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THE INTERACTION ENERGY OF 't HOOFT MONOPOLES
IN THE PRASAD - SOMMERFIELD LIMIT

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ABSTRACT

In the Prasad-Sommerfield limit of vanishing Higgs mass, the size of 't Hooft monopoles increases when they interact with each other. For monopoles of like charges, their interaction energy decreases at least with the fourth power of the distance.

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We consider 't Hooft monopoles in a classical non-Abelian gauge theory with a massless Higgs field ϕ in the adjoint representation. The theory is defined by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu} + \frac{1}{2} D^\mu \varphi \cdot D_\mu \varphi \quad (1)$$

and the Prasad-Sommerfield condition

$$\lim_{r \rightarrow \infty} |\varphi(\vec{r})| = c. \quad (2)$$

The force between those monopoles is a good testing ground for the duality conjecture of Montonen and Olive¹⁾²⁾. In support of this conjecture, Manton made plausible that the Higgs field yields an attractive force independent of the monopole charge, which just cancels the magnetic repulsion in case of like charges³⁾. Manton assumed that between the monopoles the field configuration is the same as for Dirac monopoles, whereas in their neighbourhoods the modified Bogomolny equations

$$F_{ij} = \pm \varepsilon_{ijk} (D_k + a_k) \varphi, \quad i, j, k = 1, 2, 3, \quad (3)$$

are satisfied. Here a is the monopole acceleration. The plus sign applies for monopoles of positive charge, the minus sign for antimonopoles. Let h be the length of the Higgs field, and

$$\varphi = h \hat{\varphi}. \quad (4)$$

For M Dirac monopoles, F is parallel to ϕ everywhere,

$$D \hat{\varphi} = 0, \quad (5)$$

and

$$F_{ij} \cdot \hat{\varphi} = \varepsilon_{ijk} \partial_k \sum_{m=1}^M \frac{g_m}{4\pi |\vec{r} - \vec{r}_m|}. \quad (6)$$

Here $g_m = \pm 4\pi/e$ is the charge of the m^{th} monopole, and $\vec{r}_m$ is its position. For h one has

$$h = c - \sum_{m=1}^M \frac{1}{e|\vec{r} - \vec{r}_m|}. \quad (7)$$

The monopoles should be well separated compared to their radius of order $(ec)^{-1}$.

Inserting Eqs. (5)-(7) into Eq. (3), one obtains the accelerations. Manton's assumptions are reasonable, but it is difficult to prove that Eq. (3) has (even approximate) solutions.

The results of Manton were rederived by Goldberg et al.⁴⁾ They also assumed Eqs. (5)-(7) between the monopoles, but restricted themselves to configurations with

$$A_0 = \partial_0 \varphi = \partial_0 A_i = 0. \quad (8)$$

Then they calculated the energy-momentum tensor, and in particular the momentum flux through a surface surrounding a monopole.

This approach avoids the use of Eq. (3), but it suffers from the fact that Eq. (8) will only be valid for a particular time. Moreover, the Eqs. (5)-(7) can only be considered as a first approximation, as dipole and higher terms can be expected.

Only Magruder's method⁵⁾ allows one to study the forces between monopoles without any assumptions. He proposes to minimize the total energy keeping fixed the field $\hat{\phi}$ with prescribed singularities at the monopole positions $\vec{r}_m$. At the minimum, Eq. (8) can be imposed. Thus one can define the interaction energy of the monopoles as

$$E(\vec{r}_1, \dots, \vec{r}_M) = 4\pi M c/e + \min \int \left(\frac{1}{4} F_{ij} \cdot F_{ij} + \frac{1}{2} D_i \varphi \cdot D_i \varphi \right) dV. \quad (9)$$

If all monopoles have the same charge, a more convenient formula is

$$E = \min \frac{1}{4} \int (F_{ij} \pm \epsilon_{ijk} D_k \varphi)^2 dV, \quad (10)$$

where the sign is chosen according to the charge.

