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COVARIANT QUANTIZATION OF GAUGE THEORIES
WITH OPEN GAUGE ALGEBRA

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A B S T R A C T

We show that when the algebra of gauge transformations is not closed one can still generally define an effective action that is invariant under an appropriate extension of the Becchi-Rouet-Stora transformations. Both the action and transformation rules contain terms of higher order in the ghost fields. The generalized Ward identities imply the existence of a gauge-independent S matrix. Nilpotency of extended BRS transformations is regained upon use of the full field equations of ghost and ordinary fields.

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A gauge-field theory can be specified in terms of an action which is invariant under a set of transformations characterized by arbitrary spacetime-dependent parameters. These transformations are introduced in a way which is logically unrelated to a particular action, and it is often possible to construct a variety of actions invariant under the same transformations. Since the product of two such transformations will still leave the action invariant we can naturally obtain an algebra of infinitesimal transformations which close under commutation (and/or anticommutation in the case of a graded group like one has in supergravity ¹⁾). Usually this gauge algebra has a simple structure, which is identical to that of the corresponding spacetime-independent transformations with field-independent structure functions. But this is by no means essential as one can observe for the nonAbelian gauge algebra of general coordinate transformations where the corresponding algebra of constant translations is Abelian.

However, it may happen that under (anti)commutation the gauge transformations do not lead to only those transformations that were initially introduced. This could imply that the original set of transformations was taken too small, but if this is not the case then it introduces in this way another (infinite) set of local invariance transformations which must be proportional to the field equations derived from the invariant action. The existence of such transformations is not typical for a gauge theory, but they can be constructed in any field theory. Consequently these local invariances do not require the absence of certain degrees of freedom from the invariant action, which implies that at first instance they may be ignored in quantizing the theory. Hence there are cases that the gauge algebra is not closed in the strict sense, so that closure can only be achieved by introducing these local invariances proportional to the field equations. Such algebras which only close "on the mass-shell" will be referred to as open gauge algebras. It is worth pointing out that for dimensional reasons such algebras are most likely to be encountered for gauge symmetries with an intrinsic dimension, and thus for theories with a dimensionful gauge-coupling constant such as gravity.

Although the superfluous local transformations themselves do not require the elimination of degrees of freedom in the quantization, their presence in the gauge algebra causes difficulties. It is the purpose of this letter to generally consider the covariant quantization procedure for gauge theories with an open gauge algebra. The quantization of such theories has been discussed extensively for the case of supergravity, where it was known that the gauge algebra was open ²⁾, and that the conventionally quantized action was not invariant under

BRS transformations ³⁾. It was then shown that this defect required extra terms in the quantized action quartic in the ghost fields, whereas the standard quantization only leads to terms quadratic in the ghost fields. The presence of the new terms was established in the Hamiltonian formalism (canonical quantization) ⁴⁾, and by extending the BRS transformations for the quantized action in the framework of the Lagrangian formulation (covariant quantization) ^{5,6)}. Moreover these modifications were found using standard quantization techniques in various formulations of supergravity with auxiliary fields where the algebra is closed in the strict sense ^{7,8)}. The terms quartic in the ghost fields do then occur upon elimination of the auxiliary fields after quantization. However, the result of most of these considerations does only apply to supergravity in the second-order form, and it is, for instance, not known in general under what conditions it is possible to close the algebra with the aid of auxiliary fields. On the other hand it has been shown by Fradkin and Fradkina ⁹⁾ that many-ghost interactions even higher than the quartic ones are present in the Hamiltonian expression for the S matrix in the general case.

We will present a general proof that these many-ghost interactions are also required when extending the covariant quantization procedure like it was done by Kallosh for supergravity ⁵⁾. More specifically we prove that it is possible for any gauge theory with open gauge algebra to define an effective quantum action which is invariant under the appropriate extension of the BRS transformations ¹⁰⁾. To simplify the notation we will only discuss the case of Bosonic symmetries. A more complete version of the proof will be discussed elsewhere. We start with a classical action S which is invariant under infinitesimal local gauge transformations.

$$\delta \varphi^i = R^i_{\alpha} \xi^{\alpha}, \quad (1)$$

where ξ^{α} are the transformation parameters, and the indices α and i label the symmetry generators and the fields at the various spacetime points, respectively. The quantity R^i_{α} is in general field dependent.

