.

	Left	Centre	Right	
5.a,b,c,d	$q - E(q)$	$2q$	$q - E(q)$	
5.e,f,5.a'	$q - E(q)$	$-2E(q)$	$q - E(q)$	
5.b'	$-q - E(q)$	$-2E(q)$	$-q - E(q)$	
5.c',e'	$q - E(q)$	$-2E(q)$	$-q - E(q)$	
5.d',f'	$-q - E(q)$	$-2E(q)$	$q - E(q)$	(3.1)

The other factors in all cases give $1/p \cdot r E^2(q)$ in our kinematic region. In the above there are correction terms $\sim m^2/r$ which we neglect. Changing $q \rightarrow -q$ in 5.b', we get

$$\frac{4[-q - E(q)](-2E) + 4(-q - E(q))(2q) + 4 \cdot 2q(q - E(q))}{(q - E(q))^2 (-q - E(q)) 2q (-2E)} \approx$$

$$\approx \frac{2}{(E^2 - q^2)q} \approx \frac{2}{m^2 q} \quad (3.2)$$

Thus the integral becomes

$$\frac{1}{pr} \frac{2}{m^2} \int_{-Er}^{Er} \frac{dq}{(q^2 + m^2)q} \approx \frac{2\pi i}{pr 2m^4} \quad (3.3)$$

from the pole at $q = im$. The $q = 0$ pole above gives twice this answer. The two is the difference between $\text{Im } T$ and ΔT . The q integral is convergent and effectively cut-off at $q \sim N$. The diagrams 5.a,b,c,d,e,f 5.a'b' all give logarithms separately since $q - E(q) \sim m^2/q$ $q > 0$ giving eventually $\int dq/q$. Thus the planar graphs give $2 \log r/pr$ and the crossed graphs $-2 \log r/pr$. Clearly this cannot be the full answer since the planar graphs by themselves must be covariant.

The terms to make the above covariant come from the graphs of Fig. 6 for $r \ll q \ll Nr \ll p$ ϵ small, N large. Here the denominators work in a different manner. Since q is large but $\ll p$ the four integrands are of the form

$$\begin{aligned}
 a) & \frac{1}{m^4 E(p) E(q-r) [q + E(r) - E(q-r)]} \\
 b) & \frac{1}{m^4 E(p) E(q-r) [q - E(r) - E(r-q) - 2E(q)]} \\
 c) & \frac{1}{m^4 E(p) E(q+r) [q + E(r) - E(q+r) - 2E(q)]} \\
 d) & \frac{1}{m^4 E(p) E(q+r) [q - E(r) - E(q+r)]}
 \end{aligned} \tag{3.4}$$

For $q > r$, $q - r \gg m$ this gives

$$\frac{1}{2r(q-r)} + \frac{1}{(q-r)(-2q)} + \frac{1}{(-2q)(q+r)} + \frac{1}{(-2r)(q+r)} \approx 0 \tag{3.5}$$

The first two terms, however, give $\frac{1}{r} \int_r^p dq/q \approx \log p/r$ which together with $2 \log r$ from 5.a,b,c,d,e,f of Fig. 5 give finally $\log(pr)/pr$, as required covariantly.

We leave it to the reader to check that the diagrams we have omitted in the different phase space regions and indeed the regions we omit are negligible, i.e., are $\ll 1/pr$.

An alternative way to get (3.3) is through integration on the real axis :

Since the integrand $f(q)$ fulfils $f(q) = -f(-q)$ we calculate $\frac{1}{2} \int_c f(q) dq$ where c is a contour encircling the real axis and we pick the pole at $q=0$.

