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DEEP PROCESSES IN NON-RELATIVISTIC CONFINING POTENTIALS *)

P.M. Fishbane +) and M.T. Grisaru ")
CERN - Geneva

A B S T R A C T

We study deep inelastic and hard scattering processes for non-relativistic particles confined in deep potentials. The mechanisms by which the effects of confinement disappear and the particles scatter as if free are useful in understanding the analogous results for a relativistic field theory.

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- +) Permanent address : Physics Department, University of Virginia, Charlottesville, VA 22901, USA.
- ") Permanent address : Physics Department, Brandeis University, Waltham, Ma 02154, USA.

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1. - INTRODUCTION

The great success that the parton picture and other quantum chromodynamic (QCD) inspired short distance phenomenology have had in dealing with "deep" processes has in the background the problem of the non-observation of quarks and the supposition, as yet unproved, that QCD will confine them through some arbitrarily strong force at long distances ¹⁾. One must imagine then that all the complicated dynamics associated with the non-observation of coloured states, e.g., the necessary vacuum polarizations and recombinations, do not affect the essential features of the deep or short distance processes calculated in low order perturbations of the (running) coupling constant. Unfortunately no tractable relativistic scheme for making four-dimensional models of confined systems as yet exists, not to mention a detailed understanding of the behaviour of QCD. Thus one cannot check the required behaviour.

Of all the "deep" processes - deep inelastic scattering, field angle inclusive scattering, annihilation, high mass lepton pair production, and other higher order jet production processes - the first two are not fundamentally relativistic and can be studied in potential models. Indeed here there is a complete analogue to the relativistic case in the variables used to describe the process. There is a deep limit (a Bjorken limit in the first case) where one can test whether a non-trivial scaling behaviour holds or not, whether the connection between these two processes implied by the parton model holds or not, etc. We have taken advantage of this analogy to study these processes in the context of fully confining potentials. Many of the results we find are certainly ^{2),3)} well known and hardly surprising. West ²⁾ has treated the deep inelastic scattering problem in potentials. Our approach is, however, rather different, avoiding some technical difficulties which exist in West's work, and leading to some possibly useful insights into the properties responsible for scaling.

We are able to study a large class of three-dimensional confining potentials, although it is useful also to study some specific examples, the harmonic oscillator and the square well. We find that in deep processes the confinement plays no rôle. In other words, particles scatter as if they were free, scattering incoherently, and the presence of indefinitely rising potentials which represent the necessary colour correlation force does not affect the scaling properties. We feel this lends useful - though of course incomplete - support to the perturbative calculation methods of relativistic QCD in deep regions.

The deep limit in non-relativistic scattering is reached by first taking all rest masses large and only then permitting the three-momentum and energy transfers to become large, compared to the energy level splittings of the potentials under consideration. The appropriate scaling variable comes from studying the "deep inelastic" scattering of a free particle of mass m , for which one finds a scaling function νW which is a delta function at $x=1$ when x is defined by

$$x = \frac{\vec{q}^2}{2m q_0} \quad (1.1)$$

For bound systems of more than one particle other values of x are possible between zero and one, as for the usual Bjorken scaling variable. For high energy fixed angle scattering the scaling variable is provided by the angle itself, as in the relativistic case.

In addition to reproducing the canonical results of the parton model (the logarithmic scaling violations of QCD are of course not within our province), several features of the non-relativistic calculation would seem to be of interest in their own right, for the light they shed on the physics of the processes involved. These features may continue to play a rôle when the fully relativistic problem is understood. First, the inclusive matrix elements involve the overlap of some highly excited state of the problem with the ground state (or more generally with some state at an energy level small compared with the hard energy-momentum transfer involved). This overlap restricts the wave function of the excited state to a region far from the walls, i.e., to a region where its kinetic energy is large compared to the potential. The ground state is thus a crucial ingredient in understanding how this process can lead to free particle behaviour. Secondly, while from one point of view the scaling behaviour involves the complicated addition of many non-scaling channels, it is possible to see that this is really only a consequence of a symmetry of the problem : by choosing a different basis the problem reduces to one involving only a single scaling channel. This is a concrete realization of a necessary property of the quark-parton model. In the non-relativistic problem the symmetry is only a spatial one, while presumably in the relativistic problem it is more complicated, involving the decomposition of many hadronic channels into the quark-parton basis.