To quantize we introduce a gauge condition F_{α} , a ghost field c^{α} and a conjugate ghost field $c^{\ast}_{\alpha}$ for every generator. The statistics of the ghost fields is opposite to that of the corresponding generators. The effective quantum action with quadratic gauge-fixing terms can then generally be written as

$$S_{\text{eff}} = S - \frac{1}{2} F_{\alpha}^2 + \bar{F}_i \left\{ R_{\alpha}^i + \sum_{n=2} \frac{1}{n} \bar{F}_{i_2} \dots \bar{F}_{i_n} M_{\alpha \alpha_2 \dots \alpha_n}^{i i_2 \dots i_n}(n) c^{\alpha_2} \dots c^{\alpha_n} \right\} c^{\alpha}, \quad (2)$$

where $\bar{F}_i$ is defined by

$$\bar{F}_i = \frac{\delta}{\delta \varphi^i} (c_{\alpha}^* F_{\alpha}) \quad (3)$$

Notice that the statistics of $\bar{F}_i$ is always opposite to that of the field ϕ^i . The quantities $M(n)$ to be specified later are field dependent and symmetric or antisymmetric in the field indices i_k and gauge-group indices α_k , according to the statistics of the corresponding quantities c^{α} and $\bar{F}_i$. For simplicity we will only consider Boson fields so that the $\bar{F}_i$ are anticommuting.

We will prove that one can always find quantities $M(n)$ and $X(n)$ such that the effective action (2) is invariant under the following extension of the BRS transformations:

$$\begin{aligned} \delta \varphi^i &= \left\{ R_{\alpha}^i + \sum_{n=2} \bar{F}_{i_2} \dots \bar{F}_{i_n} M_{\alpha \alpha_2 \dots \alpha_n}^{i i_2 \dots i_n}(n) c^{\alpha_2} \dots c^{\alpha_n} \right\} c^{\alpha} \Lambda \\ \delta c_{\alpha}^* &= -\Lambda F_{\alpha} \\ \delta c^{\alpha} &= \left\{ \frac{1}{2} f_{\mu\nu}^{\alpha} + \sum_{n=1} \bar{F}_{i_1} \dots \bar{F}_{i_n} X_{\mu\nu \alpha_1 \dots \alpha_n}^{\alpha i_1 \dots i_n}(n) c^{\alpha_2} \dots c^{\alpha_n} \right\} c^{\mu} c^{\nu} \Lambda \end{aligned} \quad (4)$$

where Λ is an anticommuting c-number and $f_{\mu\nu}^{\alpha}$ are the (possibly field-dependent) structure constants of the gauge algebra. To prove this, order by order in the number of gauge fields, one must show that the variation of the effective action of a given order can always be absorbed by adding a higher-order term to the field transformations or to the ghost-field transformations, which then determine the quantities M and X in the next order.

Let us first summarize the proof for the terms of lowest order in the number of ghost fields, as it was given by Kallosh ⁵⁾. She observed that since the BRS

invariance of the action was affected by terms proportional to the field equations, this effect could be compensated for by adding additional terms to the field transformations. Indeed invariance in lowest order of the action (2) requires that

$$\{R_{\alpha,j}^i R_{\beta}^j + \frac{1}{2} f_{\alpha\beta}^{\gamma} R_{\gamma}^i + M_{\alpha\beta}^{ij}(2) S_{,j}\} c^{\alpha} c^{\beta} = 0 \quad (5)$$

which merely expresses the fact that the algebra closes only up to terms proportional to the field equations. Hence the nonclosure of the algebra determines $M(2)$. The next step is to consider the Jacobi identity for three gauge transformations, which leads to the following equation

$$\begin{aligned} c^{\alpha} c^{\beta} c^{\gamma} \{ & (f_{\alpha\beta}^{\mu} f_{\mu\gamma}^{\lambda} - R_{\gamma}^k f_{\alpha\beta,k}^{\lambda}) R_{\lambda}^i + \\ & 2 (f_{\alpha\beta}^{\mu} M_{\mu\gamma}^{ij}(2) - R_{\gamma}^k M_{\alpha\beta}^{ij}(2)_{,k} + R_{\gamma,k}^{\hat{li}} M_{\alpha\beta}^{\hat{k}j\hat{l}}(2)) S_{,j} \} \\ & = 0 \end{aligned} \quad (6)$$

where the square brackets denote antisymmetrization over all independent combinations, and a hat indicates that the corresponding index is exempted from this operation.

