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GRAVITATIONAL INSTANTON CONTRIBUTION TO SPIN $3/2$ AXIAL ANOMALY

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ABSTRACT

We compute the index $I_{3/2}$ of the Rarita-Schwinger spin $3/2$ operator in the presence of the asymptotically flat (anti) self-dual gravitational instanton metric of Eguchi and Hanson. Using G index theory, we compute the Atiyah-Patodi-Singer boundary correction to the index and find that $I_{3/2} = \bar{7} 2$, confirming a less rigorous argument by Hawking and Pope.

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One of the interesting properties of the instanton solution¹⁾ of SU(2) Yang-Mills theory is its relationship to the axial anomaly. The Atiyah-Singer index theorem²⁾ says that the topological index k of the Yang-Mills instanton is precisely the number of positive minus negative chirality zero frequency modes of the Dirac equation in the presence of the instanton³⁾. It is therefore natural to think that a gravitational analogue of the Yang-Mills instanton might give a similar contribution to the spectrum of the Dirac equation. Now, Hawking's gravitational instanton⁴⁾, the self-dual Euclidean Taub-NUT metric, was shown⁵⁾ to give a vanishing Dirac index when the Atiyah-Patodi-Singer index theorem⁶⁾ for manifolds with boundary was properly applied. Another gravitational instanton, an asymptotically flat self-dual solution of Euclidean gravity, was then proposed by Eguchi and Hanson⁷⁾; but here, too, after the manifold was properly identified⁸⁾, a calculation by Atiyah⁹⁾ showed that the Dirac index was zero.

Recently, Hawking and Pope¹⁰⁾ have given a heuristic argument suggesting that gravitational instantons should contribute to the spin 3/2 Rarita-Schwinger axial anomaly rather than the spin 1/2 anomaly due to supersymmetry. Our purpose here is to provide a rigorous confirmation of this fact for the Rarita-Schwinger operator in the presence of the Eguchi-Hanson gravitational instanton. We also confirm the vanishing of the Dirac index for this metric.

We begin with the metric of Ref. 7),

$$ds^2 = dr^2 \left(1 - \left(\frac{a}{r} \right)^4 \right)^{-1} + r^2 \left(\sigma_x^2 + \sigma_y^2 + \left(1 - \left(\frac{a}{r} \right)^4 \right) \sigma_z^2 \right) \quad (1)$$

where

$$\begin{aligned} \sigma_x &= \frac{1}{2} (-\cos \psi \, d\theta - \sin \theta \sin \psi \, d\phi) \\ \sigma_y &= \frac{1}{2} (\sin \psi \, d\theta - \sin \theta \cos \psi \, d\phi) \\ \sigma_z &= \frac{1}{2} (-d\psi - \cos \theta \, d\phi) \end{aligned} \quad (2)$$

Geodesic completeness and regularity of the manifold require⁸⁾

$$\begin{aligned} 0 &\leq \theta \leq \pi \\ 0 &\leq \phi \leq 2\pi \\ 0 &\leq \psi \leq 2\pi \end{aligned} \tag{3}$$

As $r \rightarrow \infty$, the metric approaches a flat metric, except that the boundary of the manifold is $P_3(\mathbb{R}) = SO(3)$ instead of $S^3 = SU(2)$ as it would be for Euclidean space. As $r \rightarrow a$ the manifold is regular and reduces topologically to the sphere S^2 ; the Euler characteristic is therefore the same as that of S^2 ,

$$\chi = 2 \tag{4}$$

The Atiyah-Patodi-Singer index theorem⁶⁾ for manifolds with boundary expresses the index of the signature, Dirac, and Rarita-Schwinger operators, respectively, as

$$\begin{aligned} I_s &= \frac{1}{3} P_1 - \eta_s := \frac{1}{3} P_1 - \xi_s \\ I_{1/2} &= -\frac{1}{24} P_1 - \frac{1}{2} (\eta_{1/2} + h_{1/2}) := -\frac{1}{24} P_1 - \xi_{1/2} \\ I_{3/2} &= \frac{21}{24} P_1 - \frac{1}{2} (\eta_{3/2} + h_{3/2}) := \frac{21}{24} P_1 - \xi_{3/2} \end{aligned} \tag{5}$$