The plan of this paper is as follows : in the following section, we discuss deep inelastic scattering. This section is divided into several parts. First we treat the harmonic oscillator, which can be solved exactly. Then the square well is discussed, an important case because the more general confining potential, which is discussed next, can be reduced to it. In the third section we discuss the case of high energy fixed angle scattering.

2. - DEEP INELASTIC SCATTERING

For non-relativistic deep inelastic scattering, the structure function $W(\nu, \vec{q}^2)$ is found by squaring the transition amplitude (lowest order in electromagnetism) and summing over all possible states which conserve momentum. Here $\vec{q}$ is the three-momentum transfer to the bound system and ν is the laboratory energy (we generally work in this frame) carried by the off-shell spacelike photon. To set our normalization let us first calculate W for a free particle of mass m .

$$W \delta^{(3)}(\vec{P} + \vec{q} - \vec{P}') = \sum_f |\langle i | e^{i\vec{q} \cdot \vec{r}} | f \rangle|^2 \delta\left(\frac{P'^2}{2m} - \nu - \frac{P^2}{2m}\right) \quad (2.1)$$

In this expression $\vec{P}(\vec{P}')$ is the initial (final) three-momentum, $e^{i\vec{q} \cdot \vec{r}}$ is the electric current for the particle

$$\begin{aligned} \langle i | e^{i\vec{q} \cdot \vec{r}} | f \rangle &= \frac{1}{(2\pi)^3} \int d^3r e^{i\vec{P} \cdot \vec{r}} e^{i\vec{q} \cdot \vec{r}} e^{-i\vec{P}' \cdot \vec{r}} \\ &= \delta^{(3)}(\vec{P} + \vec{q} - \vec{P}') \end{aligned}$$

In squaring, one of the $\delta^{(3)}$ factors is interpreted in the usual way as a volume factor, which we drop. Then in the limit

$$\nu W = \delta\left(1 - \frac{\vec{q}^2}{2m\nu}\right) = \delta(1-x) \quad (2.2)$$

A. - Harmonic oscillator

We now imagine DI scattering from a single charged particle of mass μ in a harmonic oscillator potential. While this example is certainly well known, several of its aspects seem useful to repeat here. The Hamiltonian is $\frac{1}{2}\mu\omega^2(x_1^2 + x_2^2 + x_3^2)$, with energy levels $E_{n_1 n_2 n_3} = \omega(n_1 + n_2 + n_3 + \frac{3}{2}) = \omega(n + \frac{3}{2})$; the degeneracy of the n^{th} level is $\frac{1}{2}(n+1)(n+2)$. It is simplest to work in a Euclidean separation of the wave function,

$$\Psi_{n_1 n_2 n_3}(\vec{r}) = \prod_{i=1}^3 \Psi_{n_i}(x_i) = \prod_{i=1}^3 c_{n_i} e^{-\alpha^2 x_i^2 / 2} H_{n_i}(\alpha x_i)$$

$$\alpha = \sqrt{\mu\omega} \quad c_{n_i} = (\mu\omega/\pi)^{1/4} (2^{n_i} n_i!)^{-1/2} \quad (2.3)$$

The overlap integral $\langle i | e^{i\vec{q} \cdot \vec{r}} | f \rangle$ is now a product of three factors of the form

$$I_{n'n}(q) = \int_{-\infty}^{\infty} dx e^{iqx} e^{-\alpha^2 x^2 / 2} H_{n'}(\alpha x) H_n(\alpha x) c_{n'} c_n \quad (2.4)$$