Eq. (6) implies that the expression

$$(M_{\alpha\beta}^{ij}(2)_{,k} R_{\gamma}^k - R_{\gamma,k}^{\hat{li}} M_{\alpha\beta}^{\hat{k}j\hat{l}}(2) - f_{\alpha\beta}^{\mu} M_{\mu\gamma}^{ij}(2)) c^{\alpha} c^{\beta} c^{\gamma} \quad (7)$$

can be written in the form

$$-R_{\mu}^{\hat{li}} X_{\alpha\gamma\beta}^{\mu j\hat{l}}(1) - 2 M_{\alpha\beta\gamma}^{ijk}(3) S_{,k}$$

which defines $X(1)$ and $M(3)$, and is sufficient to prove the invariance of the effective action (2) under BRS transformations (4) through terms cubic in the ghost fields. For the second-order form of supergravity $M(3)$ turned out to vanish, whereas the terms in the BRS transformations proportional to $M(2)$ and $X(1)$ did not induce new terms in the variation of the action. This completed the proof for supergravity ⁵⁾.

As a byproduct of these arguments we also find that the following equation should hold for the structure constants

$$(f_{\alpha\beta}^{\mu} f_{\mu\gamma}^{\lambda} - R_{\gamma}^k f_{\alpha\beta,k}^{\lambda} + 2 X_{\alpha\gamma}^{\lambda} \beta^{(1)} S_{ij}) c^{\alpha} c^{\beta} c^{\gamma} = 0 \quad (8)$$

which is the counterpart of (5) for the adjoint representation.

To extend this procedure to higher orders in the ghost fields for a general theory with open gauge algebra we start by assuming that the transformations (4) leave the effective action (2) invariant up to terms proportional to $(\bar{F}_{i_1} \dots \bar{F}_{i_N})$. Hence this implies that the quantities $M(n)$ and $X(k)$ have been determined for $n < N+1$ and $k < N-1$, respectively. It is then convenient to define a quantity $N(n)$ for $n \geq 2$, which is totally antisymmetric in its indices $i_1 \dots i_n$, by

$$\begin{aligned} N^{i_1 \dots i_n}(n) = & c^{\beta} c^{\alpha_n} \dots c^{\alpha_1} \left\{ M_{\alpha_1 \dots \alpha_n}^{i_1 \dots i_n}(n), q R_{\beta}^q - R_{\beta, q}^{[i_1} M_{\alpha_1 \dots \alpha_n}^{\hat{q} i_2 \dots i_n]}(n) \right. \\ & - \frac{n}{2} f_{\alpha_1 \alpha_2}^{\mu} M_{\mu \beta \alpha_3 \dots \alpha_n}^{i_1 \dots i_n}(n) \\ & + \sum_{k=2}^{n-1} \frac{(k-1)!(n-k)!}{(n-1)!} \left(M_{\alpha_1 \dots \alpha_k}^{[i_1 \dots i_k]}(k), q M_{\beta \alpha_{k+1} \dots \alpha_n}^{\hat{q} i_{k+1} \dots i_n]}(n-k+1) \right. \\ & \left. \left. - k M_{\mu \beta \alpha_3 \dots \alpha_k}^{[i_1 \dots i_k]}(k) X_{\alpha_1 \alpha_2 \alpha_{k+1} \dots \alpha_n}^{\mu i_{k+1} \dots i_n]}(n-k) \right) \right\} \quad (9) \end{aligned}$$

Notice that for $n = 2$ this corresponds precisely to the expression (7). The invariance of the effective action (2) up to order N can now be expressed by the following equations

$$N^{i_1 \dots i_n}(n) = -c^{\beta} c^{\alpha_n} \dots c^{\alpha_1} \left\{ R_{\mu}^{[i_1} X_{\alpha_1 \beta \alpha_2 \dots \alpha_n}^{\mu i_2 \dots i_n]}(n-1) + n M_{\alpha_1 \dots \alpha_n \beta}^{i_1 \dots i_n]}(n+1) S_{ij} \right\} \quad (10)$$

for $n < N$. To prove the invariance of the action in N -th order then requires that we can find quantities $M(N+1)$ and $X(N-1)$ such that Eq. (10) is also satisfied for $n=N$.