Here P_1 is the integral of the first Pontryagin class, expressed in terms of the Riemannian curvature. The curvature of (1) is anti-self-dual with the orientation (2), and one readily obtains

$$P_1 = -\frac{1}{8\pi^2} \int_M \text{tr} (R \wedge R) = -3 \tag{6}$$

For I_s and $I_{1/2}$ the coefficients of P_1 in Eq. (5) are well known from the expansion of the L genus and the $\hat{A}$ genus respectively. For $I_{3/2}$ the coefficient $+\frac{21}{24}$ has been evaluated in Refs. 13) and 11).

η denotes the appropriate η invariant⁶⁾ on the boundary and h is the dimension of the kernel of the operators, restricted to the boundary. The additional surface corrections required in Refs 5) and 6) are absent here because the metric falls off so rapidly as $r \rightarrow \infty$.

The boundary correction term ξ , which helps to render the index an integer, can be computed directly by means of the G index theorem^{2),12)}, an equivariant generalization of the Atiyah-Singer index theorem.

In our case the boundary at infinity is the three-dimensional real projective space $P_3(\mathbb{R})$. More generally, we can define three-dimensional lens spaces $L(\theta_1, \theta_2)$ in the following way:

Consider a cyclic group of order m with generator g_0 which acts on $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ such that g_0 affects rotations by angles θ_1 and θ_2 in the two summands $\mathbb{R}^2$. This action preserves the length of vectors, and we can thus restrict it to the unit disc

$$D^4 = \{x | x \in \mathbb{R}^4, x^2 \leq 1\}$$

and the unit sphere

$$S^3 = \{x | x \in \mathbb{R}^4, x^2 = 1\}$$

Then we define

$$L(\theta_1, \theta_2) = S^3/G \quad (7)$$

the quotient space of S^3 with respect to the action of G . The only fixed point of G is the origin.

One has, of course,

$$L(\theta_1, \theta_2) = \partial(D^4/G) \quad (8)$$

$P_3(\mathbb{R})$ is obtained as the special case $\theta_1 = \theta_2 = \pi$,

Atiyah, Patodi and Singer⁶⁾ showed that the ξ invariant of the signature operator on $L(\theta_1, \theta_2)$ is given by

$$\begin{aligned}\xi_s(\theta_1, \theta_2) &= \frac{1}{m} \sum_{g \neq 1} L_s(g, D^4) \\ &= -\frac{1}{m} \sum_{k=1}^{m-1} \cot \frac{1}{2} k \theta_1 \cot \frac{1}{2} k \theta_2\end{aligned}\quad (9)$$

$L_s(g, D^4)$ denotes the integrand of the G signature theorem, a well-determined calculable differential form on the fixed point set of g , which generalizes the ordinary integrand and occurring in the Atiyah-Singer index theorem.

For $P_3(\mathbb{R})$ one has

$$\xi_s = 0, \quad I_s = 0 \quad (10)$$

This reasoning can be carried over to the Dirac operator and the Rarita-Schwinger operator. In general, the analogue of Eq. (9) should give ξ only modulo integers for these cases, but because one only has to deal with the free Dirac and Rarita-Schwinger equation on D^4 , these integers are under control, too.

One obtains

$$\xi_{1/2, 3/2}(\theta_1, \theta_2) = \frac{1}{m} \sum_{g \neq 1} L_{1/2, 3/2}(g, D^4) \quad (11)$$

From the G index theorem^{2), 12)} one finds

$$L_{1/2, 3/2}(g, X) = \frac{i^* (ch_3 \Delta_{1/2, 3/2}^+ - ch_3 \Delta_{1/2, 3/2}^-)}{ch_3(\wedge_{-1} N^g \otimes \mathbb{C}) e(TX^g)} \chi_d(TX^g \otimes \mathbb{C}) [X^g] \quad (12)$$

Here X^g denotes the fixed point set of g in X , TX^g the tangent bundle of X^g , $TX^g \otimes \mathbb{C}$ its complexification, N^g the normal bundle of X^g in TX . i^* means restriction to X^g . $\Lambda_{-1}(N^g \otimes \mathbb{C})$ is the alternating sum of the external powers of the complexified normal bundle of X^g .