For definiteness we take $n=0$ (the ground state). Equation (2.4) is easily evaluated :

$$I_{n'0}(q) = \left(\frac{q^{2n'}}{2^{n'} \alpha^{2n'} n'!} \right)^{1/2} e^{-q^2/4\alpha^2} \quad (2.5)$$

so that

$$W = \sum_{n'_1 n'_2 n'_3} \frac{1}{2^{n'} \alpha^{2n'}} \frac{q_1^{2n'_1} q_2^{2n'_2} q_3^{2n'_3}}{n'_1! n'_2! n'_3!} e^{-q^2/4\alpha^2} \delta_{n' + 3/2, \nu/\omega} \quad (2.6)$$

where the Kronecker delta expresses energy conservation. Each separate channel in the sum in (2.6) will not scale in the Bjorken limit ; but the sum is a multinomial sum, and scales as we shall see below. The result of the sum gives, including a density of states factor,

$$\nu W = \frac{\nu}{\omega n'!} \left(\frac{\vec{q}^2}{2\alpha^2} \right)^{n'} e^{-\vec{q}^2/2\alpha^2}, \quad n' \approx \frac{\nu}{\omega} \quad (2.7)$$

We eliminate $\vec{q}^2$ in (2.7) in favour of x and ν , and take the limit $\nu \rightarrow \infty$ (i.e., $\nu/\omega \rightarrow \infty$). Using Stirling's formula we find

$$\nu W \approx \left(\frac{\nu}{2\pi\omega} \right)^{1/2} (x e^{1-x})^{\nu/\omega} \quad (2.8)$$

For $x < 1$, $x e^{1-x} < 1$, and $\nu W (x < 1) \rightarrow 0$. At $x = 1$ νW is large. We find its integral over x equals 1, so that

$$\nu W \rightarrow \delta(1-x) \quad (2.9)$$

a scaling function.

From (2.6) we see that if we had chosen one eigendirection along $\vec{q}$, only one eigenchannel would have contributed to the fully scaling form (2.9).

B. - Square well

We begin by studying the one-dimensional case. We have $V(y) = 0$ for $0 \leq y \leq \pi$ and $V(y) = \infty$ otherwise,

$$\Psi_n(y) = \left(\frac{2}{\pi} \right)^{1/2} \sin n y, \quad E_n = \frac{n^2}{2m} \quad (2.10)$$

The ground state is $n=1$. Energy conservation gives $n^2 = 1 + 2m\nu = 1 + q^2/x$. The overlap integral is

$$I_n = \langle i | e^{i q y} | f \rangle = \frac{2}{\pi} \int_0^\pi dy e^{i q y} \sin y \sin n y \quad (2.11)$$

We find

$$I_n = -\frac{2}{\pi} \left[e^{i(q-n)\frac{\pi}{2}} \frac{\cos(q-n)\frac{\pi}{2}}{(q-n)^2 - 1} - e^{-i(q+n)\frac{\pi}{2}} \frac{\cos(q+n)\frac{\pi}{2}}{(q+n)^2 - 1} \right] \quad (2.12)$$

In the Bjorken limit only the first term on the right-hand side is important. Introducing instead of x the variable z by

$$n = q(1-z) \quad , \quad z = 1 - (x)^{-1/2} \quad (2.13)$$

we consider

$$\int_{-\infty}^{\infty} |I_n|^2 f(z) dz = \frac{4}{\pi^2} \int_{-\infty}^{\infty} \frac{\cos^2 \frac{\pi}{2} q z}{[q^2 z^2 - 1]^2} f(z) dz \quad (2.14)$$

where $f(z)$ is a test function. Making the change of variables $z \rightarrow z/q$ we find, in the limit $q \rightarrow \infty$

$$\int_{-\infty}^{\infty} |I_n|^2 f(z) dz \rightarrow \frac{f(0)}{q} \frac{4}{\pi^2} \int \frac{\cos^2 \frac{\pi}{2} z}{(z^2 - 1)^2} dz = \frac{f(0)}{q} \quad (2.15)$$