The main ingredient to show that this is indeed possible is provided by the Jacobi identity of two ordinary gauge transformations with a transformation defined by

$$\delta_M^{(n)} \varphi^i = M_{\alpha_1 \dots \alpha_n}^{i i_2 \dots i_n}(n) \xi_{i_2 \dots i_n}^{\alpha_1 \dots \alpha_n}, \quad (11)$$

with $\xi_{i_2 \dots i_n}^{\alpha_1 \dots \alpha_n}$ as an arbitrary transformation parameter. Notice that this transformation (11) is not an invariance of the classical action. If we now contract the Greek indices of the gauge transformations and the transformation (11) in the Jacobi identity with ghost fields then a tedious but straightforward calculation leads to the following result

$$\begin{aligned} N^{i_1 \dots i_n j}_{(n+1)} S_j = & c \gamma c^\beta c^{\alpha_n} \dots c^{\alpha_1} \left\{ \frac{1}{n} R_\mu^{[i_1} A_{\beta \gamma \alpha_1 \dots \alpha_n}^{\mu i_2 \dots i_n]}(n-1) \right. \\ & + \sum_{k=2}^n \frac{k!(n-k)!}{n!} M_{\mu \alpha_2 \dots \alpha_k}^{[i_1 \dots i_k]}(k) \left(A_{\beta \gamma \alpha_{k+1} \dots \alpha_n}^{\mu i_{k+1} \dots i_n]}(n-k) \right. \\ & \left. \left. - (n+1-k) X_{\alpha_1 \beta \alpha_{k+1} \dots \alpha_n}^{\mu i_{k+1} \dots i_n]} \hat{J}_{\gamma}^{(n+1-k)} S_j \right) \right\} \end{aligned} \quad (12)$$

where the quantities $A(n)$ are defined by

$$\begin{aligned} A_{\alpha \beta \gamma \alpha_1 \dots \alpha_n}^{\mu i_1 \dots i_n}(n) = & -R_\gamma^q X_{\alpha \beta \alpha_1 \dots \alpha_n}^{\mu i_1 \dots i_n}(n), q + R_{\gamma, q}^{[i_n} X_{\alpha \beta \alpha_1 \dots \alpha_n}^{\mu i_1 \dots i_{n-1}]} \hat{q}_{\gamma}^{(n)} \\ & + \frac{n+2}{2} f_{\alpha \beta}^\lambda X_{\lambda \gamma \alpha_1 \dots \alpha_n}^{\mu i_1 \dots i_n}(n) + f_{\lambda \gamma}^\mu X_{\alpha \beta \alpha_1 \dots \alpha_n}^{\lambda i_1 \dots i_n} - \frac{1}{2} f_{\alpha \beta, q}^\mu M_{\gamma \alpha_1 \dots \alpha_n}^{q i_1 \dots i_n}(n+1) \\ & + \sum_{k=2}^n \frac{(k-1)!(n-k+1)!}{n!} \left\{ -M_{\gamma \alpha_1 \dots \alpha_{k-1}}^{\hat{q} [i_1 \dots i_{k-1}]}(k) X_{\alpha \beta \alpha_k \dots \alpha_n}^{\mu i_k \dots i_n]}(n+1-k), q \right. \\ & \quad + M_{\gamma \alpha_1 \dots \alpha_{k-1}}^{[i_n i_1 \dots i_{k-1}]}(k), q X_{\alpha \beta \alpha_k \dots \alpha_n}^{\mu i_k \dots i_{n-1}]} \hat{q}_{\gamma}^{(n+1-k)} \\ & \quad \left. + (k+1) X_{\lambda \gamma \alpha_1 \dots \alpha_{k-1}}^{\mu [i_1 \dots i_{k-1}]}(k-1) X_{\alpha \beta \alpha_k \dots \alpha_n}^{\lambda i_k \dots i_n]}(n+1-k) \right\} \end{aligned}$$

for $n \geq 1$, and

$$A_{\alpha \beta \gamma}^{\mu}(0) = \frac{1}{2} \left(f_{\alpha \beta}^\lambda f_{\lambda \gamma}^\mu - R_\gamma^q f_{\alpha \beta, q}^\mu \right) \quad (13)$$