The generalized Chern character ch_g is defined as follows:

Consider a formal splitting of a complex bundle E into complex line bundles

$$E = \bigoplus L^j \quad (13.a)$$

and take $ch(L^j) = e^{y_j}$. Let g act on L^j by multiplication with $e^{i\alpha_j}$. Then the generalization ch_g of the Chern character ch is defined by

$$ch_g(E) = \sum_j e^{y_j + i\alpha_j} \quad (13.b)$$

For the Dirac operator the right-hand side of Eq. (12) can even be found in the literature ²⁾.

In our case X^g consists only of one point, and only the zero form part of Eq. (12) has to be kept.

The result is

$$L_{1/2}(g_0, D^4) = \frac{1}{4 \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2}} \quad (14.a)$$

hence

$$\sum_{1/2}(\theta_1, \theta_2) = \frac{1}{4m} \sum_{k=1}^{m-1} \frac{1}{\sin \frac{k\theta_1}{2} \sin \frac{k\theta_2}{2}} \quad (14.b)$$

In particular for $P_3(\mathbb{R})$

$$\xi_{1/2} = \frac{1}{8} \quad ; \quad \eta_{1/2} = \frac{1}{4} \quad (14.c)$$

So, the Dirac index becomes

$$I_{1/2} = \frac{1}{8} - \frac{1}{2} \left(\frac{1}{4} \right) = 0 \quad (15)$$

as found by Atiyah⁹⁾.

For spin 3/2 one can reduce the computation of $\xi_{3/2}$ to the spin 1/2 case by taking into account the Rarita-Schwinger bundle decomposition¹¹⁾

$$\Delta_{1/2}^{\pm} \otimes (TX \otimes \mathbb{C}) = \Delta_{3/2}^{\pm} \oplus \Delta_{1/2}^{\pm} \oplus \Delta_{1/2}^{\pm} \oplus \Delta_{1/2}^{\mp} \quad , \quad (16)$$

which exhibits the unphysical states, which have to be removed by imposing subsidiary conditions on the vector spinor. Following the reasoning of Ref. 11) and looking at Eq. (12) one notices that, due to the additivity and multiplicativity of ch_g one obtains the integrand for spin 3/2 from the spin 1/2 integrand by multiplying with $i^* ch_g (TX \otimes \mathbb{C}) - 1$, where the (-1) comes from the net subtraction of one $\Delta_{1/2}^{\pm}$ bundle, two ghosts of opposite chirality cancelling one another in the index.

Because X^g is zero dimensional one finds from Eq. (13)

$$\begin{aligned} i^* ch_g (TX \otimes \mathbb{C}) - 1 &= 2 \cos \theta_1 + 2 \cos \theta_2 - 1 \\ &= -5 \\ &\text{for } \theta_1 = \theta_2 = \pi \end{aligned} \quad (17)$$

Hence

$$\begin{aligned} \xi_{3/2}(\theta_1, \theta_2) &= \frac{1}{4m} \sum_{k=1}^{m-1} \frac{2 \cos k\theta_1 + 2 \cos k\theta_2 - 1}{\sin \frac{k\theta_1}{2} \sin \frac{k\theta_2}{2}} \\ &= -\frac{5}{8} \end{aligned} \tag{18}$$

for $P_3(\mathbb{R})$.

The spin $3/2$ index then becomes

$$I_{3/2} = -\frac{21}{8} - \frac{1}{2} \left(-\frac{5}{4}\right) = -2 \tag{19}$$

in agreement with Hawking and Pope¹⁰⁾.

A more detailed treatment including more general gravitational instanton metrics will be given elsewhere.

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