Now turn to the general ladders. The dominant time orderings are of the form shown in Fig. 7. The horizontal lines a all carry positive three momentum, i.e., are forward going while b lines are backward going. Thus q joins two lines which are going in opposite directions. Then the above integral for $p > p_1, \dots > p_n > q, r > r_1, \dots > r_m > q$ becomes

$$\int \prod_i \frac{dp_i}{p_i} \prod_j \frac{dr_j}{r_j} \frac{dq}{q} \quad (3.6)$$

where the intermediate state cutting only lines a, b gives $2q$, since q is attached to two lines travelling in opposite directions. Then we get

$$\sum_{m+n+1=N} \int_1^r \frac{dq}{q} \frac{[\ln p/q]^n}{n!} \frac{[\ln r/q]^m}{m!} = \quad (3.7)$$

$$= \int_1^r \frac{dq}{q} \frac{[\ln p/q + \ln r/q]^{N-1}}{(N-1)!} =$$

$$= \frac{1}{N! \cdot 2} [\ln pr - 2 \ln q]^N \Big|_2^r =$$

$$= \frac{1}{2 N!} [(\ln pr)^N - (\ln p/r)^N] \quad (3.8)$$

The first term is the covariant answer when we count all six diagrams as in Fig. 5. The diagram of Fig. 8 for $p \geq q \geq r$ gives for $p_1 > \dots > p_{n-1} > q$

$$\int_r^p \frac{dq}{q} \frac{[\ln p/q]^{N-1}}{(N-1)!} \approx \frac{1}{N!} [\ln p/r]^N \quad (3.9)$$

which cancels the unwanted term above.

Since all the large p_i, r_j cancel in the cuts through the central box involving q the intermediate state structure of Fig. 7 involving q is exactly as in Eq. (3.1) where we now ignore corrections of $O(\min(1/p_n, 1/r_m))$. Thus we get the same cancellations leading to $\delta(2q)$ as in (3.3) which now corresponds to putting the ladder on energy shell in the only physically allowed intermediate state.

4. - SIGNATURED MANDELSTAM DIAGRAMS

Now turn to the Mandelstam diagrams. We have seen that the signatured ladder has q of Fig. 7 smaller than any other momentum and typically $\sim 1/p_n \sim 0$. Thus we expect the dominant diagrams to be as in Fig. 9, where we only show the small momenta q_1, q_2 and the extra loop integral σ connecting the top and bottom of the graphs. We omit the large momentum rungs of the ladder. Clearly we must sum over all the time ordering and twists of the two low energy boxes, i.e., $8! \times 4$ diagrams, all of which are non-zero. However, there is a trivial trick which lets us carry out this summation. Covariantly the q_1 and q_2 integrals are entirely independent of each other. In OFPT when the two loops overlap in time we obtain denominators which mix these variables and only in summing all time orderings do we regain independence.

Now because all the big momenta cancel the denominators of cuts through q_1, q_2 loops are the same (to order $1/\text{big momentum}$) as those of the graphs of Fig. 10, where the horizontal lines are massless and the wiggly lines are mass m . Thus in Fig 10 we obtain for a, b, c $(2q_1 + O(1/p))$ $(2\sigma + O(1/p))$ $(2\sigma + 2q_2 + O(1/p))$ agreeing with Fig. 9 to order $(1/p)$ where p is the smallest big momentum in the problem.

However, the diagrams of Fig. 10 only have over-all momentum conservation but not conservation of energy in each independent process. We know that summation of all time-ordering must give individual energy conservation, i.e., $\delta(2\sigma)$ which corresponds in Fig. 10 to putting b on-shell.

To gain insight into this problem without calculating all $4 \times 8!$ diagrams consider the simple potential theory examples of Fig. 11.