If we combine the result of Eqs. (10) and (12) we find a generalization of the Jacobi identity for the structure constants (8) which contains the quantities $A(n)$:

$$\begin{aligned}
 & C^{\alpha_n} \dots C^{\alpha_2} C^{\alpha} C^{\beta} C^{\gamma} \left\{ R_{\mu}^{[i_1} (A_{\gamma\alpha\beta\alpha_2 \dots \alpha_n}^{\mu \ i_2 \dots i_n] (n-1) - n X_{\alpha\beta\alpha_2 \dots \alpha_n}^{\mu \ i_2 \dots i_n] \hat{j}} (n) S_{j} \right. \\
 & \left. + \sum_{k=0}^{n-2} \frac{k! (n-k)!}{(n-1)!} M_{\mu\alpha_2 \dots \alpha_{n-k}}^{[i_1 \dots i_{n-k}} (n-k) (A_{\gamma\alpha\beta\alpha_{n-k+1} \dots \alpha_n}^{\mu \ i_{n-k+1} \dots i_n] (k) \right. \\
 & \left. - (k+1) X_{\alpha\beta\alpha_{n-k+1} \dots \alpha_n}^{\mu \ i_{n-k+1} \dots i_n] \hat{j}} (k+1) S_{j} \right\} = 0 \\
 & (n \leq N-1) \tag{14}
 \end{aligned}$$

Hence this leads to $N-2$ identities which imply that the terms $(A(n-1) - nX(n) S_{j})$ are of a certain form for $0 \leq n \leq N-2$, which after substitution into Eq. (12) with $n=N-1$ implies that $N(N)$ satisfies a generalization of Eq. (6):

$$N^{i_1 \dots i_{N-1} j} (N) S_{j} = \frac{1}{N-1} R_{\mu}^{[i_1} C_{\gamma\alpha\beta}^{\mu \ i_2 \dots i_{N-1}] (N-2) C^{\alpha_{N-1}} \dots C^{\alpha_2} C^{\alpha} C^{\beta} C^{\gamma} \tag{15}$$

where $C(N-2)$ is equal to $A(N-2)$ modified by terms determined by the solution of Eqs. (14). Eq. (15) now implies that $N(N)$ can be written in the form

$$N^{i_1 \dots i_N} (N) = -C^{\beta} C^{\alpha_N} \dots C^{\alpha_1} \left\{ R_{\mu}^{[i_1} X_{\alpha_1\beta\alpha_2 \dots \alpha_N}^{\mu \ i_2 \dots i_N] (N-1) + N M_{\alpha_1 \dots \alpha_N}^{i_1 \dots i_N j} (N+1) S_{j} \right\} \tag{16}$$

which determines $X(N-1)$ and $M(N+1)$, and completes the proof through order N .

We have thus shown that for any gauge theory with an open gauge algebra it is possible to define an effective quantum action that is invariant under the appropriate extension of the BRS transformations. This extension of the covariant quantization procedure requires in general the presence of many-ghost interactions, and although a direct comparison is not quite possible we believe our results to be equivalent to those obtained in the Hamiltonian formalism by Fradkin and Fradkina ⁹⁾.

With these results it is straightforward to obtain the generating functional of the n -point Green's functions, as well as the corresponding generalized Ward identities. For this purpose it is, however, necessary to find an invariant functional-integration measure, which in this approach must be constructed for each specific theory separately. In connection with this we note that the Jacobian of the BRS transformation depends only on the gauge-fixing terms through the quantities $\bar{F}_1$, precisely as the transformations themselves. Another important point is that the Ward identities derived from the extended BRS

symmetry do still imply the gauge independence of the S matrix. Furthermore one can show by a rather extensive calculation that the extended BRS transformations are nilpotent upon use of the field equations as derived from the full quantum action. These field equations include those for the ghost fields as well. The aspect of nilpotency is of importance in discussing the higher-order quantum corrections ¹¹⁾.

It is remarkable that the covariant quantization procedure allows a generalization to any theory with open gauge algebra. Although at present the only known theories of this kind are models of supergravity, this fact proves that as far as the quantization is concerned no a priori conditions need to be imposed on gauge theories in order to lead to physically acceptable results. It is worth repeating that it remains questionable whether these results can be obtained in the general case through the introduction of auxiliary fields, as turned out to be possible for supergravity.

A more complete version of this proof with further results will be presented elsewhere.

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