Re-expressed in terms of x and v we have

$$|I_n|^2 \xrightarrow{v \rightarrow \infty} \left(\frac{2}{mv} \right)^{1/2} \delta(1-x) \quad (2.16)$$

We multiply by a density of states factor $dn/dE = \sqrt{m/2v}$ and finally find

$$v W = v |I_n|^2 \frac{dn}{dE} = \delta(1-x) \quad (2.17)$$

a fully scaling function.

For a three-dimensional Euclidean square well, $V(\vec{r}) = 0$ for $0 \leq r_i \leq \pi$ $i = 1, 2, 3$, we have

$$\Psi_{n_1 n_2 n_3} = \prod_{i=1}^3 \left(\frac{2}{\pi} \right)^{3/2} \sin n_i r_i$$

$$E_n = \frac{1}{2m} (n_1^2 + n_2^2 + n_3^2) = \frac{n^2}{2m} \quad (2.18)$$

and energy conservation requires $n^2 = 3 + 2mv = 3 + q^2/x$. The overlap integral is largest when each n_i is a finite fraction of the component q_i of $\vec{q}$ and the overlap integral is the product of factors like the first term on the right-hand side of Eq. (2.12). We write

$$W = \int dn_1 dn_2 dn_3 \delta \left(\frac{n_1^2 + n_2^2 + n_3^2}{2m} - \frac{q^2}{2mx} \right) \prod_{i=1}^3 |I_{n_i}|^2 \quad (2.19)$$

Making the change of variables $n_i = q(1-z_i)$ and proceeding as in Eq. (2.15), we then find, in the limit $q \rightarrow \infty$

$$v W \rightarrow \delta(1-x) \quad (2.20)$$

The spherical square well can also be worked out in principle but the calculation is not so explicit because of the presence of spherical Bessel functions. However we do not expect the qualitative features to be any different.

C. - More general confining potential

We work here with a one-dimensional confining potential $V(y)$. Given the remarks and methods above, extension to the Euclidean n dimensional case is straightforward. Since the Bjorken limit involves high excitations, we can use WKB wave functions. For convenience we write them here

$$\Psi_n(y) = \frac{C}{\sqrt{P(y)}} \sin \left(\int_b^y p dy' + \frac{\pi}{4} \right) = \frac{(-1)^n C}{\sqrt{P(y)}} \sin \left(\int_y^a p dy' + \frac{\pi}{4} \right)$$

$$P(y) = [2m(E - V(y))]^{1/2}, \quad C = \sqrt{\frac{4m}{T}}, \quad T = \int_a^b \frac{2m}{P(y)} dy \quad (2.21)$$

and a and b are the classical turning points. We need the overlap integral $\langle 0 | e^{iqy} | n \rangle = \int dy \psi_0^*(y) e^{iqy} \psi_n(y)$, where n is determined by energy conservation. The crucial feature which we require for the δ function behaviour of the modulus of this integral is provided by the ground state $\psi_0(y)$: namely ψ_0 is effectively confined to a region $-c < y < c$ (for convenience we have taken V even ; this is easily relaxed), where c is a number characterized by the particular well. So long as the well "spreads" from the low-lying levels - and from this point of view the square well is the worst case - we can expand $p(y)$ within the integral :

$$P(y) \approx \sqrt{2mE} - \frac{\tilde{V}}{2\sqrt{2mE}} \quad (2.22)$$

where $\tilde{V}$ is the maximum value of V in the region $-c < y < c$. Then

$$\langle 0 | e^{iqy} | n \rangle \approx \left(\frac{4m}{T(E)\sqrt{2mE}} \right)^{1/2} \int dy \psi_0^*(y) e^{iqy} \sin(\sqrt{2mE}y + \mathcal{C}). \quad (2.23)$$