The first term is trivial

$$\begin{aligned} T_a &= \delta(1+2-1'-2') \frac{1}{2'-1'} + \delta(1+2-1'-2') \frac{1}{1'-2'} = \\ &= \delta(1+2-1'-2') 2i\pi \delta(1'-2') \end{aligned}$$

(4.1)

where we write all denominators with the same $i\epsilon$.

$$T_b = \delta(1+1'-3-3') \left[\frac{1}{(1'-3')(3-2)} + \frac{1}{(1-2)(1-3)} + \frac{1}{(1-2)(3-2)} \right] \quad (4.2)$$

$$\begin{aligned} &= \delta(1+1'-3-3') \left[\frac{1}{(3-2)} \left[\frac{1}{(1'-3')} - \frac{1}{(3'-1')} \right] + \frac{1}{(3-2)(3'-1)} + \right. \\ &\quad \left. + \frac{1}{(1-2)(3'-1)} + \frac{1}{(1-2)(3-2)} \right] = \\ &= \delta(1+1'-3-3') \left[2i\pi \delta(1'-3') \frac{1}{3-2} + \frac{1+3+3'-1}{(3-2)(3'-1)(1-2)} \right] = \\ &= \delta(1-3) \delta(1'-3') \frac{2i\pi}{3-2} \quad (4.3) \end{aligned}$$

This is now the correct answer with independent scattering and two δ functions. We can also consider this in an instructive but different manner. Let us calculate the real part of (4.2) by dividing each term into principal values and δ function parts. We consider it as a function of $1, 2, 3, 3'$ and eliminate $1'$, and pretend we integrate over $2, 3$ with arbitrary test functions. Then the three terms may be written, schematically,

$$\begin{aligned} &\int d3 \, P \frac{1}{3-1} \int d2 \, P \frac{1}{3-2} - \pi^2 \delta(3-1) \delta(3-2) \\ &\int d2 \, P \frac{1}{1-2} \int d3 \, P \frac{1}{1-3} - \pi^2 \delta(3-1) \delta(1-2) \\ &\int d2 \, P \frac{1}{1-2} \int d3 \, P \frac{1}{3-2} - \pi^2 \delta(1-2) \delta(3-2) \quad (4.4) \end{aligned}$$

where we have only written out the terms in which we are interested. The principal value integrals have also been carefully ordered so that each integral contains only one pole. This is a crazy looking answer. The on-shell contributions give a factor of 3 which clearly is not present in the physical intermediate states. Then we write

$$\frac{1}{(3-1)(3-2)} = \left[\frac{1}{3-1} - \frac{1}{3-2} \right] \frac{1}{1-2} \quad (4.5)$$

and the first term becomes

$$\int d3 \int d2 \frac{1}{(3-1)(1-2)} - \int d3 \int d2 \frac{1}{(3-2)(1-2)} \quad (4.6)$$

These apparently cancel the other two P.P integrals except that the second term is in the wrong order, and the two-integral has two poles. However, we have the Poincaré-Bertrand formula

$$\int d3 \int d2 P \frac{1}{(3-2)(1-2)} = \int d2 P \frac{1}{1-2} \int d3 P \frac{1}{3-2} + \pi^2 \delta(1-2) \delta(3-2) \quad (4.7)$$

which gives us our correct answer. Thus even at this trivial level taking the discontinuities of this simple potential theory model is extremely delicate and depends on the orders of doing principal value integrals.

We can analyze T_0 in a similar way to obtain the extra δ function and two independent amplitudes. We may also recover the 1, -4, 2 counting ratios for the various none, single, double discontinuities which lead in our case to the Pomeron cutting rules.

Thus in our Mandelstam case we obtain $\delta(2\sigma)$ and two independent amplitudes whose signature now works independently as before. Thus in Fig. 9 we have $\sigma \sim 0$, $q_1 \sim q_2 \sim 0$ and the big momenta above and below are independently integrated up to p, r , respectively. Thus we obtain for m_1, m_2 above q_1, q_2 in the inside and outside ladders and n_1, n_2 below

$$\int_1^p \frac{dp_1}{p_1} \int_1^r \frac{dr_1}{r_1} \frac{[\ln p_1]^{m_1+m_2}}{(m_1+m_2)!} \frac{[\ln r_1]^{n_1+n_2}}{(n_1+n_2)!} C_{n_1}^{n_1+n_2} C_{m_1}^{m_1+m_2}$$