Magruder used trial configurations to obtain upper bounds on the interaction energy, but they did not take into account correctly the deformations of the monopoles. Consequently the bounds were rather crude. It is easy to see that the monopoles have to be deformed. Consider, for example, monopoles of light charge. Between them, Eqs. (5)-(8) yield the energy momentum tensor

$$\begin{aligned} T^{00} &= 2 T_m^{00}, \\ T^{i0} &= T^{ij} = 0 \quad \text{for } i, j = 1, 2, 3, \end{aligned} \quad (11)$$

where T_m^{00} is the magnetic energy density. Thus the integral over T^{00} between the monopoles decreases with the monopole distance, and for unchanged monopole mass one would expect a force twice as large as the magnetic repulsion alone. On the other hand, no momentum is transported. This difficulty vanishes if one notes that according to Eq. (7) the presence of further monopoles reduces the effective value of c near the m^{th} monopole to

$$c_m = c - \sum_{l \neq m} \frac{1}{e|\vec{r}_l - \vec{r}_m|}. \quad (12)$$

Thus the radius of the monopole will increase by a factor of c/c_m , and its mass will decrease by the same ratio. Because of this effect, the Higgs field yields an attraction.

Let us choose the trial configuration for Eqs. (9)-(10) accordingly. To simplify the notation, we write

$$\varphi = A_4 = D_4, \quad (13)$$

such that

$$D_i \varphi = F_{i4}. \quad (14)$$

Let A^m be a potential for an isolated monopole centred at $\vec{r}_m$, with scale c_m given by Eq. (12). Let A^0 be a potential for the Dirac field given by Eqs. (5)-(7), and use the trial function

$$A(\vec{r}) = \sum_{m=1}^M \alpha_m(\vec{r}) A^m(\vec{r}) + \left(1 - \sum_{m=1}^M \alpha_m(\vec{r})\right) A^0(\vec{r}), \quad (15)$$

where

$$\alpha_m(\vec{r}) = \alpha(\vec{r} - \vec{r}_m) \quad (16)$$

and

$$\alpha(\vec{r}) = \begin{matrix} 1 \\ (1 - \exp(-|r - r_a|^{-1})) \exp(-|r - r_b|^{-1}) \\ 0 \end{matrix} \left. \vphantom{\begin{matrix} 1 \\ (1 - \exp(-|r - r_a|^{-1})) \exp(-|r - r_b|^{-1}) \\ 0 \end{matrix}} \right\} \text{for } \begin{cases} r \leq r_a \\ r_a \leq r \leq r_b \\ r_b \leq r \end{cases} \quad (17)$$

r_a and r_b will be fixed later. The monopoles have to be sufficiently far apart, such that

$$\alpha_l \alpha_m = 0 \quad \text{for } l \neq m. \quad (18)$$

Then

$$F_{ij} = (1 - \sum_m \alpha_m) F_{ij}^0 + \sum_m \left(\alpha_m F_{ij}^m + (2i\alpha^m (A^m - A^0)_j - (i \leftrightarrow j)) + \alpha^m (\alpha^m - 1) (A^m - A^0)_i \times (A^m - A^0)_j \right). \quad (19)$$

Let s be the minimum distance between the monopoles. To study the behaviour of E for large s , we have to move r_a slowly out of the central regions of the monopoles, where the non-Abelian effects are important. Thus we put

$$r_b = r_a + 1, \quad (20)$$

and

$$r_a \sim s^\epsilon, \quad (21)$$

where ϵ may be arbitrarily small.

It is easy to choose gauges such that

$$A^m(\vec{r}) - A^0(\vec{r}) = O(s^{-2}) \text{ for } r_a \leq |\vec{r} - \vec{r}_m| \leq r_b. \quad (22)$$

The conventional choices for the A^m and A^0 in Refs. 2) and 3) are sufficient. If all monopoles have positive charge, F^0 and all the F^m are self-dual. Thus one obtains from Eq. (10)

$$E(\vec{r}_1, \dots, \vec{r}_M) = O(s^{-4+\epsilon}) \quad (23)$$

for arbitrarily small ϵ .

For monopoles of unlike charges one only obtains

$$E \leq \sum_{m \neq l} \frac{g_m g_l}{4\pi |\vec{r}_m - \vec{r}_l|} + O(s^{-2+\epsilon}). \quad (24)$$

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