$\mathcal{C}$ is a constant which supplies only a phase. Finally we need only note that in the region $-c \leq y \leq c$ where $\psi_0^*(y)$ is essentially non-zero, we can expand it in a Fourier series with only the first few terms contributing appreciably. (In general the evaluation of matrix elements by the WKB method and the above replacement require some care ⁴⁾ because of the possibility of cancellations due to rapid variation in the remainder of the integral. However, in the Bjorken limit the dominant contribution will come precisely from values of $q \sim \sqrt{2mE}$ which lead to little variation in the integrand and our procedure is justified.) The problem is then reduced to the square well with the initial state represented by a superposition of low-lying states. An analysis along the lines that lead to Eq. (2.22) gives now

$$|\langle 0 | e^{iqy} | n \rangle|^2 \approx \frac{2m}{T(E)\sqrt{2mE}} \cdot \frac{1}{q} \delta(1-x) \quad (2.24)$$

Therefore

$$\begin{aligned}
 W &= \sum_n |\langle 0 | e^{iqy} | n \rangle|^2 \delta(E - \frac{q^2}{2m_x}) \\
 &= \int dE \frac{dn}{dE} \frac{2m}{T(E)q^2} \delta(1-x) \delta(E - \frac{q^2}{2m_x})
 \end{aligned}
 \tag{2.25}$$

The level density dn/dE is found for large n , i.e., in the Bjorken limit by the quantization condition

$$\begin{aligned}
 n(E)\pi &= \int_{-c}^c \sqrt{2m(E - V(y))} dy \\
 \frac{dn}{dE} &= \frac{1}{2\pi} T(E)
 \end{aligned}
 \tag{2.26}$$

This factor just cancels the similar denominator factor in Eq. (2.19), leaving

$$\nu W = \delta(1-x)
 \tag{2.27}$$

the desired scaling result.

D. - Many-particle systems

We here comment briefly on the form of the structure function we found above. Defining the scaling variable by $x = \vec{q}^2 / (2m_{\text{tot}}\nu)$, we first imagine that rather than one particle in a well, we have two particles of mass m_1 and m_2 at positions r_1 and r_2 bound together. The weak probe introduces an operator $O = e^{iq \cdot r_1} \pm e^{iq \cdot r_2}$, depending on the charge. By straightforward separation of variables into centre-of-mass and relative co-ordinates we find (in the lab frame)

$$\nu W = \delta(x - \frac{4m_r}{4m_r + m_t})
 \tag{2.28}$$

where $m_r (m_t)$ is the reduced (total) mass. The δ function fixes x at $x = \frac{1}{2}$ for $m_1 = m_2$ (the maximum x) while for $m_1 \gg m_2$, $x \rightarrow (4m_2/m_1)$. For n equal mass particles bound with confining forces, $x_{\max} = 1/n$.

Interactions between particles in the system (spectator interactions) can also introduce changes in the form of the structure functions. Let us consider two particles in a well, positions r_1 and r_2 , and include now a further interaction between them,

$$V_{\text{tot}} = V(r_1) + V(r_2) + U(r_1 - r_2) \quad (2.29)$$

We could then imagine doing perturbation theory in $U(r_1 - r_2)$ for the wave function $\psi_{n_1}(r_1, r_2)$ which appears in $\langle \psi_0 | (e^{iq \cdot r_1} + e^{iq \cdot r_2}) | \psi_{n_1} \rangle$:

$$\psi_{n_1}(r_1, r_2) = \psi_{n_1}(r_1) \psi_{n_2}(r_2) + \sum_{m_1, m_2} \frac{\langle m_1, m_2 | U(r_1 - r_2) | n_1, n_2 \rangle}{E_{n_1} + E_{n_2} - E_{m_1} - E_{m_2}} \psi_{m_1}(r_1) \psi_{m_2}(r_2) \quad (2.30)$$

where $E_{n_1} + E_{n_2} = q^2/x$. ψ_0 would be expanded in a similar way.