Here p_1, r_1 are always the biggest loop momentum. On summation, this gives

$$\int_1^p \frac{dp_1}{p_1} \int_1^r \frac{dr_1}{r_1} \frac{[\log p_1 r_1]^{N_1+N_2}}{N_1! N_2!}$$

where N_1, N_2 are the number of rungs in each ladder minus 2.

$$\frac{1}{(N_1+N_2+2)(N_1+N_2+1)} \frac{[\ln pr]^{N_1+N_2+2}}{N_1! N_2!}$$

and we have dropped all the non-covariant terms $(\log p)^N$, etc. These, as in the ladder case, are cancelled by the cases where the q_1, q_2 loops are attached to the r line. We must, of course, add the other orderings where p, r flow along the other routes.

5. - CONCLUSIONS

Both Mandelstam diagrams considered in this paper (the unsigned one of Section 2 and the signed one of Section 4) are purely imaginary asymptotically.

In OFPT the presence of a factor i requires that some intermediate state is on the energy shell. In the case of the doubly twisted diagram we have shown in Section 2 that this factor can be obtained by putting on the energy shell a state in which a ladder is "hanging" in the only globally non-vanishing TOC that has an imaginary part (the real part of this specific TOC is two powers of s leading over the covariant result, and it is cancelled by summing to it other TOC that, however, are purely real).

The covariant suppression of "hanging chains" is achieved summing to the previous contribution a number of vanishing TOC. This chain of cancellations ends with the TOC in which the two ladders do not overlap in time. This last TOC is the only one that contains the "physical" four-particle state that gives the complete covariant answer by itself if we put it on the energy shell ¹⁾. Therefore we see that the relation between TOC and unitarity is far from straightforward. The situation is even more complicated for the case of the signed ladder (Section 4). The covariant answer contains a factor i^3 . This would lead one to expect that in OFPT we would re-obtain this factor by putting three states on the energy shell. This expectation is fulfilled only for the sum of the $8!$ non-vanishing TOC considered in Section 4 (and all three states on the energy shell contain hanging chains). However, the individual TOC gets an imaginary contribution also from integration regions in which only one state is on the energy shell, and it happens, in the class of reference frames considered in Section 4, that the sum of these contributions vanishes (this would not be the case in the laboratory frame). We also remark that the global vanishing of TOC in which the two ladders are not sufficiently nested happens through a cancellation in which both kinds of terms are essential. Hence we come to the conclusion that the vanishing of a wrong TOC is not associated to a relation between physical state discontinuities. Also in the case of the signed ladder, if we want to make contact with states that appear in the physical unitarity we are forced to go through a cancellation chain analogous to the one discussed in Section 2. The chain ends with non-overlapping time orderings which are non-zero only when certain intermediate states, which this time are the physical ones, are put on the energy shell. The procedure leads for the signed diagram to the two possible time orderings a), b) of Fig. 12. As far as we can see, the procedure does not establish uniquely the relative weight of a) and b) but only their sum. One possible way to fix this relative weight is the following : the elastic amplitude and its unitarity cuts are correctly reproduced by the effective S matrix $e^{i\lambda(a+a^+)}$ where a^+ , a are creation and annihilation operators of chains and the zero-chain state should be identified with the four-particle intermediate state. If we insist that a) and b) correspond to the fourth order terms in the expansion of the effective S matrix, $\langle 0|aa^+aa^+|0 \rangle$ and $\langle 0|aaa^+a^+|0 \rangle$, respectively, we should obviously attribute a weight 2 to b) and 1 to a) for a total of $3\lambda^4/4!$. The well-known AGK coefficients that relate the zero, one and two chain discontinuities are easily obtained in this eikonal formulation by taking the λ^4 terms in the expansion of

$\langle 0|S^+S|0 \rangle$ upon insertion of the corresponding intermediate state.
These weights cannot be read off directly from the diagrams of Fig. 12.

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FIG.1

FIG.2

FIG. 3

FIG. 4

FIG. 5

FIG. 6

FIG. 7

FIG. 8

a

b

c

FIG.11

a

b

FIG. 12