An analysis of the resulting matrix elements can be carried out, to lowest order in the interaction U . We find terms which lead to a $\delta(1-x)$ form, and other terms which contain a spread in x . However, their contribution to vW falls off rapidly with q so that in the Bjorken limit the result of Eq. (2.22) is still valid.

3. - FIXED ANGLE SCATTERING

This is the only other hard process with a non-relativistic analogue. The interaction of two permanently bound systems is treated through the lowest order in the interaction, here taken to be a Coulomb-like interaction. If one imagines this to be an analogue to QCD, then the "colour neutralization" must take place through at least one more order of interaction. But this neutralization can occur through (and would be dynamically dominated by) an arbitrarily soft interaction which will by virtue

of its softness change none of the lowest order hard scattering results. (Of course it is presumably true that the complicated dynamics of QCD is responsible for the confinement itself. Our separation of a hard piece responsible for the scattering and its perturbative treatment is an analogue to the asymptotic freedom of QCD.)

Each system is taken as a two-particle system, the two particles of opposite charges : r_1 and r_2 (r_3 and r_4), charges $+e$ and $-e$, respectively, form bound system 1 (2), with total momentum $\vec{P}_1$ ($\vec{P}_2$) before the collision and $\vec{P}'_1$ ($\vec{P}'_2$) after. The scattering amplitude is

$$M \propto \langle \Psi_f | V | \Psi_i \rangle = \langle \Psi_{1f} \Psi_{2f} | V | \Psi_{1i} \Psi_{2i} \rangle \quad (3.1)$$

where the Coulomb-like interaction is in momentum space

$$V \sim \frac{1}{q^2} \left(e^{i\vec{q} \cdot (\vec{r}_1 - \vec{r}_3)} - e^{i\vec{q} \cdot (\vec{r}_1 - \vec{r}_4)} - e^{i\vec{q} \cdot (\vec{r}_2 - \vec{r}_3)} + e^{i\vec{q} \cdot (\vec{r}_2 - \vec{r}_4)} \right) \quad (3.2)$$

Making the usual variable separation

$$\begin{aligned} r_1 &= R_1 + \frac{1}{2} \rho_1 & r_2 &= R_1 - \frac{1}{2} \rho_1 \\ r_3 &= R_2 + \frac{1}{2} \rho_2 & r_4 &= R_2 - \frac{1}{2} \rho_2 \end{aligned} \quad (3.3)$$

we have

$$\begin{aligned} \Psi_{1i} &= e^{i\vec{P}_1 \cdot R_1} \varphi_i^{(0)}(\rho_1) & \Psi_{2i} &= e^{i\vec{P}_2 \cdot R_2} \varphi_2^{(0)}(\rho_2) \\ \Psi_{1f} &= e^{i\vec{P}'_1 \cdot R_1} \varphi_i^{(n)}(\rho_1) & \Psi_{2f} &= e^{i\vec{P}'_2 \cdot R_2} \varphi_2^{(n)}(\rho_2) \end{aligned} \quad (3.4)$$

We have taken the initial state to be the ground state (indicated by the superscript on the internal part of the wave function φ), although any fixed level will do. The final internal states are arbitrary subject to the condition ($\epsilon_n^{(i)}$ is the n^{th} energy level of system i)

$$Z = \sum_{i=1}^2 \left(\frac{1}{2m_i^{(i)}} \vec{P}_i^2 + \epsilon_i^{(i)} \right) - \sum_{i=1}^2 \left(\frac{1}{2m_i^{(i)}} \vec{P}_i'^2 \right) - \epsilon_n^{(1)} - \epsilon_m^{(2)} = 0 \quad (3.5)$$

It is straightforward to use (3.2)-(3.4) in (3.1) :

$$\begin{aligned} M = \frac{1}{2} \delta^{(3)}(\vec{P}_2 - \vec{P}_2' - \vec{q}) \int d^3 p_1 d^3 p_2 \varphi_1^{(n)*}(p_1) \varphi_2^{(m)*}(p_2) \\ \cdot \left[\cos \frac{1}{2} \vec{q} \cdot (\vec{p}_1 - \vec{p}_2) - \cos \frac{1}{2} \vec{q} \cdot (\vec{p}_1 + \vec{p}_2) \right] \varphi_1^{(o)}(p_1) \varphi_2^{(o)}(p_2) \end{aligned} \quad (3.6)$$

The integrals in this equation are all familiar ones, and we can use methods of Section 2. For large q we can use the WKB forms for the $\varphi_1^{(n)*}$ and $\varphi_2^{(m)*}$ factors, expanded as in Eq. (2.17) because $\varphi_1^{(o)}$ has a spatial restriction. This gives (one three-dimensional δ function has become a volume)

$$|M|^2 \propto \frac{1}{q^4} \delta^{(3)}(\vec{P}_2 - \vec{P}_2' - \vec{q}) \frac{m_r^{(1)}}{T_1 |P_1|} \frac{m_r^{(2)}}{T_2 |P_2|} \frac{1}{q} \delta(1-x_1) \frac{1}{q} \delta(1-x_2) \quad (3.7)$$

where we have defined the x_i by

$$\begin{aligned} \sqrt{2m_r^{(1)} \epsilon_n^{(1)}} &= |P_1| = x_1 q \\ \sqrt{2m_r^{(2)} \epsilon_m^{(2)}} &= |P_2| = x_2 q \end{aligned} \quad (3.8)$$

To compute the cross-section of the inclusive process we integrate over n and m and the $\vec{P}_1'$ subject to energy conservation :

$$d\sigma \propto \int d^3 P_1' d^3 P_2' \sum_{n,m} |M|^2 \delta(Z) \quad (3.9)$$

We have as in Section 2, Eq. (2.20)

$$\sum_{n,m} = \int d\epsilon^{(1)} d\epsilon^{(2)} T_1 T_2 = \int dx_1 dx_2 \frac{x_1 q^2}{m_r^{(1)}} \frac{x_2 q^2}{m_r^{(2)}} T_1 T_2 \quad (3.10)$$

The T_i factors cancel in Eq. (3.9) and the integrations over the x_i can be done with the $\delta(1-x_i)$ factors ; in addition $\int d^3 P_2' \delta^{(3)}(\vec{P}_2 - \vec{P}_2' - \vec{q}) = 1$.

$$d\sigma \propto \int d^3 P_1' \frac{1}{q^4} \delta \left(\frac{P_1'^2}{2m_t^{(1)}} + \frac{P_2'^2}{2m_t^{(2)}} - \frac{P_1'^2}{2m_t^{(1)}} - \frac{(P_2 - q)^2}{2m_t^{(2)}} - \frac{q^2}{2m_r^{(1)}} - \frac{q^2}{2m_r^{(2)}} \right) \quad (3.11)$$

We can compare this to the same process for free particles. It is simplest for this comparison to go to the lab frame, $\vec{P}_2 = 0$, and to use for the fixed angle process $|\vec{q}| \equiv y |\vec{P}_1|$. Equation (3.11) then takes the form

$$d\sigma \propto \int d^3 P_1' \frac{2m_t^{(1)}}{q^4} \delta(P_1'^2 + \xi^2 q^2) \quad (3.12)$$

where

$$\xi = \frac{m_t^{(1)}}{m_t^{(2)}} - \frac{1}{y} + m_t^{(1)} \left(\frac{1}{m_r^{(1)}} + \frac{1}{m_r^{(2)}} \right)$$

If, say, we study the scattering of m_1 and m_2 , free particles, then we find for the same cross-section the same form, with the trivial change $m_t^{(i)} \rightarrow m_i$ and the reduced mass terms of ξ absent. We have thus established that the bound particles scatter in this process as if they were free.

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